[illegible]

Stewart, is scarce.
with Gregory's Geometry

3/-   24/.

Stewart's Theorems & Gregory's
Practical Geometry
by McLaurin.
in one vol.

SOME

GENERAL THEOREMS

Of considerable use in the

HIGHER PARTS

OF

MATHEMATICS.

By MATTHEW STEWART Minister at *Rosneath*.

EDINBURGH:

Printed by W. SANDS, A. MURRAY, and J. COCHRAN,

Sold by said W. SANDS in the Parliament-close, and by J. and P. KNAPTON, at the Crown in Ludgate street, *London*.

MDCCXLVI.

PREFACE.

THE theorems contained in the following sheets are given without being demonstrated, excepting the first five: And as they are entirely new, save one or two at most, the author expects their being published even in this way may be agreeable to those that are not unaccustomed to speculations of this kind. Such will easily allow, that to explain, in a proper way, so many theorems, so general, and of so great difficulty as most of these are, would require a greater expence of time and thought than can be expected soon from one in the author's situation. He therefore thought it was better they should appear in the way they now are, than lie by him till an uncertain hereafter. If any give themselves the trouble to explain some of these theorems, they will find their time and pains sufficiently rewarded, by the discovery of several new and curious propositions that otherwise might have escaped their observation.

EDINBURGH, Oct. 1.
1746.

Pag. lin. ERRATA

Pag.	lin.	
3	7	for [illegible] : read [illegible]
4	5	for [illegible] : read [illegible]
14	25	for [illegible], read [illegible]
17	20	for [illegible], read [illegible]
25	3	for [illegible] by the same number, read [illegible] by the number of the sides of the figure, together with [illegible] the multiple of the square of [illegible] by the same number
[illegible]	25	for [illegible], read [illegible] together with [illegible] the multiple of the square of [illegible] by the number of the sides of the figure, &
112	23	for [illegible], read [illegible]
119	14	for [illegible], read [illegible]
130	2	for the lines, read of the lines
139	1 & 3	for [illegible], read [illegible]
141	15	for [illegible] there, read [illegible] always taken at [illegible] : [illegible] the right lines given in position there
152	last	for 54, read 55

SOME

GENERAL THEOREMS

Of considerable use in

The higher parts of Mathematics.

PROPOSITION I. *Fig.* 1.

I*F from* A, *the vertex of any triangle* ABC, *there be drawn* AD *to any point* D *in the base, and* DE, DF *be drawn parallel to* AC, AB *meeting* AB, AC *in* E, F, *the sum of the rectangles* BAE, CAF *will be equal to the square of* AD *together ~~with the rectangle BDC.~~*

About the triangle ABC let there be a circle described, and let AD meet the circle in G; join BG, CG; and from the point E draw EH making the angle AHE equal to the angle ABG, and produce AB to any point K.

Because the angles AHE, ABG are equal, the points E, B, G, H are in a circle; therefore the rectangle BAE is equal to the rectangle GAH. The angle EHD will alſo be equal to the angle GBK, that is, to the angle ACG. And becauſe AC, DE are parallel, the angles EDH, GAC will be equal; therefore the triangles EDH, GAC will be ſimilar; and therefore AC will be to AG, as DH to DE: Therefore the rectangle contained by AC, DE, that is, the rectangle CAF, is equal to the rectangle contained by AG, DH. But becauſe the rectangle BAE is equal to the rectangle GAH, and likewiſe the rectangle CAF equal to the rectangle contained by AG, DH, the ſum of the rectangles BAE, CAF will be equal to the rectangle GAD, that is, equal to the rectangle ADG together with the ſquare of AD. But the rectangle ADG is [35. 3.] equal to the rectangle BDC; therefore the ſum of the rectangles BAE, CAF is equal to the ſquare of AD together with the rectangle BDC. *Q. E. D.*

PROPOSITION II. *Fig.* 2. 3.

In the right line AB *take any point* C *between the points* A, B; *and from the points* A, B, C *let there be drawn right lines to any*

any point D; *the ſquare of* AD *together with the ſpace to which the ſquare of* BD *has the ſame ratio that* BC *has to* CA, *will be equal to the rectangle* BAC *together with the ſpace to which the ſquare of* CD *has the ſame ratio that* BC *has to* BA.

1. When the point D [*Fig.* 1.] is not in the line AB.

Draw AE, DF parallel to CD, AB meeting BD, AE in E, F.

Becauſe the ſquare of BD is to the rectangle BDE as BD to DE, that is, as BC to CA, the rectangle BDE will be the ſpace to which the ſquare of BD has the ſame ratio that BC has to CA. And becauſe the ſquare of AF, that is, the ſquare of CD, is to the rectangle EAF as AF to AE, that is, as BD to BE, or BC to BA, the rectangle EAF will be the ſpace to which the ſquare of CD has the ſame ratio that BC has to BA. But [1.] the ſquare of AD together with the rectangle BDE, is equal to the rectangle BAC together with the rectangle EAF; therefore the ſquare of AD together with the ſpace to which the ſquare of BD has the ſame ratio that BC has to CA; is equal to the rectangle BAC together with the ſpace to which the ſquare of CD has the ſame ratio that BC has to BA. *Q. E. D.*

2. When the point D [*Fig.* 2.] is in the line AB,

Draw CE perpendicular to AB, and let CE be equal to AC; join AE, BE; draw BF parallel to CE meeting AE in F; and draw DG parallel to CE or BF meeting AE, BE in G, H; and join GC, HC.

Becauſe AC is equal to CE, AD will be equal to DG; therefore the ſquare of AD will be equal to twice the triangle ADG. And becauſe the ſquare of BD is to the rectangle BDH, that is, twice the triangle BDH, as BC to CE, or CA, twice the triangle BDH will be the ſpace to which the ſquare of BD has the ſame ratio that BC has to CA. Again, becauſe AC, CE are equal, the rectangle BAC will be equal to twice the triangle AEB; and becauſe EG is to EF, that is, CD to CB, as GH to BF, or AB, CD will be to GH as BC to AB: Therefore the ſquare of CD will be to the rectangle contained by CD, GH, that is, twice the triangle

angle GCH, or GEH, as BC to AB. Therefore twice the triangle GEH will be the ſpace to which the ſquare of CD has the ſame ratio that BC has to AB. But it is evident, that twice the ſum of the triangles ADG, BDH is equal to twice the ſum of the triangles AEB, GEH; therefore the ſquare of AD together with the ſpace to which the ſquare of BD has the ſame ratio that BC has to CA, is equal to the rectangle BAC together with the ſpace to which the ſquare of CD has the ſame ratio that BC has to AB. *Q. E. D.*

Corollary. If from the vertex of any triangle there be drawn a line biſecting the baſe, the ſum of the ſquares of the ſides of the triangle will be equal to twice the ſquare of the line biſecting the baſe together with the ſum of the ſquares of the ſegments of the baſe.

PROPOSITION III.

Theorem I. *Fig.* 4.

Let there be any regular figure ABC *circumſcribed about a circle, and from any point* D *within*

within the figure let there be drawn DE, DF, DG *perpendicular to the ſides of the figure; the ſum of the perpendiculars* DE, DF, DG *will be equal to the multiple of the ſemidiameter of the circle by the number of the ſides of the figure.*

Join DA, DB, DC. The figure ABC will be divided into as many triangles as there are ſides in the figure; and becauſe every one of the triangles is equal to half the rectangle contained by the baſe and the perpendicular drawn from the vertex to the baſe, and all the baſes are equal, becauſe the figure is regular; therefore the ſum of all the triangles will be equal to half the rectangle contained by the ſum of the perpendiculars and one of the ſides of the figure; and therefore twice the figure will be equal to the rectangle contained by the ſum of the perpendiculars and one of the ſides of the figure. But the rectangle contained by the ſemidiameter of the circle and the ſum of the ſides of the figure, is equal to twice the figure: Therefore the rectangle contained by the ſum of the perpendiculars DE, DF, DG and one of the ſides of the figure, is equal to the rect-angle

angle contained by the ſemidiameter of the circle and the ſum of the ſides of the figure; and therefore the ſum of the perpendiculars DE, DF, DG will be to the ſemidiameter of the circle, as the ſum of the ſides of the figure to one of the ſides of the figure, that is, as the number of the ſides of the figure to one. Therefore the ſum of the perpendiculars DE, DF, DG is equal to the multiple of the ſemidiameter of the circle by the number of the ſides of the figure. *Q. E. D.*

LEMMA I. *Fig.* 5.

Let there be any circle ABC, *and let* AD *be a tangent to the circle in the point* A; *from the point* A *let there be drawn* AB *to any point* B *in the circle, and let* BD *be perpendicular to* AD; *the ſquare of* AB *will be equal to the rectangle contained by* BD *and the diameter.*

Let AC be the diameter of the circle, and join BC. Becauſe the angles ACB, BAD are [32. 3.] equal, and the angles ABC, ADB likewiſe

likewiſe equal, becauſe both right, the triangles ABC, ADB will be ſimilar; therefore AC will be to AB as AB to BD: Therefore the ſquare of AB is equal to the rectangle contained by BD, AC. *Q. E. D.*

PROPOSITION IV.

THEOREM II. *Fig.* 6.7.

Let the circumference of a circle be divided into any number of equal parts in the points A, B, C, *&c. and from the points* A, B, C, *&c. let there be drawn right lines to any point* D; *the ſum of the ſquares of* AD, BD, CD, *&c. will be equal to the multiple of the ſquare of the line drawn from the centre of the circle to the point* D *by the number of the points* A, B, C, *&c. together with the multiple of the ſquare of the ſemidiameter by the ſame number.*

1. When the point D [*Fig.* 6.] is in the circumference of the circle, it is to be ſhewn, that the ſum of the ſquares of AD, BD, CD, *&c.* is equal to twice the multiple of the ſquare

ſquare of the ſemidiameter by the number of the points A, B, C, &c.

Let there be a regular figure circumſcribed about the circle, touching the circle in the points A, B, C, &c. and draw DE, DF, DG perpendicular to the ſides of the figure. Becauſe the ſquare of AD is [*Lem.* 1.] equal to the rectangle contained by DE and the diameter, and likewiſe the ſquare of BD equal to the rectangle contained by DF and the diameter, and ſo on; it is evident, that the ſum of the ſquares of AD, BD, CD, &c. will be equal to the rectangle contained by the ſum of the perpendiculars DE, DF, DG, &c. and the diameter. But becauſe [3.] the ſum of the perpendiculars DE, DF, DG, &c. is equal to the multiple of the ſemidiameter by the number of the ſides of the circumſcribed figure, that is, by the number of the points A, B, C, &c. the rectangle contained by the ſum of the perpendiculars DE, DF, DG, &c. and the diameter, will be equal to twice the multiple of the ſquare of the ſemidiameter by the number of the points A, B, C, &c. Therefore the ſum of the ſquares of AD, BD, CD, &c. will be equal to twice the multiple of the ſquare of the ſemidiameter by the number of the points A, B, C, &c. *Q. E. D.*

2. When the point D [*Fig.* 7.] is not in the circumference of the circle, it is to be ſhewn, that the ſum of the ſquares of AD, BD, CD, &c. is equal to the multiple of the ſquare of the line drawn from the centre of the circle to the point D by the number of the points A, B, C, &c. together with the multiple of the ſquare of the ſemidiameter by the ſame number.

Let E be the centre of the circle, and join DE; let DE meet the circle in the point F on the other ſide of the centre E, and join AE, BE, CE, &c. AF, BF, CF, &c. The ſquare of AD together with the ſpace to which the ſquare of AF has the ſame ratio that EF has to ED, will [2.] be equal to the rectangle EDF together with the ſpace to which the ſquare of AE, or EF; has the ſame ratio that EF has to FD, that is, together with the rectangle EFD: And therefore the ſquare of AD together with the ſpace to which the ſquare of AF has the ſame ratio that EF has to ED, will be equal to the ſquare of DF. The ſame way it is ſhown, that the ſquare of BD

BD together with the ſpace to which the ſquare of BF has the ſame ratio that EF has to ED, is equal to the ſquare of DF, and ſo on: Therefore the ſum of the ſquares of AD, BD, CD, &c. together with the ſpace to which the ſum of the ſquares of AF, BF, CF, &c. has the ſame ratio that EF has to ED, will be equal to the multiple of the ſquare of DF by the number of the points A, B, C, &c. But becauſe the ſum of the ſquares of AF, BF, CF, &c. is equal [by the firſt part of this] to twice the multiple of the ſquare of EF by the number of the points A, B, C, &c. the ſpace to which the ſum of the ſquares of AF, BF, CF, &c. has the ſame ratio that EF has to ED, will be equal to twice the multiple of the rectangle FED by the number of the points A, B, C, &c.: Therefore the ſum of the ſquares of AD, BD, CD, &c. together with twice the multiple of the rectangle FED by the number of the points A, B, C, &c. is equal to the multiple of the ſquare of DF by the ſame number: And therefore the ſum of the ſquares of AD, BD, CD, &c. is equal to the multiple of the ſum of the ſquares of DE, EF by the num-

 ber

ber of the points A, B, C, *&c.* *Q. E. D.*

Cor. I. Let there be two circles having the ſame centre, and let the circumference of one of the circles be divided into any number of equal parts, and from the points of diviſion let there be drawn right lines to any point in the circumference of the other, the ſum of the ſquares of theſe lines will always be the ſame.

Cor. II. Let there be two regular figures inſcribed in a circle, and from all the angles of both figures let there be drawn right lines to any point, the ſum of the ſquares of the lines drawn from the angles of the one, will be to the ſum of the ſquares of the lines drawn from the angles of the other, as the number of the ſides of the one to the number of the ſides of the other.

LEMMA II. *Fig.* 8, 9.

Let there be any number of right lines AB, AC, AD, AE, &c. *interſecting each other in the point* A, *and making all the angles about the point* A *equal; let there be any circle paſſing through the point* A; *the circumference*

cumference of the circle will be divided by the lines intersecting each other in the point A into as many equal parts as there are lines.

1. When the circle does not touch any of the lines intersecting each other in the point A. [*Fig.* 8.]

Let AB, AC, AD, AE, &c. meet the circle in B, C, D, E, &c. Because the angles BAC, CAD, DAE, &c. are equal, the segments BC, CD, DE, &c. will be equal. Let BE be the segment in which the point A is; draw BD, ED to any point D in the circle; the angle BDE will be equal to the angle adjacent to the angle BAE, that is, to the angle BAF, or BAC; therefore the segment BAE is equal to the segment BC.

2. When the circle touches one of the lines intersecting each other in the point A. [*Fig.* 9.]

Let it touch AB, and let AC, AD, AE meet the circle in C, D, E. Because the angle CAD is equal to the angle DAE, CD will be equal to DE, &c. Join CD; and because the angle ADC is equal to the angle CAB, that is, to the angle CAD, or DAE, the

the ſegment AC will be equal to the ſegment CD, or DE. The ſame way it is ſhown, that the ſegment AE is equal to the ſegment DE, or DC. Therefore the Lemma is evident. *Q. E. D.*

PROPOSITION V.

THEOREM III. *Fig.* 10. 11.

Let there be any regular figure circumſcribed about a circle, and from any point let there be drawn perpendiculars to the ſides of the figure, and likewiſe a right line to the centre of the circle; twice the ſum of the ſquares of the perpendiculars to the ſides of the figure, will be equal to the multiple of the ſquare of the line drawn to the centre by the number of the ſides of the figure, together with twice the multiple of the ſquare of the ſemidiameter by the ſame number.

1. When the number of the ſides of the figure circumſcribed about the circle is even, [*Fig.* 10.]

Let ABCDEF, &c. be any regular figure of an even number of ſides circumſcribed about circle, and from any point G let there be drawn GH, GK, GL, GM, GN, GO perpendicular to the ſides of the figure, and let *a* be the centre of the circle, and join G*a*; twice the ſum of the ſquares of GH, GK, GL, GM, GN, GO, &c. will be equal to the multiple of the ſquare of G*a* by the number of the ſides of the figure, together with twice the multiple of the ſquare of the ſemidiameter of the circle by the ſame number.

Let the circumſcribed figure touch the circle in P, Q, R, S, T, V, &c. and join GP, GQ, GR, GS, GT, GV, &c. join *a*P, *a*Q, *a*R, &c. and draw GX, GY, GZ, &c. perpendicular to *a*P, *a*Q, *a*R, &c.

Becauſe the number of the ſides of the circumſcribed figure is even, it is plain, that *a*P, *a*Q, *a*R, &c. will paſs through the oppoſite points of contact, that is, through the points S, T, V; and therefore the number of lines interſecting each other in the point *a* will be half the number of the ſides of the figure, and all the angles round the point *a* will be equal. Becauſe the ſum of the ſquares of GH, GX is equal to the ſquare of GP, and the ſum of the ſquares of GK, GY equal to the

the square of GQ, and so on; it is evident, that the sum of the squares of GH, GK, GL, GM, GN, GO, &c. together with twice the sum of the squares of GX, GY, GZ, &c. is equal to the sum of the squares of GP, GQ, GR, GS, GT, GV, &c, that is, [4.], equal to the multiple of the square of G*a* by the number of the sides of the figure together with the multiple of the square of the semidiameter of the circle by the same number: Therefore twice the sum of the squares of GH, GK, GL, GM, GN, GO, &c. together with four times the sum of the squares of GX, GY, GZ, &c. will be equal to twice the multiple of the square of G*a* by the number of the sides of the figure together with twice the multiple of the square of the semidiameter of the circle by the same number. Again, Because the angles GX*a*, GZ*a*, GY*a* are right, the points X, Y, Z will be in the circumference of the circle whose diameter is G*a*; and because the circle passes through the point *a*, the circumference will be divided into equal parts in the points X, Y, Z as many in number as there are right lines *a*P, *a*Q, *a*R, &c. [*Lem.* 2.]. Bisect G*a* in *b*; the sum of the squares of GX, GY, GY, GZ, &c. will [4.] be equal to twice the multiple of the ſquare of Gb by the number of the lines aP, aQ, aR, &c. that is, (becauſe the number of the lines aP, aQ, aR, &c. is equal to half the number of the ſides of the figure), equal to the multiple of the ſquare of Gb by the number of the ſides of the circumſcribed figure; and therefore four times the ſum of the ſquares of GX, GY, GZ, &c. will be equal to the multiple of the ſquare of aG by the number of the ſides of the figure. Therefore twice the ſum of the ſquares of GH, GK, GL, GM, GN, GO, &c. together with the multiple of the ſquare of Ga by the number of the ſides of the circumſcribed figure, will be equal to twice the multiple of the ſquare of Ga by the number of the ſides of the figure together with twice the multiple of the ſquare of the ſemidiameter by the ſame number: And therefore the ſum of the ſquares of GH, GK, GL, GM, GN, GO, &c. will be equal to the multiple of the ſquare of Ga by the number of the ſides of the figure together with twice the multiple of the ſquare of the ſemidiameter by the ſame number.

2. When the number of the ſides of the figure

gure circumſcribed about the circle is odd. [*Fig.* 11.]

Let ABCDE, *&c.* be any regular figure of an odd number of ſides circumſcribed about a circle, and from any point F let there be drawn FG, FH, FK, FL, FM, *&c.* perpendicular to the ſides of the figure, and let *a* be the centre of the circle, and join F*a*; twice the ſum of the ſquares of FG, FH, FK, FL, FM, *&c.* will be equal to the multiple of the ſquare of F*a* by the number of the ſides of the figure together with twice the multiple of the ſquare of the ſemidiameter of the circle by the ſame number.

Let the circumſcribed figure touch the circle in the points N, O, P, Q, R, *&c.* and join FN, FO, FP, FQ, FR, *&c.* join *a*N, *a*O, *a*P, *a*Q, *a*R, *&c.* and draw FS, FT, FV, FX, FY, *&c.* perpendicular to *a*N, *a*O, *a*P, *a*Q, *a*R, *&c.* Becauſe the ſum of the ſquares of FG, FS is equal to the ſquare of FN, and the ſum of the ſquares of FH, FT equal to the ſquare of FO, and ſo on; it is evident, that the ſum of the ſquares of FG, FH, FK, FL, FM, *&c.* together with the ſum of the ſquares of FS, FT, FV, FX, FY, *&c.* is equal to the ſum

ſum of the ſquares of FN, FO, FP, FQ, FR, &c. that is, [4.] equal to the multiple of the ſquare of F*a* by the number of the ſides of the figure together with the multiple of the ſquare of the ſemidiameter by the ſame number; therefore twice the ſum of the ſquares of FG, FH, FK, FL, FM, &c. together with twice the ſum of the ſquares of FS, FT, FV, FX, FY, &c. is equal to twice the multiple of the ſquare of F*a* by the number of the ſides of the figure together with twice the multiple of the ſquare of the ſemidiameter of the circle by the ſame number. Again, Becauſe the angles FS*a*, FT*a*, FV*a*, FX*a*, FY*a*, &c. are right, the points S, T, V, X, Y, &c. will be in the circumference of the circle whoſe diameter is F*a*; and becauſe the circle paſſes through the point *a*, and the lines *a*N, *a*O, *a*P, *a*Q, *a*R, &c. make all the angles round the point *a* equal, the circumference of the circle will be divided into equal parts in the points S, T, V, X, Y, &c. as many in number as there are right lines *a*N, *a*O, *a*P, *a*Q, *a*R, &c. [*Lem.* 2.]. Biſect F*a* in *b*; the ſum of the ſquares of FS, FT, FV, FX, FY, &c, will be equal to twice the multiple of the ſquare of F*b* by the number of the lines *a*N, *a*O, *a*P, *a*Q, *a*R, &c. [4.] that is, by the number of the fides of the circumfcribed figure; and therefore twice the fum of the fquares of FS, FT, FV, FX, FY, &c. will be equal to the multiple of the fquare of F*a* by the number of the fides of the figure. Therefore twice the fum of the fquares of FG, FH, FK, FL, FM, &c. together with the multiple of the fquare of F*a* by the number of the fides of the figure, will be equal to twice the multiple of the fquare of F*a* by the number of the fides of the figure together with twice the multiple of the fquare of the femidiameter of the circle by the fame number: And therefore twice the fum of the fquares of FG, FH, FK, FL, FM, &c. will be equal to the multiple of the fquare of F*a* by the number of the fides of the figure together with twice the multiple of the fquare of the femidiameter of the circle by the fame number. *Q. E. D.*

Cor. I. Let there be any regular figure circumfcribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the fides of the figure; twice the fum of the fquares of the perpendiculars

diculars will be equal to thrice the multiple of the square of the semidiameter of the circle by the number of the sides of the figure.

COR. II. Let there be two circles having the same centre, and from any point in the circumference of the one let there be drawn perpendiculars to the sides of any regular figure circumscribed about the other; the sum of the squares of these perpendiculars will always be the same.

COR. III. Let there be two regular figures circumscribed about a circle, and from any point let there be drawn perpendiculars to the sides of both figures; the sum of the squares of the perpendiculars drawn to the sides of the one, will be to the sum of the squares of the perpendiculars drawn to the sides of the other, as the number of the sides of the one to the number of the sides of the other.

PROPOSITION VI. *Fig.* 12. 13.

Let A, B *be two points in the semidiameter of a circle whose centre is* C, *and let the rectangle* ACB *be equal to the square of the semidiameter; bisect* AB *in* D, *and draw*

DE perpendicular to AB; *from the point* A *draw* AF *to any point* F *in the circle, and draw* FE *perpendicular to* DE; *the ſquare of* AF *will be equal to twice the rectangle contained by* AC, FE.

Let CG be equal to AC, and join GF; let EF meet the circle in H, and join AH, GH, AE, CE, CF, CH; and let CE meet the circle in K, L.

The ſquare of CD is equal to the rectangle ACB together with the ſquare of AD, that is, equal to the ſquare of the ſemidiameter together with the ſquare of AD. Add the ſquare of DE to both; and the ſquare of CE will be equal to the ſquare of the ſemidiameter together with the ſquare of AE. Take away the ſquare of the ſemidiameter from both; and the ſquare of AE will be equal to the rectangle KEL, that is, equal to the rectangle FEH: And therefore FE is to AE, as AE to EH; Therefore the triangles AEF, AHE are ſimilar, and the angle EAF will be equal to the angle AHE, that is, equal to the angle HAG. Again, Becauſe the angle ACF is equal to the angle CFH, that is, equal to the

the angle CHF, or GCH, the angle ACH will be equal to the angle GCF; and becauſe AC, CH are equal to GC, CF, the triangles ACH, GCF will be every way equal, and the angle CGF will be equal to the angle HAG, that is, equal to the angle EAF; and becauſe the angles EFA, FAG are equal, the triangles AEF, FAG will be ſimilar; and therefore EF will be to AF, as AF to AG: Therefore the ſquare of AF is equal to the rectangle contained by EF, AG, that is, equal to twice the rectangle contained by FE, AC. *Q. E. D.*

PROPOSITION. VII.

THEOREM IV. *Fig.* 14. 15.

Let there be any circle whoſe centre is A, *and let* BCD *be a ſegment of the circle, and* BD *the chord of the ſegment; about the ſegment let there be any equilateral figure circumſcribed touching the circle in the points* E, F, G, &c. *and let the two ſides of the figure next to* BD *meet* BD *in* H, K; *biſect the ſegment* BCD *in* F, *and join* AF; *in* AF *take the point* L *on the ſame ſide the*

centre

centre A *with the point* F, *and let the sum of the sides of the figure circumscribed about the segment be to* HK *as the semi-diameter to* AL; *draw* LM *perpendicular to* AL *meeting the circle in* M. *If from the points* E, F, G, &c. *the points of contact of the circumscribed figure, and the point* L, *there be drawn right lines to any point* N, *the sum of the squares of* EN, FN, GN, &c. *will be equal to the multiple of the sum of the squares of* LM, LN *by the number of the sides of the figure.*

1. When the point N is in the circumference of the circle. [*Fig.* 14.]

In AF take the point O, and let the rectangle LAO be equal to the square of the semidiameter of the circle, and let OP be perpendicular to AF; draw NP perpendicular to OP; bisect LO in Q, and let QR parallel to OP meet NP in R; let NP, AO meet BD in S, T, and join AH, AK, NH, NK; and join likewise AM; and draw NV, NX, NY, &c. perpendicular to the sides of the figure meeting the sides of the figure in V, X, Y, &c.

Because

24

Fig. 9

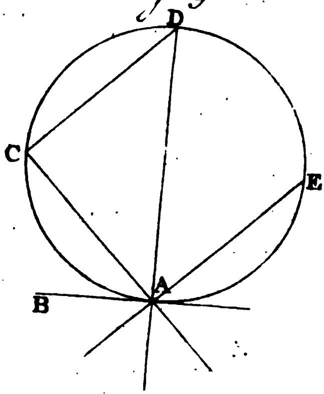
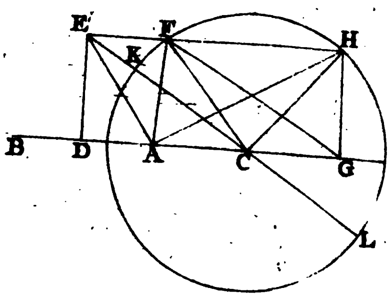


Fig. 12



Becauſe the rectangle LAO is equal to the ſquare of the ſemidiameter of the circle, that is, equal to the ſquare of AF; AO will be to AF, as AF to AL, that is, as the ſum of the ſides of the figure circumſcribed about the ſegment to HK; and therefore the rectangle contained by AO, HK will be equal to the rectangle contained by AF and the ſum of the ſides of the figure, that is, will be equal to twice the figure AHEFGKA. And becauſe the rectangle contained by AT, HK is equal to twice the triangle AHK, the rectangle contained by OT, HK will be equal to twice the figure HEFGKH, that is, the rectangle contained by PS, HK will be equal to twice the figure HEFGKH. Again, Becauſe the rectangle contained by NS, HK is equal to twice the triangle NHK, the rectangle contained by NP, HK will be equal to twice the figure NHEFGKN. But the rectangle contained by the ſum of the perpendiculars NV, NX, NY, &c. and one of the ſides of the figure, is equal to twice the figure NHEFGKN; therefore the rectangle contained by NP, HK is equal to the rectangle contained by the ſum of NV, NX, NY, &c.

and one of the ſides of the figure: And therefore NP will be to one of the ſides of the figure, as the ſum of the perpendiculars NV, NX, NY to HK, that is, the multiple of NP by the number of the ſides of the figure, will be to the ſum of the ſides of the figure, as the ſum of the perpendiculars NV, NX, NY, &c. to HK: Therefore the multiple of NP by the number of the ſides of the figure, will be to the ſum of the perpendiculars NV, NX, NY, as the ſum of the ſides of the figure to HK, that is, as AF to AL, or twice AF to twice AL: Therefore twice the multiple of the rectangle contained by NP, AL by the number of the ſides of the figure, is equal to the rectangle contained by the ſum of the perpendiculars NV, NX, NY, &c. and twice AF. But becauſe [*Lem.* 1.] the ſquare of NE is equal to the rectangle contained by NV and twice AF, and the ſquare of NF equal to the rectangle contained by NX and twice AF, and the ſquare of NG equal to the rectangle contained by NY and twice AF, and ſo on; the ſum of the ſquares of NE, NF, NG, &c. will be equal to the rectangle contained by the ſum of the perpendiculars

NV,

NV, NX, NY, &c. and twice AF, that is, will be equal to twice the multiple of the rectangle contained by NP, AL by the number of the sides of the figure. Again, Because the rectangle OAL is equal to the square of AM, the rectangle OLA will be equal to the square of LM, that is, twice the rectangle contained by PR, AL will be equal to the square of LM. And because [6.] twice the rectangle contained by NR, AL is equal to the square of LN, twice the rectangle contained by NP, AL will be equal to the sum of the squares of LM, LN: Therefore twice the multiple of the rectangle contained by NP, AL by the number of the sides of the figure, will be equal to the multiple of the sum of the squares of LM, LN by the same number: And therefore the sum of the squares of NE, NF, NG, &c. will be equal to the multiple of the sum of the squares of LM, LN by the number of the sides of the figure. *Q. E. D.*

2. When the point N is not in the circumference of the circle. [*Fig.* 15.]

Join NA, and let NA meet the circle in the point O on the other side the centre A with the point N; and join EO, FO, GO, &c. LO; join likewise EA, FA, GA, &c. and join AM; let LO meet the circle in P, and draw NQ parallel to AL meeting OL in Q.

Because [2.] the ſquare of EN together with the ſpace to which the ſquare of EO has the ſame ratio that OA has to AN, is equal to the rectangle ANO together with the ſpace to which the ſquare of AE has the ſame ratio that AO has to ON, and the ſquare of AE is equal to the ſum of the ſquares of AL, LM; the ſquare of EN together with the ſpace to which the ſquare of EO has the ſame ratio that OA has to AN, will be equal to the rectangle ANO together with the ſpace to which the ſquare of AL has the ſame ratio that AO has to ON together with the ſpace to which the ſquare of LM has the ſame ratio. But [2.] the rectangle ANO together with the ſpace to which the ſquare of AL has the ſame ratio that AO has to NO, is equal to the ſquare of NL together with the ſpace to which the ſquare of LO has the ſame ratio that OA has to AN. Therefore the ſquare of EN together with the ſpace to which the

ſquare

ſquare of EO has the ſame ratio that OA has to AN, is equal to the ſquare of NL together with the ſpace to which the ſquare of LO has the ſame ratio that OA has to AN together with the ſpace to which the ſquare of LM has the ſame ratio that OA has to ON. Becauſe the ſquare of LO is to the rectangle OLQ as OL to LQ, that is, as OA to AN, the rectangle OLQ will be the ſpace to which the ſquare of OL has the ſame ratio that OA has to AN. And becauſe the rectangle OLP is to the rectangle contained by LP, OQ as OL to OQ, that is, as OA to ON, and the ſquare of LM is equal to the rectangle OLP, the ſquare of LM will be to the rectangle contained by LP, OQ as OA to ON: Therefore the rectangle contained by LP, OQ will be the ſpace to which the ſquare of LM has the ſame ratio that OA has to ON. And therefore the ſquare of EN together with the ſpace to which the ſquare of EO has the ſame ratio that OA has to AN, is equal to the ſquare of NL together with the rectangle OLQ together with the rectangle contained by LP, OQ. But becauſe the rectangle contained by LP, OQ is equal to the rectangle OLP together

ther with the rectangle contained by LP, LQ, therefore the rectangle OLQ together with the rectangle contained by LP, OQ is equal to the rectangle OLP together with the rectangle contained by OP, LQ, that is, equal to the ſquare of LM together with the rectangle contained by OP, LQ. Therefore the ſquare of NE together with the ſpace to which the ſquare of EO has the ſame ratio that OA has to AN, is equal to the ſum of the ſquares of LM, LN together with the rectangle contained by OP, LQ. The ſame way it is ſhewn, that the ſquare of FN together with the ſpace to which the ſquare of FO has the ſame ratio that OA has to AN, is equal to the ſum of the ſquares of LM, LN together with the rectangle contained by OP, LQ; and likewiſe, that the ſquare of GN together with the ſpace to which the ſquare of GO has the ſame ratio that OA has to AN, is equal to the ſum of the ſquares of LM, LN together with the rectangle contained by OP, LQ; and ſo on. Therefore the ſum of the ſquares of EN, FN, GN, *&c.* together with the ſpace to which the ſum of the ſquares of EO, FO, GO, *&c.* has the ſame ratio that OA has to

AN,

AN, is equal to the multiple of the ſum of the ſquares of LM, LN by the number of the ſides of the figure together with the multiple of the rectangle contained by OP, LQ by the ſame number.

Again, Becauſe the ſum of the ſquares of EO, FO, GO, *&c.* is (by the firſt part of this) equal to the multiple of the ſum of the ſquares of LM, LO by the number of the ſides of the figure, and the ſquare of LM is equal to the rectangle OLP; the ſum of the ſquares of EO, FO, GO, *&c.* will be equal to the multiple of the rectangle LOP by the number of the ſides of the figure. And becauſe OA is to AN as OL to LQ, that is, as the rectangle LOP to the rectangle contained by OP, LQ, that is, as the multiple of the rectangle LOP by the number of the ſides of the figure to the multiple of the rectangle contained by OP, LQ by the ſame number, and the ſum of the ſquares of EO, FO, GO, *&c.* is equal to the multiple of the rectangle LOP by the number of the ſides of the figure; the ſum of the ſquares of EO, FO, GO, *&c.* will be to the multiple of the rectangle contained by OP, LQ as OA to AN. And therefore the mul-

tiple

tiple of the rectangle contained by OP, LQ by the number of the sides of the figure, will be the space to which the sum of the squares of EO, FO, GO, &c. has the same ratio that OA has to AN: Therefore the sum of the squares of EN, FN, GN, &c. together with the multiple of the rectangle contained by OP, LQ by the number of the sides of the figure, is equal to the multiple of the sum of the squares of LM, LN by the number of the sides of the figure together with the multiple of the rectangle contained by OP, LQ by the same number. And therefore the sum of the squares of EN, FN, GN, &c. will be equal to the multiple of the sum of the squares of LM, LN by the number of the sides of the figure. *Q. E. D.*

PROPOSITION VIII.

THEOREM V. *Fig.* 16.

Let there be any circle whose centre is A, *and let* BCD *be a semicircle, and* BD *the diameter of the circle; about the semicircle let there*

there be any regular figure described, and let the sides of the figure next to BD *meet* BD *in* E, F; *bisect the semicircle in* G, *and join* AG; *and in* AG *take the point* H *on the same side the centre* A *with the point* G, *and let* AG *be to* AH *as the sum of the sides of the figure to* EF; *and let the rectangle* HAK *be equal to the square of the semidiameter, and let* HL *be equal to* AH: *If from any point* M *there be drawn* MN, MO, MP, &c. *perpendicular to the sides of the figure circumscribed about the semicircle, and likewise there be drawn* ML *to the point* L; *twice the sum of the squares of the perpendiculars* MN, MO, MP, &c. *will be equal to the multiple of the square of* ML *by the number of the sides of the figure together with the multiple of the rectangle* KLA *by the same number.*

Let the figure touch the semicircle in the points Q, R, S, *&c.* and join AQ, AR, AS, *&c.*; draw MT, MV, MX, *&c.* perpendicular to AQ, AR, AS; join MA, MH, and draw HY perpendicular to AH meeting the circle in Y, and join MQ, MR, MS, &c.

Becauſe [7.] the ſum of the ſquares of MQ, MR, MS, &c. is equal to the multiple of the ſum of the ſquares of HM, HY by the number of the ſides of the figure circumſcribed about the ſemicircle, twice the ſum of the ſquares of MQ, MR, MS, &c. will be equal to twice the multiple of the ſum of the ſquares of HM, HY by the number of the ſides of the figure: Therefore twice the ſum of the ſquares of MQ, MR, MS, &c. together with twice the multiple of the ſquare of AH by the number of the ſides of the figure, is equal to twice the multiple of the ſum of the ſquares of HM, HA by the number of the ſides of the figure together with twice the multiple of the ſquare of HY by the ſame number. And becauſe twice the ſum of the ſquares of HM, HA is equal to the ſum of the ſquares of ML, MA, twice the multiple of the ſum of the ſquares of HM, HA by the number of the ſides of the figure, will be equal to the multiple of the ſum of the ſquares of ML, MA by the ſame number: Therefore twice the ſum of the ſquares

ſquares of MQ, MR, MS, &c. together with twice the multiple of the ſquare of AH by the number of the ſides of the figure, is equal to the multiple of the ſum of the ſquares of ML, MA by the ſame number. But becauſe twice the ſum of the ſquares of MN, MO, MP, &c. together with twice the ſum of the ſquares of MT, MV, MX, &c. is equal to twice the ſum of the ſquares of MQ, MR, MS, &c.; therefore twice the ſum of the ſquares of MN, MO, MP together with twice the ſum of the ſquares of MT, MV, MX, &c. is equal to the multiple of the ſum of the ſquares of ML, MA by the number of the ſides of the figure together with twice the multiple of the ſquare of HY by the ſame number.

Again, Becauſe the angles MTA, MVA, MXA are right, the points T, V, X will be in the circumference of the circle whoſe diameter is AM; and becauſe AQ, AR, AS, &c. make all the angles about the point A equal, the circumference of this circle will be divided into equal parts in the points T, V, X, &c. [*Lem.* 2.] as many in number as there are lines AQ, AR, AS, &c. that is, into as many equal parts as there are ſides in the circumſcribed figure: Therefore twice the ſum of the ſquares of MT, MV, MX, *&c.* will be equal to the multiple of the ſquare of MA by the number of the ſides of the figure: Therefore twice the ſum of the ſquares of MN, MO, MP, *&c.* together with the multiple of the ſquare of MA by the number of the ſides of the figure together with twice the multiple of the ſquare of AH by the ſame number, is equal to the multiple of the ſum of the ſquares of ML, MA by the number of the ſides of the figure together with twice the multiple of the ſquare of HY by the ſame number: And therefore twice the ſum of the ſquares of MN, MO, MP, *&c.* together with twice the multiple of the ſquare of AH by the number of the ſides of the figure, is equal to the multiple of the ſquare of ML by the number of the ſides of the figure together with twice the multiple of the ſquare of HY by the ſame number.

Again, Becauſe the rectangle HAK is equal to the ſquare of the ſemidiameter of the circle, that is, equal to the ſum of the ſquares of AH, HY; the rectangle KHA, that is, the

rectangle

rectangle KHL, will be equal to the ſquare of HY: And therefore twice the multiple of the rectangle KHL by the number of the ſides of the figure, will be equal to twice the multiple of the ſquare of HY by the ſame number: Therefore twice the ſum of the ſquares of MN, MO, MP, &c. together with twice the multiple of the ſquare of AH or HL by the number of the ſides of the figure, is equal to the multiple of the ſquare of ML by the number of the ſides of the figure together with twice the multiple of the rectangle KHL by the ſame number. Therefore twice the ſum of the ſquares of MN, MO, MP, &c. is equal to the multiple of the ſquare of ML by the number of the ſides of the figure together with the multiple of the rectangle KLA by the ſame number. *Q. E. D.*

Cor. Let there be any equilateral figure inſcribed in a ſemicircle; a point is given, ſuch, that if from any point there be drawn perpendiculars to the ſides of the figure, and likewiſe a right line to the given point, twice the ſum of the ſquares of the perpendiculars will be equal to the multiple of the ſquare of the line drawn to the given

point

point by the number of the ſides of the figure together with a given ſpace.

Let ABCD [*Fig.* 17.] be an equilateral figure inſcribed in a ſemicircle; let AD be the diameter, and F the centre; biſect the ſemicircle in G, and join FG; let FH be perpendicular to AB one of the ſides of the figure; in FG take the point K, and let FH be to FK as the ſum of the ſides of the figure ABCD to AD; let KL be equal to FK, and let the rectangle KFM be equal to the ſquare of FH: If from any point N there be drawn NO, NP, NQ, *&c.* perpendicular to AB, BC, CD, *&c.* the ſides of the figure, and likewiſe there be drawn NL to the point L, twice the ſum of the ſquares of NO, NP, NQ, *&c.* will be equal to the multiple of the ſquare of NL by the number of the ſides of the figure together with the multiple of the rectangle MLF by the ſame number.

The ſecond and fourth theorems are but particular caſes of one more general; which is this.

PRO-

PROPOSITION IX.

THEOREM VI.

Let there be any number of given points; a point may be found, ſuch, that if from all the given points there be drawn right lines to the point found, and from all the given points and the point found there be drawn right lines to any point, the ſum of the ſquares of the lines drawn from the given points, will be equal to the ſum of the ſquares of the lines drawn from the given points to the point found together with the multiple by the number of the given points of the ſquare of the line drawn from the point found.

For example: Let the number of the given points be three, and the theorem will be as follows.

Let there be three given points; a point may be found, ſuch, that if from the three given points there be drawn right lines to the point found, and from the three given points and the point found there be drawn right lines to any point, the ſum of the ſquares of the lines drawn from the three given points, will be equal to the ſum of the ſquares of the lines drawn from the three given points to the point found together with thrice the ſquare of the line drawn from the point found.

This theorem may be made more general thus.

PROPOSITION X.

THEOREM VII.

Let there be any number of given points A, B, C, &c. [Fig. 18.] *and let* a, b, c, &c. *be given magnitudes as many in number as there are given points; a point* X *may be found, ſuch, that if from the given points* A, B, C, &c. *there be drawn right lines to the point* X, *and from the given points and the point* X *there be drawn right lines to any point* Y, *the ſquare of* AY, *together with the ſpace to which the ſquare of* BY *has the ſame ratio that* a *has to* b, *together with the ſpace to which the ſquare of* CY *has the ſame*

same ratio that a has to c, and so on; will be equal to the square of AX, *together with the space to which the square of* BX *has the same ratio that a has to b, together with the space to which the square of* CX *has the same ratio that a has to c, and so on, together with the space to which the square of* XY *has the same ratio that a has to the sum of a, b, c,* &c.

Let the number of the given points be three, and let *a*, *b*, *c* be equal to 1, 2, 3; and the theorem will be as follows.

Let A, B, C be three given points; a point X may be found, such, that if from the points A, B, C there be drawn right lines to the point X, and from the points A, B, C and the point X there be drawn right lines to any point Y, the square of AY together with twice the square of BY together with thrice the square of CY, will be equal to the square of AX together with twice the square of BX together with thrice the square of CX together with six times the square of XY.

Cor. I. Let there be any number of circles given by position, and about each of the circles

let there be an equilateral figure circumſcribed; a point may be found, ſuch, that if from any point there be drawn perpendiculars to the ſides of all the figures, and likewiſe there be drawn a right line to the point found, twice the ſum of the ſquares of the perpendiculars will be equal to the multiple of the ſquare of the line drawn to the point found by the number of the ſides of all the figures together with a given ſpace.

Suppoſe, for example, two circles to be given by poſition, and about one of the circles let there be an equilateral triangle circumſcribed, and about the other let there be a ſquare circumſcribed; a point may be found, ſuch, that if from any point there be drawn perpendiculars to the ſides of the triangle and the ſquare, and likewiſe there be drawn a right line to the point found, twice the ſum of the ſquares of the perpendiculars will be equal to ſeven times the ſquare of the line drawn to the point found together with a given ſpace.

Cor. II. Let there be any number of ſemicircles given by poſition, and about each of the ſemicircles let there be an equilateral figure circumſcribed; a point may be found, ſuch,


E
D

let there be an equilateral figure circumſcribed; a point may be found, ſuch, that if from any point there be drawn perpendiculars to the ſides of all the figures, and likewiſe there be drawn a right line to the point found, twice the ſum of the ſquares of the perpendiculars will be equal to the multiple of the ſquare of the line drawn to the point found by the number of the ſides of all the figures together with a given ſpace.

Suppoſe, for example, two circles to be given by poſition, and about one of the circles let there be an equilateral triangle circumſcribed, and about the other let there be a ſquare circumſcribed; a point may be found, ſuch, that if from any point there be drawn perpendiculars to the ſides of the triangle and the ſquare, and likewiſe there be drawn a right line to the point found, twice the ſum of the ſquares of the perpendiculars will be equal to ſeven times the ſquare of the line drawn to the point found together with a given ſpace.

Cor. II. Let there be any number of ſemicircles given by poſition, and about each of the ſemicircles let there be an equilateral figure circumſcribed; a point may be found,

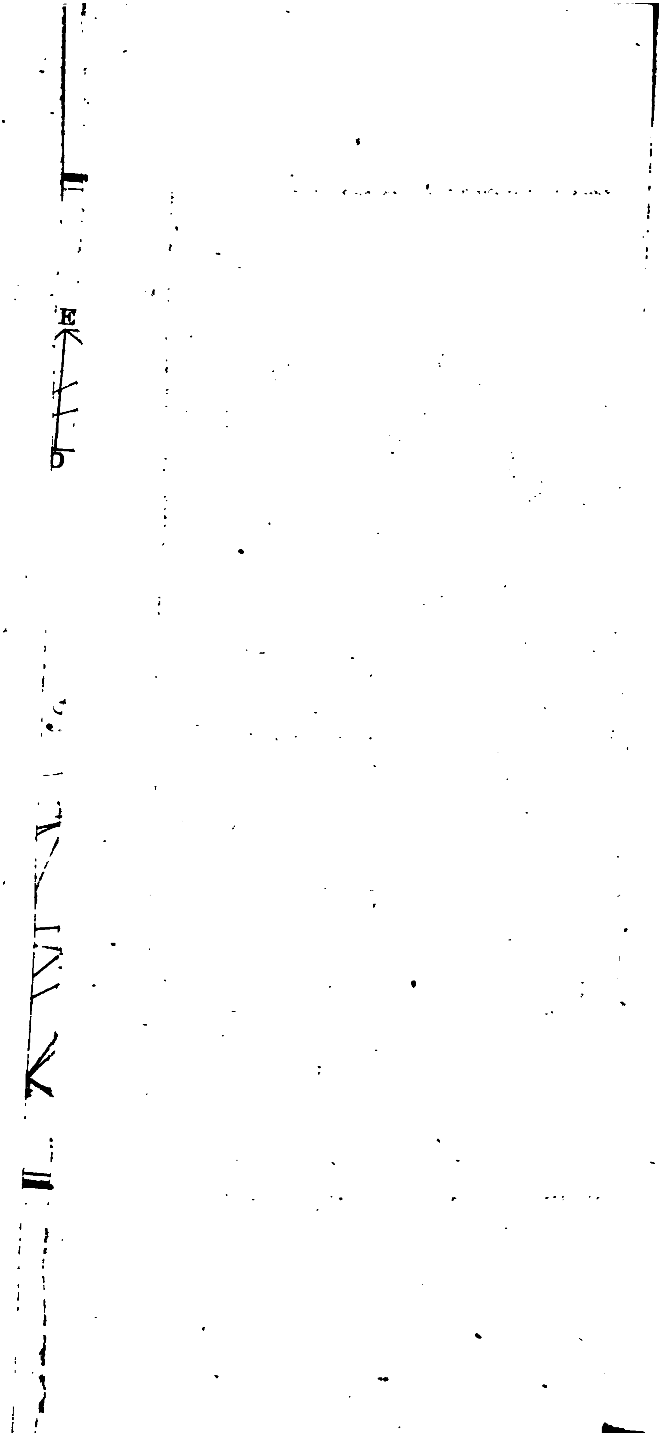
ſuch,


E
D

let there be an equilateral figure circumſcribed; a point may be found, ſuch, that if from any point there be drawn perpendiculars to the ſides of all the figures, and likewiſe there be drawn a right line to the point found, twice the ſum of the ſquares of the perpendiculars will be equal to the multiple of the ſquare of the line drawn to the point found by the number of the ſides of all the figures together with a given ſpace.

Suppoſe, for example, two circles to be given by poſition, and about one of the circles let there be an equilateral triangle circumſcribed, and about the other let there be a ſquare circumſcribed; a point may be found, ſuch, that if from any point there be drawn perpendiculars to the ſides of the triangle and the ſquare, and likewiſe there be drawn a right line to the point found, twice the ſum of the ſquares of the perpendiculars will be equal to ſeven times the ſquare of the line drawn to the point found together with a given ſpace.

Cor. II. Let there be any number of ſemicircles given by poſition, and about each of the ſemicircles let there be an equilateral figure circumſcribed; a point may be found, ſuch,


E
D

ſuch, that if from any point there be drawn perpendiculars to the ſides of all the figures, and likewiſe there be drawn a right line to the point found, twice the ſum of the ſquares of the perpendiculars will be equal to the multiple of the ſquare of the line drawn to the point found by the number of the ſides of all the figures together with a given ſpace.

Suppoſe, for example, two ſemicircles to be given by poſition, and about each of the ſemicircles let there be an equilateral figure circumſcribed, and let the number of the ſides of the one be three, and the number of the ſides of the other be four; a point may be found, ſuch, that if from any point there be drawn perpendiculars to the ſides of both the figures, and likewiſe there be drawn a right line to the point found, twice the ſum of the ſquares of the perpendiculars drawn to the ſides of the figures, will be equal to ſeven times the ſquare of the line drawn to the point found together with a given ſpace.

Cor. III. Let there be any number of circles given by poſition, and likewiſe any number of ſemicircles given by poſition; and about each of the circles let there be an equilateral figure circumſcribed, and likewiſe about each of the ſemicircles let there be an equilateral figure circumſcribed: a point may be found, ſuch, that if from any point there be drawn perpendiculars to the ſides of all the figures, and likewiſe a right line to the point found, twice the ſum of the ſquares of the perpendiculars will be equal to the multiple of the ſquare of the line drawn to the point found by the number of the ſides of all the figures.

Suppoſe, for example, two circles to be given by poſition, and likewiſe two ſemicircles to be given by poſition; and about each of the circles let there be an equilateral figure circumſcribed, and likewiſe about each of the ſemicircles let there be an equilateral figure circumſcribed; and let the number of the ſides of the figure circumſcribed about one of the circles be three, and the number of the ſides of the figure circumſcribed about the other be four; and let the number of the ſides of the figure circumſcribed about one of the ſemicircles be three, and the number of the ſides of the figure circumſcribed about the other be five: a point may be found, ſuch, that

that if from any point there be drawn perpendiculars to the ſides of all the figures, and likewiſe there be drawn a right line to the point found, twice the ſum of the ſquares of the perpendiculars drawn to the ſides of the figures, will be equal to fifteen times the ſquare of the line drawn to the point found together with a given ſpace.

From the two laſt theorems the two following theorems may be eaſily derived.

PROPOSITION. XI.

THEOREM VIII.

Let there be any number of given points, two points may be found, ſuch, that if from all the given points and the two points found there be drawn right lines to any point, twice the ſum of the ſquares of the lines drawn from the given points, will be equal to the multiple by the number of the given points of the ſum of the ſquares of the lines drawn from the two points found.

Let the number of the given points be three, and the theorem will be as follows.

Let

Let there be three given points, two points may be found, ſuch, that if from the three given points and the two points found there be drawn right lines to any point, twice the ſum of the ſquares of the lines drawn from the three given points, will be equal to ſix times the ſum of the ſquares of the lines drawn from the two points found; and ſo on.

PROPOSITION XII.

THEOREM IX.

Let there be any number of given points, and let a, b, c, &c. be given magnitudes, as many in number as there are given points; two points may be found, ſuch, that if from all the given points and the two points found there be drawn right lines to any point, the ſquare of the line drawn from one of the given points, together with the ſpace to which the ſquare of the line drawn from another of the given points has the ſame ratio that a has to b, together with the ſpace to which the ſquare of the line drawn from another of the given points has the ſame ratio that a has to c, *to c, and ſo on, will be equal to the ſpace to which the ſum of the ſquares of the lines drawn from the two points found has the ſame ratio that twice a has to the ſum of a, b, c,* &c.

Let the number of the given points be three, and let *a, b, c* be equal to 1, 2, 3; and the theorem will be as follows.

Let there be three given points, two points may be found, ſuch, that if from the three given points and the two points found there be drawn right lines to any point, the ſquare of the line drawn from one of the given points together with twice the ſquare of the line drawn from another of the given points together with thrice the ſquare of the line drawn from the third given point, will be equal to the ſpace to which the ſum of the ſquares of the lines drawn from the two points found has the ſame ratio that two has to ſix, that is, will be equal to thrice the ſum of the ſquares of the lines drawn from the two points found; and ſo on.

PRO-

PROPOSITION XIII.

THEOREM X.

Let there be any number of right lines given by position, and parallel to each other; a right line may be found parallel to the lines given by position, such, that if from any point there be drawn a perpendicular to the right lines given by position and to the line found, the sum of the squares of the lines intercepted between the point and the right lines given by position, will be equal to the multiple of the square of the line intercepted between the point and the right line found, by the number of the right lines given by position, together with a given space.

Let the number of the lines given by position be three, and the theorem will be as follows.

Let there be three lines given by position, and parallel to each other; a line may be found parallel to the right lines given by position, such, that if from any point there be drawn a perpendicular to the right lines given by position

tion and to the line found, the sum of the squares of the lines intercepted between the point and the three lines given by position, will be equal to thrice the square of the line intercepted between the point and the line found, together with a given space. And so on.

PROPOSITION XIV.

THEOREM XI.

Let there be any number of right lines intersecting each other in one point, and making all the angles round the point of intersection equal; and from any point let there be drawn perpendiculars to the right lines, and likewise let there be drawn a right line to the point of intersection: twice the sum of the squares of the perpendiculars will be equal to the multiple of the square of the line drawn to the point of intersection by the number of the lines.

Let the number of the lines be three, and the theorem will be as follows.

Let there be three right lines intersecting each other in a point, and making all the angles round the point of interſection equal; and from any point let there be drawn perpendiculars to the three lines, and likewiſe let there be drawn a right line to the point of interſection: twice the ſum of the ſquares of the perpendiculars will be equal to thrice the ſquare of the line drawn to the point of interſection.

Again, Let there be five right lines interſecting each other in one point, and making all the angles round the point of interſection equal; and from any point let there be drawn perpendiculars to the right lines, and likewiſe let there be drawn a right line to the point of interſection: twice the ſum of the ſquares of the perpendiculars will be equal to five times the ſquare of the line drawn to the point of interſection. And ſo on.

This theorem is very eaſily deduced from *Prop.* 4. by *Lem.* 2.

Cor. If there be any number of right lines interſecting each other in a given point, and making all the angles round the point of interſection equal; and from a point there be drawn perpendiculars to the right lines, and the ſum of the

the squares of the perpendiculars be equal to a given space; the point from which the perpendiculars are drawn will be in the circumference of a given circle.

PROPOSITION XV.

THEOREM XII.

Let there be any number of right lines given by position intersecting each other in a point; two right lines may be found that will be given by position, such, that if from any point there be drawn perpendiculars to all the right lines given by position, and likewise perpendiculars to the two right lines found, twice the sum of the squares of the perpendiculars drawn to the right lines given by position, will be equal to the multiple of the sum of the squares of the perpendiculars drawn to the two lines found by the number of the lines given by position.

Let the number of the lines given by position be three, and the theorem will be as follows.

Let there be three right lines given by position

ſition interſecting each other in a point; two right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the three lines given by poſition and to the two lines found, twice the ſum of the ſquares of the perpendiculars drawn to the three lines given by poſition, will be equal to thrice the ſum of the ſquares of the perpendiculars drawn to the two lines found. And ſo on.

PROPOSITION XVI.

THEOREM XIII.

Let there be any number of right lines given by poſition, that are neither all parallel, nor interſecting each other in one point; two right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe perpendiculars to the two right lines found, twice the ſum of the ſquares of the perpendiculars drawn to the right lines given by poſition, will be equal to the multiple of the ſum of the

the squares of the perpendiculars drawn to the two right lines found by the number of the right lines given by position, together with a given space.

Let the number of the lines given by position be three, and the theorem will be as follows.

Let there be three right lines given by position, that are neither all parallel, nor intersecting each other in one point; two right lines may be found that will be given by position, such, that if from any point there be drawn perpendiculars to the three lines given by position and to the two lines found, twice the sum of the squares of the perpendiculars drawn to the three lines given by position, will be equal to thrice the sum of the squares of the perpendiculars drawn to the two lines found, together with a given space. And so on.

COR. If the right lines given by position be so situated, as to form an equilateral figure circumscribed either about a circle or semicircle; or, if the number of the lines given by position be even, and each two and two of the lines intersect each other at right angles;

the

any point there be drawn right lines in given angles to all the right lines given by poſition, and likewiſe there be drawn perpendiculars to the two lines found, the ſum of the ſquares of the lines drawn in given angles to the right lines given by poſition, will be equal to the ſpace to which the ſum of the ſquares of the perpendiculars drawn to the two lines found has a given ratio.

PROPOSITION XX.

THEOREM XVII.

Let there be any number of right lines given by poſition, that are neither all parallel, nor interſecting each other in one point, and let a, b, c, &c. *be given magnitudes as many in number as there are right lines given by poſition; two right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the two right lines found, the ſquare of the perpendicular drawn to one of the lines given by poſition*

position, together with the space to which the square of the perpendicular drawn to another of the lines given by position has the same ratio that a has to b, together with the space to which the square of the perpendicular drawn to another of the lines given by position has the same ratio that a has to c, and so on, will be equal to the space to which the sum of the squares of the perpendiculars drawn to the two lines found has the same ratio that twice a has to the sum of a, b, c, &c. together with a given space.

Let the number of the lines given by position be three, and let *a*, *b*, *c* be equal to 1, 2, 3; and the theorem will be as follows.

Let there be three right lines given by position, that are neither all parallel, nor intersecting each other in one point; two right lines may be found that will be given by position, such, that if from any point there be drawn perpendiculars to the three lines given by position and to the two lines found, the square of the perpendicular drawn to one of the right lines given by position, together with twice the square of the perpendicular drawn

to

to another of the lines given by poſition, together with thrice the ſquare of the perpendicular drawn to the third line given by poſition, will be equal to thrice the ſum of the ſquares of the perpendiculars drawn to the two lines found, together with a given ſpace. And ſo on.

Cor. Let there be any number of right lines given by poſition, that are neither all parallel, nor interſecting each other in one point; two right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn right lines in given angles to all the right lines given by poſition, and likewiſe there be drawn perpendiculars to the two lines found, the ſum of the ſquares of the lines drawn in given angles to the right lines given by poſition, will be equal to the ſpace to which the ſum of the ſquares of the perpendiculars drawn to the two lines found has a given ratio, together with a given ſpace.

PRO-

PROPOSITION XXI.

THEOREM XVIII.

Let there be any number of right lines given by position, and let a, b, c, &c. be given magnitudes as many in number as there are right lines given by position; three right lines may be found that will be given by position, such, that if from any point there be drawn perpendiculars to all the right lines given by position, and likewise there be drawn perpendiculars to the three lines found, the square of the perpendicular drawn to one of the lines given by position, together with the space to which the square of the perpendicular drawn to another of the lines given by position has the same ratio that a has to b, together with the space to which the square of the perpendicular drawn to another of the lines given by position has the same ratio that a has to c, and so on, will be equal to the space to which the sum of the squares of the perpendiculars drawn to the three lines found has the same ratio that thrice a has to the sum of a, b, c, &c.

Let the number of the lines given by position

tion be four, and let a, b, c, d be equal to 1, 2, 3, 4; and the theorem will be as follows.

Let there be four right lines given by position; three right lines may be found that will be given by position, ſuch, that if from any point there be drawn perpendiculars to the four lines given by position, and likewiſe there be drawn perpendiculars to the three lines found, the ſquare of the perpendicular drawn to one of the lines given by position, together with twice the ſquare of the perpendicular drawn to another of the lines given by poſition, together with thrice the ſquare of the perpendicular drawn to another of the lines given by poſition, together with four times the ſquare of the perpendicular drawn to the fourth line given by poſition, will be equal to the ſpace to which the ſum of the ſquares of the perpendiculars drawn to the three lines given by poſition has the ſame ratio that three has to ten.

Cor. Let there be any number of right lines given by poſition; three right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn right lines

lines in given angles to all the right lines given by position, and likewise there be drawn perpendiculars to the three lines found, the sum of the squares of the lines drawn in given angles to the right lines given by position, will be equal to the space to which the sum of the squares of the perpendiculars drawn to the three lines found has a given ratio.

PROPOSITION XXII.

THEOREM XIX.

Let there be any regular figure of a greater number of sides than three circumscribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the sides of the figure; twice the sum of the cubes of the perpendiculars, will be equal to five times the multiple of the cube of the semidiameter of the circle by the number of the sides of the figure.

Suppose, for example, a square to be circumscribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the ſquare; the ſum of the cubes of the perpendiculars, will be equal to ten times the cube of the ſemidiameter of the circle.

Again, Suppoſe a pentagon to be circumſcribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the pentagon; twice the ſum of the cubes of the perpendiculars will be equal to twenty five times the cube of the ſemidiameter of the circle. And ſo on.

PROPOSITION XXIII.

THEOREM XX.

Let there be any regular figure circumſcribed about a circle of a greater number of ſides than three, and from any point within the figure let there be drawn perpendiculars to the ſides of the figure, and likewiſe let there be drawn a right line to the centre of the circle; twice the ſum of the cubes of the perpendiculars drawn to the ſides of the figure, will be equal to twice the multiple of the cube

cube of the ſemidiameter of the circle by the number of the ſides of the figure, together with thrice the multiple by the ſame number of the ſolid, whoſe baſe is the ſquare of the line drawn to the centre, and altitude the ſemidiameter of the circle.

Suppoſe, for example, a ſquare to be circumſcribed about a circle, and from any point within the ſquare let there be drawn perpendiculars to the ſides of the ſquare, and likewiſe let there be drawn a right line to the centre of the circle; the ſum of the cubes of the perpendiculars drawn to the ſides of the ſquare, will be equal to four times the cube of the ſemidiameter of the circle, together with ſix times the ſolid, whoſe baſe is the ſquare of the line drawn to the centre, and altitude the ſemidiameter of the circle.

Again, Let there be a pentagon circumſcribed about a circle, and from any point within the pentagon let there be drawn perpendiculars to the ſides of the pentagon, and likewiſe let there be drawn a right line to the centre of the circle; twice the ſum of the cubes of the perpendiculars drawn to the ſides of the pen-

 tagon,

For example, Let the number of the ſides of the figure be five; and the theorem will be as follows.

Let there be any five-ſided figure given by poſition; four right lines may be found, ſuch, that if from any point within the figure there be drawn perpendiculars to the ſides of the figure, and likewiſe there be drawn perpendiculars to the four lines found, four times the ſum of the cubes of the perpendiculars drawn to the ſides of the figure, will be equal to five times the ſum of the cubes of the perpendiculars drawn to the four lines found.

Again, Let the number of the ſides of the figure be ſix; and the theorem will be as follows.

Let there be a ſix-ſided figure given by poſition; four right lines may be found that will be given by poſition, ſuch, that if from any point within the figure there be drawn perpendiculars to the ſides of the figure, and likewiſe there be drawn perpendiculars to the four lines found, four times the ſum of the cubes of the perpendiculars drawn to the ſides of the figure, will be equal to ſix times the ſum of the cubes of the perpendiculars drawn to the four lines found. And so on.

PROPOSITION XXV.

THEOREM XXII.

Let there be any figure given by position of a greater number of sides than three, and let a, b, c, &c. *be given magnitudes as many in number as there are sides in the figure; four right lines may be found that will be given by position, such, that if from any point there be drawn perpendiculars to the sides of the figure, and likewise there be drawn perpendiculars to the four lines found, the cube of the perpendicular drawn to one of the sides of the figure, together with the solid to which the cube of the perpendicular drawn to another of the sides of the figure has the same ratio that a has to b, together with the solid to which the cube of the perpendicular drawn to another of the sides of the figure has the same ratio that a has to c, and so on, will be equal to the solid to which the sum of the*

the cubes of the perpendiculars drawn to the four lines found has the ſame ratio that four times a has to the ſum of a, b, c, &c.

For example, Let the number of the ſides of the figure be four, and let *a*, *b*, *c*, *d* be equal to 1, 2, 3, 5; and the theorem will be as follows.

Let there be any quadrilateral figure given by poſition; four right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the ſides of the quadrilateral figure, and likewiſe there be drawn perpendiculars to the four lines found, the cube of the perpendicular drawn to one of the ſides of the figure, together with twice the cube of the perpendicular drawn to another of the ſides of the figure, together with thrice the cube of the perpendicular drawn to another of the ſides of the figure, and ſo on, will be equal to the ſolid to which the ſum of the cubes of the perpendiculars drawn to the four lines found has the ſame ratio that four has to eleven.

Again, Let the number of the ſides of the figure

figure be five, and let *a*, *b*, *c*, *d*, *e* be equal to 1, 3, 5, 7, 9; and the theorem will be as follows.

Let there be any five-ſided figure given by poſition; four right lines may be found that will be given by poſition, ſuch, that if from any point. there be drawn perpendiculars to the ſides of the figure, and likewiſe there be drawn perpendiculars to the four lines found, the cube of the perpendicular drawn to one of the ſides of the figure, together with thrice the cube of the perpendicular drawn to another of the ſides of the figure, together with five times the cube of the perpendicular drawn to another of the ſides of the figure, and ſo on, will be equal to the ſolid to which the ſum of the cubes of the perpendiculars drawn to the four lines found has the ſame ratio that four has to twenty five. And ſo on.

COR. Let there be any figure of a greater number of ſides than three; four lines may be found that will be given by poſition, ſuch, that if from any point within the figure there be drawn right lines in given angles to the ſides of the figure, and likewiſe there be drawn perpendiculars to the four lines found, the ſum of the cubes of the lines drawn in given angles to the ſides of the figure, will be to the ſum of the cubes of the perpendiculars drawn to the four lines found in a given ratio.

PROPOSITION XXVI.

THEOREM XXIII.

Let there be any regular figure inſcribed in a circle, and from all the angles of the figure let there be drawn right lines to any point in the circumference of the circle; the ſum of the fourth powers of the chords will be equal to 6 times the multiple of the fourth power of the ſemidiameter of the circle by the number of the ſides of the figure.

Suppoſe, for example, an equilateral triangle to be inſcribed in a circle, and from the angles of the triangle let there be drawn right lines to any point in the circumference of the circle; the ſum of the fourth powers of the chords will be equal to 18 times the fourth power of the ſemidiameter of the circle.

Again, Suppoſe a ſquare to be inſcribed in a a circle, and from all the angles of the ſquare let there be drawn right lines to any point in the circumference of the circle; the ſum of the fourth powers of the chords, will be equal to 24 times the fourth power of the ſemidiameter of the circle. And ſo on.

PROPOSITION XXVII.

THEOREM XXIV.

Let there be any regular figure inſcribed in a circle, and from all the angles of the figure and the centre of the circle let there be drawn right lines to any point; the ſum of the fourth powers of the lines drawn from the angles of the figure, will be equal to the multiple by the number of the ſides of the figure of the fourth power of the ſemidiameter of the circle, together with 4 times the multiple by the ſame number of the fourth power of the line whoſe ſquare is equal to the rectangle contained by the ſemidiameter and the line drawn from the centre, together with the multiple by the ſame number of the

 fourth

fourth power of the line drawn from the centre.

Suppose, for example, an equilateral triangle to be inscribed in a circle, and from all the angles of the triangle and the centre of the circle let there be drawn right lines to any point; the sum of the fourth powers of the lines drawn from the angles of the triangle, will be equal to 3 times the fourth power of the semidiameter of the circle, together with 12 times the fourth power of the line whose square is equal to the rectangle contained by the semidiameter and the line drawn from the centre, together with 3 times the fourth power of the line drawn from the centre.

Again, Suppose a square to be inscribed in a circle, and from all the angles of the square and the centre of the circle let there be drawn right lines to any point; the sum of the fourth powers of the lines drawn from the angles of the square, will be equal to 4 times the fourth power of the semidiameter of the circle, together with 16 times the fourth power of the line whose square is equal to the rectangle contained by the semidiameter and the line drawn

drawn from the centre, together with 4 times the fourth power of the line drawn from the centre.

Cor. I. Let there be two circles about the ſame centre, and from all the angles of any regular figure inſcribed in one of the circles; let there be drawn right lines to any point in the circumference of the other; the ſum of the fourth powers of theſe lines will be invariable.

Cor. II. Let there be two regular figures inſcribed in a circle, and from all the angles of both figures let there be drawn right lines to any point; the ſum of the fourth powers of the lines drawn from all the angles of one of the figures, will be to the ſum of the fourth powers of the lines drawn from the angles of the other figure, as the number of the ſides of the one to the number of the ſides of the other.

Suppoſe, for example, an equilateral triangle and a ſquare to be inſcribed in a circle, and from all the angles of both figures let there be drawn right lines to any point; the ſum of the fourth powers of the lines drawn from the angles of the triangle, will be to the ſum of the any point there be drawn right lines in given angles to all the right lines given by position, and likewise there be drawn perpendiculars to the two lines found, the sum of the squares of the lines drawn in given angles to the right lines given by position, will be equal to the space to which the sum of the squares of the perpendiculars drawn to the two lines found has a given ratio.

PROPOSITION XX.

THEOREM XVII.

Let there be any number of right lines given by position, that are neither all parallel, nor interjecting each other in one point, and let a, b, c, &c. be given magnitudes as many in number as there are right lines given by position; two right lines may be found that will be given by position, such, that if from any point there be drawn perpendiculars to the right lines given by position, and likewise there be drawn perpendiculars to the two right lines found, the square of the perpendicular drawn to one of the lines given by position

...m the centre, together with 4 times ...power of the line drawn from the

...et there be two circles about the ...nd from all the angles of any ...scribed in one of the circles ...n right lines to any point in ...of the other; the sum of ...of these lines will be inva-

...be two regular figures in- ...l from all the angles of ...e drawn right lines to ...the fourth powers of ...ll the angles of one ...o the sum of the ...drawn from the ...s the number of ...mber of the sides

...eral tri- ...cle.

...n ...n; ...the ...the ...e cir-

...IX.

...I.

...*of a greater* ...*cumscribed a-* ...*point let there* ...*the sides of the* ...*line to the centre* ...*sum of the fourth* ...*ars, will be equal to* ...*the number of the* ...*e fourth power of the* ...*rcle, together with 24* ...*the same number of the* ...*line whose square is equal* the fourth powers of the lines drawn from the angles of the ſquare, as three to four. And ſo on.

PROPOSITION XXVIII.

THEOREM XXV.

Let there be any regular figure of a greater number of ſides than four circumſcribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the figure; 8 times the ſum of the fourth powers of the perpendiculars, will be equal to 35 times the multiple by the number of the ſides of the figure of the fourth power of the ſemidiameter of the circle.

Suppoſe, for example, a pentagon to be circumſcribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the pentagon; 8 times the ſum of the fourth powers of the perpendiculars, will be equal to 175 times the fourth power of the ſemidiameter of the circle.

Again,

Again, Let there be a hexagon circumſcribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the hexagon; 4 times the ſum of the fourth powers of the perpendiculars, will be equal to 105 times the fourth power of the ſemidiameter of the circle. And ſo on.

PROPOSITION XXIX.

THEOREM XXVI.

Let there be any regular figure of a greater number of ſides than four circumſcribed about a circle, and from any point let there be drawn perpendiculars to the ſides of the figure, and likewiſe a right line to the centre of the circle; 8 *times the ſum of the fourth powers of the perpendiculars, will be equal to* 8 *times the multiple by the number of the ſides of the figure of the fourth power of the ſemidiameter of the circle, together with* 24 *times the multiple by the ſame number of the fourth power of the line whoſe ſquare is equal to* *to the rectangle contained by the semidiameter and the line drawn to the centre, together with 3 times the multiple of the fourth power of the line drawn to the centre of the circle by the number of the sides of the figure.*

Suppose, for example, a pentagon to be circumscribed about a circle, and from any point let there be drawn perpendiculars to the sides of the pentagon, and likewise a line to the centre of the circle; 8 times the sum of the fourth powers of the perpendiculars drawn to the sides of the pentagon, will be equal to 40 times the fourth power of the semidiameter of the circle, together with 120 times the fourth power of the line whose square is equal to the rectangle contained by the semidiameter and the line drawn to the centre of the circle, together with 15 times the fourth power of the line drawn to the centre. And so on.

Cor. I. Let there be two circles about the same centre, and about one of the circles let there be any regular figure of a greater number of sides than four circumscribed; if from any point in the circumference of the other there be drawn perpendiculars to the sides of the

the figure, the ſum of the fourth powers of the perpendiculars will be invariable.

Cor. II. Let there be two regular figures circumſcribed about a circle, and let the number of the ſides of each figure be greater than four, and from any point let there be drawn perpendiculars to the ſides of both figures; the ſum of the fourth powers of the perpendiculars drawn to the ſides of one of the figures, will be to the ſum of the fourth powers of the perpendiculars drawn to the ſides of the other, as the number of the ſides of the one to the number of the ſides of the other.

Suppoſe, for example, a pentagon and hexagon to be circumſcribed about a circle, and from any point let there be drawn perpendiculars to the ſides of both figures; the ſum of the fourth powers of the perpendiculars drawn to the ſides of the pentagon, will be to the ſum of the fourth powers of the perpendiculars drawn to the ſides of the hexagon, as 5 to 6. And ſo on.

PROPOSITION XXX.

THEOREM XXVII.

Let there be any number of given points; two right lines may be found that will be given by position, and likewise a point may be found, such, that if from all the given points and the point found there be drawn right lines to any point, and from the point to which the right lines are drawn there be drawn perpendiculars to the two right lines found, the sum of the fourth powers of the lines drawn from the given points, will be equal to the multiple by the number of the given points of the fourth power of the line drawn from the point found, together with the fourth power of the line whose square is a mean proportional between the sum of the squares of the perpendiculars and a certain given space, together with the fourth power of a certain given line.

Let the number of the given points be two; and the theorem will be as follows.

Let

Let there be two given points; two right lines may be found that will be given by poſition, and likewiſe a point may be found, ſuch, that if from the two given points and the point found there be drawn right lines to any point, and from the point to which the lines are drawn there be drawn perpendiculars to the two right lines found, the ſum of the fourth powers of the lines drawn from the two given points, will be equal to twice the fourth power of the line drawn from the point found, together with the fourth power of the line whoſe ſquare is a mean proportional between the ſum of the ſquares of the perpendiculars and a certain given ſpace, together with the fourth power of a certain given line.

Let the number of the given points be three; and the theorem will be as follows.

Let there be three points given; two right lines may be found that will be given by poſition, and likewiſe a point may be found, ſuch, that if from all the three given points and the point found there be drawn right lines to any point, and from the point to which the right lines are drawn there be drawn perpen-

fourth power of the line drawn from the centre.

Suppoſe, for example, an equilateral triangle to be inſcribed in a circle, and from all the angles of the triangle and the centre of the circle let there be drawn right lines to any point; the ſum of the fourth powers of the lines drawn from the angles of the triangle, will be equal to 3 times the fourth power of the ſemidiameter of the circle, together with 12 times the fourth power of the line whoſe ſquare is equal to the rectangle contained by the ſemidiameter and the line drawn from the centre, together with 3 times the fourth power of the line drawn from the centre.

Again, Suppoſe a ſquare to be inſcribed in a circle, and from all the angles of the ſquare and the centre of the circle let there be drawn right lines to any point; the ſum of the fourth powers of the lines drawn from the angles of the ſquare, will be equal to 4 times the fourth power of the ſemidiameter of the circle, together with 16 times the fourth power of the line whoſe ſquare is equal to the rectangle contained by the ſemidiameter and the line

drawn

drawn from the centre, together with 4 times the fourth power of the line drawn from the centre.

Cor. I. Let there be two circles about the ſame centre, and from all the angles of any regular figure inſcribed in one of the circles; let there be drawn right lines to any point in the circumference of the other; the ſum of the fourth powers of theſe lines will be invariable.

Cor. II. Let there be two regular figures inſcribed in a circle, and from all the angles of both figures let there be drawn right lines to any point; the ſum of the fourth powers of the lines drawn from all the angles of one of the figures, will be to the ſum of the fourth powers of the lines drawn from the angles of the other figure, as the number of the ſides of the one to the number of the ſides of the other.

Suppoſe, for example, an equilateral triangle and a ſquare to be inſcribed in a circle, and from all the angles of both figures let there be drawn right lines to any point; the ſum of the fourth powers of the lines drawn from the angles of the triangle, will be to the ſum of

the

the fourth powers of the lines drawn from the angles of the ſquare, as three to four. And ſo on.

PROPOSITION XXVIII.

THEOREM XXV.

Let there be any regular figure of a greater number of ſides than four circumſcribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the figure; 8 times the ſum of the fourth powers of the perpendiculars, will be equal to 35 times the multiple by the number of the ſides of the figure of the fourth power of the ſemidiameter of the circle.

Suppoſe, for example, a pentagon to be circumſcribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the pentagon; 8 times the ſum of the fourth powers of the perpendiculars, will be equal to 175 times the fourth power of the ſemidiameter of the circle.

Again,

Again, Let there be a hexagon circumſcribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the hexagon; 4 times the ſum of the fourth powers of the perpendiculars, will be equal to 105 times the fourth power of the ſemidiameter of the circle. And ſo on.

PROPOSITION XXIX.

THEOREM XXVI.

Let there be any regular figure of a greater number of ſides than four circumſcribed about a circle, and from any point let there be drawn perpendiculars to the ſides of the figure, and likewiſe a right line to the centre of the circle; 8 *times the ſum of the fourth powers of the perpendiculars, will be equal to* 8 *times the multiple by the number of the ſides of the figure of the fourth power of the ſemidiameter of the circle, together with* 24 *times the multiple by the ſame number of the fourth power of the line whoſe ſquare is equal to* *to the rectangle contained by the semidiameter and the line drawn to the centre, together with 3 times the multiple of the fourth power of the line drawn to the centre of the circle by the number of the sides of the figure.*

Suppose, for example, a pentagon to be circumscribed about a circle, and from any point let there be drawn perpendiculars to the sides of the pentagon, and likewise a line to the centre of the circle; 8 times the sum of the fourth powers of the perpendiculars drawn to the sides of the pentagon, will be equal to 40 times the fourth power of the semidiameter of the circle, together with 120 times the fourth power of the line whose square is equal to the rectangle contained by the semidiameter and the line drawn to the centre of the circle, together with 15 times the fourth power of the line drawn to the centre. And so on.

Cor. I. Let there be two circles about the same centre, and about one of the circles let there be any regular figure of a greater number of sides than four circumscribed; if from any point in the circumference of the other there be drawn perpendiculars to the sides of the

the

the figure, the ſum of the fourth powers of the perpendiculars will be invariable.

COR. II. Let there be two regular figures circumſcribed about a circle, and let the number of the ſides of each figure be greater than four, and from any point let there be drawn perpendiculars to the ſides of both figures; the ſum of the fourth powers of the perpendiculars drawn to the ſides of one of the figures, will be to the ſum of the fourth powers of the perpendiculars drawn to the ſides of the other, as the number of the ſides of the one to the number of the ſides of the other.

Suppoſe, for example, a pentagon and hexagon to be circumſcribed about a circle, and from any point let there be drawn perpendiculars to the ſides of both figures; the ſum of the fourth powers of the perpendiculars drawn to the ſides of the pentagon, will be to the ſum of the fourth powers of the perpendiculars drawn to the ſides of the hexagon, as 5 to 6. And ſo on.

PROPOSITION XXX.

THEOREM XXVII.

Let there be any number of given points; two right lines may be found that will be given by position, and likewise a point may be found, such, that if from all the given points and the point found there be drawn right lines to any point, and from the point to which the right lines are drawn there be drawn perpendiculars to the two right lines found, the sum of the fourth powers of the lines drawn from the given points, will be equal to the multiple by the number of the given points of the fourth power of the line drawn from the point found, together with the fourth power of the line whose square is a mean proportional between the sum of the squares of the perpendiculars and a certain given space, together with the fourth power of a certain given line.

Let the number of the given points be two; and the theorem will be as follows.

Let

Let there be two given points; two right lines may be found that will be given by poſition, and likewiſe a point may be found, ſuch, that if from the two given points and the point found there be drawn right lines to any point, and from the point to which the lines are drawn there be drawn perpendiculars to the two right lines found, the ſum of the fourth powers of the lines drawn from the two given points, will be equal to twice the fourth power of the line drawn from the point found, together with the fourth power of the line whoſe ſquare is a mean proportional between the ſum of the ſquares of the perpendiculars and a certain given ſpace, together with the fourth power of a certain given line.

Let the number of the given points be three; and the theorem will be as follows.

Let there be three points given; two right lines may be found that will be given by poſition, and likewiſe a point may be found, ſuch, that if from all the three given points and the point found there be drawn right lines to any point, and from the point to which the right lines are drawn there be drawn perpen-

diculars to the two right lines found, the ſum of the fourth powers of the lines drawn from the three given points, will be equal to thrice the fourth power of the line drawn from the point found, together with the fourth power of the line whoſe ſquare is a mean proportional between the ſum of the ſquares of the perpendiculars and a certain given ſpace, together with the fourth power of a certain given line.

This theorem may be made more general thus.

PROPOSITION XXXI.

THEOREM XXVIII.

Let there be any number of given points, and let a, b, c, &c. be given magnitudes as many in number as there are given points; two right lines may be found that will be given by poſition, and likewiſe a point may be found, ſuch, that if from all the given points and the

the point found there be drawn right lines to any point, and from the point to which the lines are drawn there be drawn perpendiculars to the two right lines found, the fourth power of the line drawn from one of the given points, together with the power to which the fourth power of the line drawn from another of the given points has the same ratio that a has to b, together with the power to which the fourth power of the line drawn from another of the given points has the same ratio that a has to c, and so on, will be equal to the power to which the fourth power of the line drawn from the point found has the same ratio that a has to the sum of a, b, c, &c. *together with the fourth power of the line whose square is a mean proportional between the sum of the squares of the perpendiculars and a certain given space, together with the fourth power of a certain given line.*

For example, Let the number of the given points be two, and let *a*, *b* be equal to 1, 2; and the theorem will be as follows.

Let there be two given points; two right lines may be found that will be given by posi-
tion,

tion, and likewiſe a point may be found, ſuch, that if from the two given points there be drawn right lines to any point, and from the point to which the right lines are drawn there be drawn perpendiculars to the two right lines found, the fourth power of the line drawn from one of the given points together with twice the fourth power of the line drawn from the other given point, will be equal to thrice the fourth power of the line drawn from the point found, together with the fourth power of the line whoſe ſquare is a mean proportional between the ſum of the ſquares of the perpendiculars drawn to the two lines found and a certain given ſpace, together with the fourth power of a certain given line.

Again, Let the number of the given points be three, and let *a*, *b*, *c* be equal to 1, 2, 3; and the theorem will be as follows.

Let there be three given points; two right lines may be found that will be given by poſition, and likewiſe a point may be found, ſuch, that if from all the given points and the point found there be drawn right lines to any point, and from the point to which the lines are drawn

drawn there be drawn perpendiculars to the two right lines found, the fourth power of the line drawn from one of the given points, together with twice the fourth power of the line drawn from another of the given points, together with thrice the fourth power of the line drawn from the laſt of the given points, will be equal to ſix times the fourth power of the line drawn from the point found, together with the fourth power of the line whoſe ſquare is a mean proportional between the ſum of the ſquares of the perpendiculars drawn to the two lines found and a certain given ſpace, together with the fourth power of a certain given line.

PROPOSITION XXXII.

THEOREM XXIX.

Let there be any number greater than three of given points; three points may be found, ſuch, that if from all the given points and the three points found there be drawn right lines to any point, thrice the ſum of the fourth powers of the lines drawn from the given points,

points, will be equal to the multiple by the number of the given points of the ſum of the fourth powers of the lines drawn from the three points found.

Let the number of the given points be four; and the theorem will be as follows.

Let there be four points given; three points may be found, ſuch, that if from the four given points and the three points found there be drawn right lines to any point, thrice the ſum of the fourth powers of the lines drawn from the four given points, will be equal to four times the ſum of the fourth powers of the lines drawn from the three points found.

Again, Let the number of the given points be five; and the theorem will be as follows.

Let there be five given points; three points may be found, ſuch, that if from the five given points and the three points found there be drawn right lines to any point, thrice the ſum of the fourth powers of the lines drawn from the five given points, will be equal to five times the ſum of the fourth powers of the lines drawn from the three points found. And ſo on.

PROPOSITION XXXIII.

THEOREM XXX.

Let there be any number greater than two of given points, and let a, b, c, d, &c. be given magnitudes as many in number as there are given points; three points may be found, such, that if from all the given points and the three points found there be drawn right lines to any point, the fourth power of the line drawn from one of the given points, together with the power to which the fourth power of the line drawn from another of the given points has the same ratio that a has to b, together with the power to which the fourth power of the line drawn from another of the given points has the same ratio that a has to c, and so on, will be to the sum of the fourth powers of the lines drawn from the three points found, as the sum of a, b, c, &c. to thrice a.

For example, Let the number of the given points be three, and let a, b, c be equal to 1, 2, 4; and the theorem will be as follows.

Let there be three given points; three points may be found, ſuch, that if from the three given points and the three points found there be drawn right lines to any point, the fourth power of the line drawn from one of the given points, together with twice the fourth power of the line drawn from another of the given points, together with four times the fourth power of the line drawn from the third given point, will be to the ſum of the fourth powers of the lines drawn from the three points found, as ſeven to three.

Again, Let the number of the given points be four, and let *a*, *b*, *c*, *d* be equal to 1, 2, 4, 6; and the theorem will be as follows.

Let there be four given points; three points may be found, ſuch, that if from the four given points and the three points found there be drawn right lines to any point, the fourth power of the line drawn from one of the given points, together with 2 times the fourth power of the line drawn from another of the given points, together with 4 times the fourth power of the line drawn from another of the given points, together with 6 times the fourth power of the line

Let there be two given points; two right lines may be found that will be given by position, and likewise a point may be found, such, that if from the two given points and the point found there be drawn right lines to any point, and from the point to which the lines are drawn there be drawn perpendiculars to the two right lines found, the sum of the fourth powers of the lines drawn from the two given points, will be equal to twice the fourth power of the line drawn from the point found, together with the fourth power of the line whose square is a mean proportional between the sum of the squares of the perpendiculars and a certain given space, together with the fourth power of a certain given line.

Let the number of the given points be three; and the theorem will be as follows.

Let there be three points given; two right lines may be found that will be given by position, and likewise a point may be found, such, that if from all the three given points and the point found there be drawn right lines to any point, and from the point to which the right lines are drawn there be drawn perpen-

round the point of interſection equal, and from any point let there be drawn perpendiculars to the right lines, and likewiſe let there be drawn a right line to the point of interſection; 8 times the ſum of the fourth powers of the perpendiculars, will be equal to 9 times the fourth power of the line drawn to the point of interſection.

Again, Let the number of the lines be five; and the theorem will be as follows.

Let there be five right lines interſecting each other in a point, and making all the angles round the point of interſection equal, and from any point let there be drawn perpendiculars to the right lines, and likewiſe let there be drawn a right line to the point of interſection; 8 times the ſum of the fourth powers of the perpendiculars, will be equal to 15 times the fourth power of the line drawn to the point of interſection. And ſo on.

Cor. Let there be any number of right lines interſecting each other in a given point, and making all the angles round the point of interſection equal, and from a point let there be drawn perpendiculars to the right lines, and the

the ſum of the fourth powers of the perpendiculars be invariable; the point from which the perpendiculars are drawn, will be in the circumference of a given circle.

PROPOSITION XXXV.

THEOREM XXXII.

Let there be any number greater than three of right lines given by poſition, that are either all parallel to each other, or all interſecting each other in one point; three right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the three lines found, 3 times the ſum of the fourth powers of the perpendiculars drawn to the right lines given by poſition, will be equal to the multiple of the ſum of the fourth powers of the perpendiculars drawn to the three lines found by the number of the lines given by poſition.

tion, and likewiſe a point may be found, ſuch, that if from the two given points there be drawn right lines to any point, and from the point to which the right lines are drawn there be drawn perpendiculars to the two right lines found, the fourth power of the line drawn from one of the given points together with twice the fourth power of the line drawn from the other given point, will be equal to thrice the fourth power of the line drawn from the point found, together with the fourth power of the line whoſe ſquare is a mean proportional between the ſum of the ſquares of the perpendiculars drawn to the two lines found and a certain given ſpace, together with the fourth power of a certain given line.

Again, Let the number of the given points be three, and let *a*, *b*, *c* be equal to 1, 2, 3; and the theorem will be as follows.

Let there be three given points; two right lines may be found that will be given by poſition, and likewiſe a point may be found, ſuch, that if from all the given points and the point found there be drawn right lines to any point, and from the point to which the lines are drawn

drawn there be drawn perpendiculars to the two right lines found, the fourth power of the line drawn from one of the given points, together with twice the fourth power of the line drawn from another of the given points, together with thrice the fourth power of the line drawn from the laſt of the given points, will be equal to ſix times the fourth power of the line drawn from the point found, together with the fourth power of the line whoſe ſquare is a mean proportional between the ſum of the ſquares of the perpendiculars drawn to the two lines found and a certain given ſpace, together with the fourth power of a certain given line.

PROPOSITION XXXII.

THEOREM XXIX.

Let there be any number greater than three of given points; three points may be found, ſuch, that if from all the given points and the three points found there be drawn right lines to any point, thrice the ſum of the fourth powers of the lines drawn from the given points,

points, will be equal to the multiple by the number of the given points of the ſum of the fourth powers of the lines drawn from the three points found.

Let the number of the given points be four; and the theorem will be as follows.

Let there be four points given; three points may be found, ſuch, that if from the four given points and the three points found there be drawn right lines to any point, thrice the ſum of the fourth powers of the lines drawn from the four given points, will be equal to four times the ſum of the fourth powers of the lines drawn from the three points found.

Again, Let the number of the given points be five; and the theorem will be as follows.

Let there be five given points; three points may be found, ſuch, that if from the five given points and the three points found there be drawn right lines to any point, thrice the ſum of the fourth powers of the lines drawn from the five given points, will be equal to five times the ſum of the fourth powers of the lines drawn from the three points found. And ſo on.

PROPOSITION XXXIII.

THEOREM XXX.

Let there be any number greater than two of given points, and let a, b, c, d, &c. be given magnitudes as many in number as there are given points; three points may be found, such, that if from all the given points and the three points found there be drawn right lines to any point, the fourth power of the line drawn from one of the given points, together with the power to which the fourth power of the line drawn from another of the given points has the same ratio that a has to b, together with the power to which the fourth power of the line drawn from another of the given points has the same ratio that a has to c, and so on, will be to the sum of the fourth powers of the lines drawn from the three points found, as the sum of a, b, c, &c. to thrice a.

For example, Let the number of the given points be three, and let *a*, *b*, *c* be equal to 1, 2, 4; and the theorem will be as follows.

Let there be three given points; three points may be found, fuch, that if from the three given points and the three points found there be drawn right lines to any point, the fourth power of the line drawn from one of the given points, together with twice the fourth power of the line drawn from another of the given points, together with four times the fourth power of the line drawn from the third given point, will be to the fum of the fourth powers of the lines drawn from the three points found, as feven to three.

Again, Let the number of the given points be four, and let *a*, *b*, *c*, *d* be equal to 1, 2, 4, 6; and the theorem will be as follows.

Let there be four given points; three points may be found, fuch, that if from the four given points and the three points found there be drawn right lines to any point, the fourth power of the line drawn from one of the given points, together with 2 times the fourth power of the line drawn from another of the given points, together with 4 times the fourth power of the line drawn from another of the given points, together with 6 times the fourth power of the line

line drawn from the fourth given point, will be to the sum of the fourth powers of the lines drawn from the three points found, as thirteen to three. And so on.

PROPOSITION XXXIV.

THEOREM XXXI.

Let there be any number greater than two of right lines intersecting each other in a point, and making all the angles round the point of intersection equal, and from any point let there be drawn perpendiculars to the right lines, and likewise let there be drawn a right line to the point of intersection; 8 times the sum of the fourth powers of the perpendiculars drawn to the right lines, will be equal to 3 times the multiple of the fourth power of the line drawn to the point of intersection by the number of the right lines.

Let the number of the lines be three; and the theorem will be as follows.

Let there be three right lines intersecting each other in a point, and making all the angles

 round

round the point of interſection equal, and from any point let there be drawn perpendiculars to the right lines, and likewiſe let there be drawn a right line to the point of interſection; 8 times the ſum of the fourth powers of the perpendiculars, will be equal to 9 times the fourth power of the line drawn to the point of interſection.

Again, Let the number of the lines be five; and the theorem will be as follows.

Let there be five right lines interſecting each other in a point, and making all the angles round the point of interſection equal, and from any point let there be drawn perpendiculars to the right lines, and likewiſe let there be drawn a right line to the point of interſection; 8 times the ſum of the fourth powers of the perpendiculars, will be equal to 15 times the fourth power of the line drawn to the point of interſection. And ſo on.

Cor. Let there be any number of right lines interſecting each other in a given point, and making all the angles round the point of interſection equal, and from a point let there be drawn perpendiculars to the right lines, and the

the ſum of the fourth powers of the perpendiculars be invariable; the point from which the perpendiculars are drawn, will be in the circumference of a given circle.

PROPOSITION XXXV.

THEOREM XXXII.

Let there be any number greater than three of right lines given by poſition, that are either all parallel to each other, or all interſecting each other in one point; three right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the three lines found, 3 times the ſum of the fourth powers of the perpendiculars drawn to the right lines given by poſition, will be equal to the multiple of the ſum of the fourth powers of the perpendiculars drawn to the three lines found by the number of the lines given by poſition.

Let the number of the lines be four; and the theorem will be as follows.

Let there be four right lines given by position, that are either all parallel to each other, or all interfecting each other in a point; three right lines may be found that will be given by position, fuch, that if from any point there be drawn perpendiculars to the four lines given by position, and likewife there be drawn perpendiculars to the three lines found, 3 times the fum of the fourth powers of the perpendiculars drawn to the four lines given by pofition, will be equal to 4 times the fum of the fourth powers of the perpendiculars drawn to the three lines found.

Again, Let the number of the lines be five; and the theorem will be as follows.

Let there be five right lines given by pofition, that are either all parallel to each other, or all interfecting each other in one point; three right lines may be found that will be given by pofition, fuch, that if from any point there be drawn perpendiculars to the five right lines given by pofition, and likewife there be drawn perpendiculars to the three lines found, 3 times

the

the ſum of the fourth powers of the perpendiculars drawn to the five lines given by poſition, will be equal to 5 times the ſum of the fourth powers of the perpendiculars drawn to the three lines found. And ſo on.

PROPOSITION XXXVI.

Theorem XXXIII.

Let there be any number greater than two of right lines given by poſition, that are either all parallel, or all interſecting each other in one point, and let a, b, c, d, &c. be given magnitudes as many in number as there are right lines given by poſition; three right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the three lines found, the fourth power of the perpendicular drawn to one of the lines given by poſition, together with the power to which the fourth power of the perpendicular drawn to another of the lines given

ven by position has the same ratio that a has to b, together with the power to which the fourth power of the perpendicular drawn to another of the lines given by position has the same ratio that a has to c, and so on, will be to the sum of the fourth powers of the perpendiculars drawn to the three lines found, as the sum of a, b, c, d, &c. to thrice a.

Let the number of the right lines given by position be three, and let a, b, c be equal to 1, 2, 5; and the theorem will be as follows.

Let there be three right lines given by position, that are either all parallel to each other, or all interfecting each other in one point; three right lines may be found that will be given by position, such, that if from any point there be drawn perpendiculars to the three right lines given by position, and likewise there be drawn perpendiculars to the three right lines found, the fourth power of the perpendicular drawn to one of the lines given by position, together with 2 times the fourth power of the perpendicular drawn to another of the lines given by position, together with 5 times the fourth power of the perpendicular drawn to the third line

line given by position, will be to the ſum of the fourth powers of the perpendiculars drawn to the three lines found, as eight to three.

Again, Let the number of the lines given by poſition be four, and let *a*, *b*, *c*, *d* be equal to 1, 2, 3, 4; and the theorem will be as follows.

Let there be four right lines given by poſition, that are either all parallel to each other, or all interſecting each other in one point; three right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the four right lines given by poſition, and likewiſe there be drawn perpendiculars to the three lines found, the fourth power of the perpendicular drawn to one of the lines given by poſition, together with 2 times the fourth power of the perpendicular drawn to another of the lines given by poſition, together with 3 times the fourth power of the perpendicular drawn to another of the lines given by poſition, together with 4 times the fourth power of the perpendicular drawn to the fourth line given by poſition, will be to the ſum of the fourth powers of the perpendiculars drawn to the three lines found, as ten to three. And so on.

Cor. Let there be any number greater than two of right lines given by position, that are either all parallel to each other, or all intersecting each other in one point; three right lines may be found that will be given by position, such, that if from any point there be drawn right lines in given angles to the right lines given by position, and likewise there be drawn perpendiculars to the three lines found, the sum of the fourth powers of the lines drawn in given angles to the right lines given by position, will be to the sum of the fourth powers of the perpendiculars drawn to the three lines found in a given ratio.

PROPOSITION XXXVII.

Theorem XXXIV.

Let there be any number greater than five of right lines given by position, that are neither all parallel to each other, nor all intersecting each other in one point; five right lines may be found that will be given by position, such,

ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the five lines found, five times the ſum of the fourth powers of the perpendiculars drawn to the right lines given by poſition, will be equal to the multiple of the ſum of the fourth powers of the perpendiculars drawn to the five lines found by the number of the right lines given by poſition.

Let the number of the lines given by poſition be ſix; and the theorem will be as follows.

Let there be ſix right lines given by poſition, that are neither all parallel to each other, nor all interſecting each other in one point; five right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the ſix lines given by poſition, and likewiſe there be drawn perpendiculars to the five lines found, five times the ſum of the fourth powers of the perpendiculars drawn to the ſix lines given by poſition, will be equal to ſix times the ſum of the fourth powers of the perpendiculars drawn to the five lines found.

Again, Let the number of the lines given by poſition be ſeven; and the theorem will be as follows.

Let there be ſeven right lines given by poſition, that are neither all parallel to each other, nor interſecting each other in one point; five right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the ſeven lines given by poſition, and likewiſe there be drawn perpendiculars to the five lines found, five times the ſum of the fourth powers of the perpendiculars drawn to the ſeven lines given by poſition, will be equal to ſeven times the ſum of the fourth powers of the perpendiculars drawn to the five lines found. And ſo on.

PROPOSITION XXXVIII.

THEOREM XXXV.

Let there be any number greater than four of right lines given by poſition, that are neither all parallel to each other, nor interſecting each

each other in one point, and let a, b, c, d, &c. be given magnitudes as many in number as there are right lines given by position; five right lines may be found that will be given by position, such, that if from any point there be drawn perpendiculars to all the right lines given by position, and likewise there be drawn perpendiculars to the five lines found, the fourth power of the perpendicular drawn to one of the lines given by position, together with the power to which the fourth power of the perpendicular drawn to another of the lines given by position has the same ratio that a has to b, together with the power to which the fourth power of the perpendicular drawn to another of the lines given by position has the same ratio that a has to c, and so on, will be to the sum of the fourth powers of the perpendiculars drawn to the five lines found, as the sum of a, b, c, d, &c. to five times a.

Let the number of the lines given by position be five, and let *a*, *b*, *c*, *d*, *&c.* be equal to 1, 2, 3, 4, 6; and the theorem will be as follows.

Let

Let there be five right lines given by position, that are neither all parallel to each other, nor interfecting each other in one point; five right lines may be found that will be given by position, fuch, that if from any point there be drawn perpendiculars to the five right lines given by position, and likewife there be drawn perpendiculars to the five lines found, the fourth power of the perpendicular drawn to one of the lines given by position, together with 2 times the fourth power of the perpendicular drawn to another of the lines given by position, together with 3 times the fourth power of the perpendicular drawn to another of the lines given by position, and fo on, will be to the fum of the fourth power of the perpendiculars drawn to the five lines found, as fixteen to five.

Again, Let the number of the lines given by pofition be fix, and let *a*, *b*, *c*, *d*, &c. be equal to 1, 3, 5, 7, 9, 11; and the theorem will be as follows.

Let there be fix right lines given by pofition, that are neither all parallel to each other, nor all interfecting each other in one

point;

point; five right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the ſix lines given by poſition, and likewiſe there be drawn perpendiculars to the five lines found, the fourth power of the perpendicular drawn to one of the lines given by poſition, together with 3 times the fourth power of the perpendicular drawn to another of the lines given by poſition, together with 5 times the fourth power of the perpendicular drawn to another of the lines given by poſition, and ſo on, will be to the ſum of the fourth powers of the perpendiculars drawn to the five lines found, as thirty ſix to five. And ſo on.

Cor. Let there be any number greater than four of right lines given by poſition, that are neither all parallel to each other, nor all interſecting each other in one point; five right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn right lines in given angles to all the lines given by poſition, and likewiſe there be drawn perpendiculars to the five lines found, the ſum of the fourth powers of the lines drawn in given angles to the right lines given by poſition, will

will be to the ſum of the fourth powers of the perpendiculars drawn to the five lines found in a given ratio.

PROPOSITION XXXIX.

THEOREM XXXVI.

Let there be any regular figure circumſcribed about a circle; and let the number of the ſides of the figure be m, and let n be any number leſs than m; let r be the ſemidiameter of the circle; and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the figure, the ſum of the n powers of the perpendiculars will be equal to $m \times \frac{1.3.5.7 \ldots 2n-1}{1.2.3.4 \ldots n} \times r^n$

The numbers in the numerator are to be continued till the laſt number be $2n-1$, and are to be continually multiplied into one another; the numbers in the denominator are to be continued till the laſt number be n, and are to be continually multiplied into one another.

For

For example, Let $m = 6$, and $n = 5$; and the theorem will be as follows.

Let there be a hexagon circumſcribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the hexagon, and let r be the ſemidiameter of the circle; the ſum of the fifth powers of the perpendiculars will be equal to $6 \times \frac{1.3.5.7.9}{1.2.3.4.5} \times r^5 = \frac{189}{4} r^5$.

Again, Let $m = 8$, and $n = 6$; and the theorem will be as follows.

Let there be an octagon circumſcribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the octagon, and let r be the ſemidiameter of the circle; the ſum of the ſixth powers of the perpendiculars will be equal to $8 \times \frac{1.3.5.7.9.11}{1.2.3.4.5.6} \times r^6 = \frac{231}{2} r^6$. And ſo on.

PROPOSITION XL.

THEOREM XXXVII.

Let there be any regular figure circumſcribed about a circle, and let m be the number of

 the

the sides of the figure; let n be any number less than m, and let r be the semidiameter of the circle; and from any point (within the figure if n be an odd number, but if even, from any point either within or without) let there be drawn perpendiculars to the sides of the figure; and likewise let there be drawn a right line to the centre of the circle, and let v be the line drawn to the centre; let a be the coefficient of the third term of a binomial raised to the n power, b the coefficient of the fifth term, c the coefficient of the seventh term, and so on; let $A = a \times \frac{1}{2}$, $B = b \times \frac{1.3}{2.4}$, $C = c \times \frac{1.3.5}{2.4.6}$, *and so on: the sum of the n powers of the perpendiculars will be equal to* $mr^n + mAv^2 r^{n-2} + mBv^4 r^{n-4} + mCv^6 r^{n-6} +$ *&c.*

For example, Let $m = 6$, and $n = 5$; and because the coefficients of the terms of a binomial raised to the fifth power are 1, 5, 10, 10, 5, 1, therefore $a = 10$, $b = 5$; and therefore $A = 5$, $B = \frac{15}{8}$. Therefore the theorem will be as follows.

Let there be a hexagon circumscribed about a a circle, and from a point within the hexagon let there be drawn perpendiculars to the ſides of the hexagon, and likewiſe a right line to the centre of the circle, and let r be the ſemidiameter of the circle, and v the line drawn to the centre; the ſum of the fifth powers of the perpendiculars drawn to the ſides of the hexagon will be equal to $6r^5 + 30r^3v^2 + \frac{45}{4}rv^4$

Again, Let $m = 8$, and $n = 6$; and becauſe the coefficients of the terms of a binomial raiſed to the ſixth power are 1, 6, 15, 20, 15, 6, 1, therefore $a = 15$, $b = 15$, $c = 1$; and therefore $A = \frac{15}{2}$, $B = \frac{45}{8}$, $C = \frac{5}{16}$. Therefore the theorem will be as follows.

Let there be an octagon circumſcribed about a circle, and from any point let there be drawn perpendiculars to the ſides of the octagon, and likewiſe a right line to the centre of the circle, and let r be the ſemidiameter of the circle, and v the line drawn to the centre; the ſum of the ſixth powers of the perpendiculars drawn to the ſides of the octagon, will be equal to $8r^6 + 60r^4v^2 + 45r^2v^4 + \frac{5}{2}v^6$. And ſo on.

Cor. I. Let there be any regular figure

circumſcribed about a circle, and let *m* be the number of the ſides of the figure, and *n* any number leſs than *m*; and from a point (within the figure if *n* be an odd number, but if *n* be even, from any point either within or without) let there be drawn perpendiculars to the ſides of the figure, and the ſum of the *n* powers of the perpendiculars be invariable; the point from which the perpendiculars are drawn, will be in the circumference of a given circle.

Cor. II. Let there be two regular figures circumſcribed about a circle, and let the number of the ſides of each figure be greater than *n*; and from any point (within both figures if *n* be an odd number, but if *n* be even, from any point either within or without) let there be drawn perpendiculars to the ſides of both figures; the ſum of the *n* powers of the perpendiculars drawn to the ſides of one of the figures, will be to the ſum of the *n* powers of the perpendiculars drawn to the ſides of the other figure, as the number of the ſides of the one to the number of the ſides of the other.

PRO-

PROPOSITION XLI.

THEOREM XXXVIII.

Let there be any regular figure inscribed in a circle, and let the number of the sides of the figure be m; and let n be any number less than m; let r be the semidiameter of the circle; and from all the angles of the figure let there be drawn right lines to any point in the circumference of the circle: the sum of the 2n powers of the chords, will be equal to $m \times \frac{1.3.5.7 \ldots 2n-1}{1.2.3.4 \ldots n} \times 2^n r^{2n}$.

The numbers in the numerator are to be continued till the last number be $2n-1$, and are to be continually multiplied into one another; the numbers in the denominator are to be continued till the last number be n, and are to be continually multiplied into one another.

For example, Let $m = 4$, and $n = 3$; and the theorem will be as follows.

Let there be a square inscribed in a circle, and from all the angles of the square let there be drawn right lines to any point in the circumference

Again, Let $m = 5$, and $n = 4$; and becauſe the coefficients of the terms of a binomial raiſed to the fourth power are 1, 4, 6, 4, 1, therefore $a = 4$, $b = 6$, $c = 4$, $d = 1$; and therefore $a^2 = 16$, $b^2 = 36$, $c^2 = 16$, $d^2 = 1$. Therefore the theorem will be as follows.

Let there be a pentagon inſcribed in a circle, and from all the angles of the pentagon and the centre of the circle let there be drawn right lines to any point; let r be the ſemidiameter of the circle, and v the line drawn to the centre: the ſum of the fourth powers of the lines drawn from the angles of the pentagon, will be equal to $5r^8 + 80v^2r^6 + 180v^4r^4 + 80v^6r^2 + 5v^8$. And ſo on.

Cor. I. Let there be any regular figure inſcribed in a circle, and let m be the number of the ſides of the figure, and n any number leſs than m; and from all the angles of the figure let there be drawn right lines to a point, and the ſum of the $2n$ powers of the lines drawn from the angles of the figure be invariable; the point to which the lines are drawn, will be in the circumference of a given circle.

Cor. II. Let there be two regular figures inſcribed

inſcribed in a circle, and let the number of the ſides of each figure be greater than n; and from all the angles of both figures let there be drawn right lines to any point: the ſum of the $2n$ powers of the lines drawn from the angles of one of the figures, will be to the ſum of the $2n$ powers of the lines drawn from the angles of the other, as the number of the ſides of the one to the number of the ſides of the other.

PROPOSITION XLIII.

THEOREM XL.

Let there be any number of given points, and let m be their number; let n be any number leſs than $m-1$: there may be found $n+1$ points, ſuch, that if from all the given points and the points found there be drawn right lines to any point, the ſum of the $2n$ powers of the lines drawn from the given points, will be to the ſum of the $2n$ powers of the lines drawn from the points found, as m to $n+1$.

For example, Let $n=3$; and the theorem will be as follows.

Let there be any number greater than 4 of given points; and let m be the number of the given points: 4 points may be found, ſuch, that if from all the given points and the 4 points found there be drawn right lines to any point, the ſum of the ſixth powers of the lines drawn from the given points, will be to the ſum of the ſixth powers of the lines drawn from the 4 points found, as m to 4.

Again, Let $n = 4$; and the theorem will be as follows.

Let there be any number greater than 5 of given points; and let m be their number: 5 points may be found, ſuch, that if from the given points and the 5 points found there be drawn right lines to any point, the ſum of the eighth powers of the lines drawn from the given points, will be to the ſum of the eighth powers of the lines drawn from the 5 points found, as m to 5. And ſo on.

PRO-

PROPOSITION XLIV.

THEOREM XLI.

Let there be any number of given points, and let m be their number; let n be any number less than m, and let a, b, c, &c. be given magnitudes as many in number as there are given points: there may be found $n+1$ points, such, that if from all the given points and the points found there be drawn right lines to any point, the 2n power of the line drawn from one of the given points, together with the power to which the 2n power of the line drawn from another of the given points has the same ratio that a has to b, together with the power to which the 2n power of the line drawn from another of the given points has the same ratio that a has to c, and so on, will be to the sum of the 2n powers of the lines drawn from the points found, as the sum of a, b, c, &c. to $\overline{n+1} \times a$.

For example, Let $n = 3$; and the theorem will be as follows.

 Let

Let there be any number greater than 3 of given points; and let *a*, *b*, *c*, &c. be given magnitudes as many in number as there are given points: 4 points may be found, ſuch, that if from all the given points and the 4 points found there be drawn right lines to any point, the ſixth power of the line drawn from one of the given points, together with the power to which the ſixth power of the line drawn from another of the given points has the ſame ratio that *a* has to *b*, together with the power to which the ſixth power of the line drawn from another of the given points has the ſame ratio that *a* has to *c*, and ſo on, will be to the ſum of the ſixth powers of the lines drawn from the four points found, as the ſum of *a*, *b*, *c*, &c. to 4 *a*.

Again, Let $n = 4$; and the theorem will be as follows.

Let there be any number greater than 4 of given points; and let *a*, *b*, *c*, &c. be given magnitudes as many in number as there are given points: 5 points may be found, ſuch, that if from the given points and the 5 points found there be drawn right lines to any point, the eighth power of the line drawn from one

of

of the given points, together with the power to which the eighth power of the line drawn from another of the given points has the ſame ratio that *a* has to *b*, together with the power to which the eighth power of the line drawn from another of the given points has the ſame ratio that *a* has to *c*, and ſo on, will be to the ſum of the eighth powers of the lines drawn from the 5 points found, as the ſum of *a*, *b*, *c*, &c. to 5 *a*.

PROPOSITION XLV.

THEOREM XLII.

Let there be any number of right lines interſecting each other in a point, and making all the angles round the point of interſection equal; let m be the number of the lines, and let n be any number leſs than m; and from any point let there be drawn perpendiculars to the right lines, and likewiſe a right line to the point of interſection; let v be the line drawn to the point of interſection: the ſum of the 2n powers of the perpendiculars

diculars drawn to the right lines will be equal to $m \times \frac{1.3.5.7 \ldots 2n-1}{1.2.3.4 \ldots n.2^n} \times v^{2n}$.

The numbers in the numerator are to be continued till the laſt number be equal to $2n-1$, and are to be continually multiplied into one another; the numbers in the denominator are to be continued till the laſt number be equal to n, and are to be continually multiplied into one another, and their product by 2^n.

For example, Let $n = 3$; and the theorem will be as follows.

Let there be any number greater than 3 of right lines interſecting each other in a point, and making all the angles round the point of interſection equal; and let m be the number of the lines; and from any point let there be drawn perpendiculars to the right lines, and likewiſe let there be drawn a line to the point of interſection, and let v be the line drawn to the point of interſection: the ſum of the ſixth powers of the perpendiculars drawn to the right lines, will be equal to $m \times \frac{1.3.5}{1.2.3.2^3} \times v^6 = m \times \frac{5}{16} v^6$.

Again,

Again, Let $n = 4$; and the theorem will be as follows.

Let there be any number greater than 5 of right lines interſecting each other in a point, and making all the angles round the point of interſection equal; and let the number of the lines be m; and from any point let there be drawn perpendiculars to the right lines, and likewiſe let there be drawn a right line to the point of interſection, and let v be the line drawn to the point of interſection: the ſum of the eighth powers of the perpendiculars drawn to the right lines, will be equal to $m \times \frac{1.3.5.7}{1.2.3.4.2^4} \times nv^8 = m \times \frac{35}{128} v^8$. And ſo on.

COR. If there be any number of lines interſecting each other in a given point, and making all the angles round the point of interſection equal; and m be the number of the lines, and n any number leſs than m; and from a point there be drawn perpendiculars to the right lines; and the ſum of the $2n$ powers of the perpendiculars be invariable: the point from which the perpendiculars are drawn, will be in the circumference of a given circle.

PRO-

PROPOSITION XLVI.

THEOREM XLIII.

Let there be any number of right lines given by position, that are either all parallel to each other, or interſecting each other in one point; and let m *be the number of the lines, and* n *any number leſs than* $m-1$: *there may be found* $n+1$ *right lines that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the lines found, the ſum of the* $2n$ *powers of the perpendiculars drawn to the right lines given by poſition, will be to the ſum of the* $2n$ *powers of the perpendiculars drawn to the lines found, as* m *to* $n+1$.

For example, Let $n=3$; and the theorem will be as follows.

Let there be any number greater than 4 of right lines given by poſition, that are either all parallel to each other, or interſecting each other in one point; and let m be the number

of

of the right lines given by poſition: there may be found 4 right lines that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the lines found, the ſum of the ſixth powers of the perpendiculars drawn to the right lines given by poſition, will be to the ſum of the ſixth powers of the perpendiculars drawn to the lines found, as m to 4.

Again, Let $n=4$; and the theorem will be as follows.

Let there be any number greater than 5 of right lines given by poſition, that are either all parallel to each other, or interſecting each other in one point; and let m be the number of the right lines given by poſition: there may be found 5 right lines that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the 5 lines found, the ſum of the eighth powers of the perpendiculars drawn to the right lines given by poſition, will be to the ſum of the eighth powers of the perpendiculars drawn to the 5 lines found, as m to 5. And ſo on.

PROPOSITION XLVII.

THEOREM XLIV.

Let there be any number of right lines given by position, that are either all parallel to each other, or intersecting each other in one point; let m be the number of the right lines given by position, and let n be any number less than m; let a, b, c, &c. be given magnitudes as many in number as there are right lines given by position: there may be found $n+1$ right lines that will be given by position, such, that if from any point there be drawn perpendiculars to the right lines given by position, and likewise there be drawn perpendiculars to the lines found, the 2n power of the perpendicular drawn to one of the lines given by position, together with the power to which the 2n power of the perpendicular drawn to another of the lines given by position has the same ratio that a has to b, together with the power to which the 2n power of the perpendicular drawn to another of the lines given by position has the same ratio that a has to c, and so on, will be to the

the sum of the 2n powers of the perpendiculars drawn to the lines found, as the sum of a, b, c, &c. *to* $\overline{n+1} \times a$.

For example, Let $n = 3$; and the theorem will be as follows.

Let there be any number greater than 3 of right lines given by position, that are either all parallel to each other, or intersecting each other in one point; and let *a, b, c, &c.* be given magnitudes as many in number as there are right lines given by position: 4 right lines may be found that will be given by position, such, that if from any point there be drawn perpendiculars to the right lines given by position, and likewise there be drawn perpendiculars to the 4 lines found; the sixth power of the perpendicular drawn to one of the right lines given by position, together with the power to which the sixth power of the perpendicular drawn to another of the lines given by position has the same ratio that *a* has to *b*, together with the power to which the sixth power of the perpendicular drawn to another of the lines given by position has the same ratio that *a* has to *c*, and

Let there be any number greater than 3 of given points; and let *a*, *b*, *c*, &c. be given magnitudes as many in number as there are given points: 4 points may be found, ſuch, that if from all the given points and the 4 points found there be drawn right lines to any point, the ſixth power of the line drawn from one of the given points, together with the power to which the ſixth power of the line drawn from another of the given points has the ſame ratio that *a* has to *b*, together with the power to which the ſixth power of the line drawn from another of the given points has the ſame ratio that *a* has to *c*, and ſo on, will be to the ſum of the ſixth powers of the lines drawn from the four points found, as the ſum of *a*, *b*, *c*, &c. to 4 *a*.

Again, Let $n = 4$; and the theorem will be as follows.

Let there be any number greater than 4 of given points; and let *a*, *b*, *c*, &c. be given magnitudes as many in number as there are given points: 5 points may be found, ſuch, that if from the given points and the 5 points found there be drawn right lines to any point, the eighth power of the line drawn from one

of

of the given points, together with the power to which the eighth power of the line drawn from another of the given points has the ſame ratio that *a* has to *b*, together with the power to which the eighth power of the line drawn from another of the given points has the ſame ratio that *a* has to *c*, and ſo on, will be to the ſum of the eighth powers of the lines drawn from the 5 points found, as the ſum of *a*, *b*, *c*, &c. to 5 *a*.

PROPOSITION XLV.

THEOREM XLII.

Let there be any number of right lines interſecting each other in a point, and making all the angles round the point of interſection equal; let m be the number of the lines, and let n be any number leſs than m; and from any point let there be drawn perpendiculars to the right lines, and likewiſe a right line to the point of interſection; let v be the line drawn to the point of interſection: the ſum of the 2n powers of the perpendiculars

diculars drawn to the right lines will be equal to $m \times \frac{1.3.5.7\ldots 2n-1}{1.2.3.4\ldots n.2^n} \times v^{2n}$.

The numbers in the numerator are to be continued till the laſt number be equal to $2n-1$, and are to be continually multiplied into one another; the numbers in the denominator are to be continued till the laſt number be equal to n, and are to be continually multiplied into one another, and their product by 2^n.

For example, Let $n=3$; and the theorem will be as follows.

Let there be any number greater than 3 of right lines interſecting each other in a point, and making all the angles round the point of interſection equal; and let m be the number of the lines; and from any point let there be drawn perpendiculars to the right lines, and likewiſe let there be drawn a line to the point of interſection, and let v be the line drawn to the point of interſection: the ſum of the ſixth powers of the perpendiculars drawn to the right lines, will be equal to $m \times \frac{1.3.5}{1.2.3.2^3} \times v^6 = m \times \frac{5}{16} v^6$.

Again,

Again, Let $n = 4$; and the theorem will be as follows.

Let there be any number greater than 5 of right lines interſecting each other in a point, and making all the angles round the point of interſection equal; and let the number of the lines be m; and from any point let there be drawn perpendiculars to the right lines, and likewiſe let there be drawn a right line to the point of interſection, and let v be the line drawn to the point of interſection: the ſum of the eighth powers of the perpendiculars drawn to the right lines, will be equal to $m \times \frac{1.3.5.7}{1.2.3.4.2^4} \times nv^8 = m \times \frac{35}{128} v^8$. And ſo on.

Cor. If there be any number of lines interſecting each other in a given point, and making all the angles round the point of interſection equal; and m be the number of the lines, and n any number leſs than m; and from a point there be drawn perpendiculars to the right lines; and the ſum of the $2n$ powers of the perpendiculars be invariable: the point from which the perpendiculars are drawn, will be in the circumference of a given circle.

PRO-

PROPOSITION XLVI.

THEOREM XLIII.

Let there be any number of right lines given by position, that are either all parallel to each other, or interſecting each other in one point; and let m be the number of the lines, and n any number leſs than $m-1$: there may be found $n+1$ right lines that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the lines found, the ſum of the $2n$ powers of the perpendiculars drawn to the right lines given by poſition, will be to the ſum of the $2n$ powers of the perpendiculars drawn to the lines found, as m to $n+1$.

For example, Let $n=3$; and the theorem will be as follows.

Let there be any number greater than 4 of right lines given by poſition, that are either all parallel to each other, or interſecting each other in one point; and let m be the number of of the right lines given by poſition: there may be found 4 right lines that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the lines found, the ſum of the ſixth powers of the perpendiculars drawn to the right lines given by poſition, will be to the ſum of the ſixth powers of the perpendiculars drawn to the lines found, as m to 4.

Again, Let $n = 4$; and the theorem will be as follows.

Let there be any number greater than 5 of right lines given by poſition, that are either all parallel to each other, or interſecting each other in one point; and let m be the number of the right lines given by poſition: there may be found 5 right lines that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the 5 lines found, the ſum of the eighth powers of the perpendiculars drawn to the right lines given by poſition, will be to the ſum of the eighth powers of the perpendiculars drawn to the 5 lines found, as m to 5. And ſo on.

PROPOSITION XLVII.

THEOREM XLIV.

Let there be any number of right lines given by position, that are either all parallel to each other, or intersecting each other in one point; let m be the number of the right lines given by position, and let n be any number less than m; let a, b, c, &c. *be given magnitudes as many in number as there are right lines given by position: there may be found n + 1 right lines that will be given by position, such, that if from any point there be drawn perpendiculars to the right lines given by position, and likewise there be drawn perpendiculars to the lines found, the 2n power of the perpendicular drawn to one of the lines given by position, together with the power to which the 2n power of the perpendicular drawn to another of the lines given by position has the same ratio that a has to b, together with the power to which the 2n power of the perpendicular drawn to another of the lines given by position has the same ratio that a has to c, and so on, will be to the*

the ſum of the $2n$ *powers of the perpendiculars drawn to the lines found, as the ſum of* a, b, c, &c. *to* $\overline{n+1} \times a$.

For example, Let $n = 3$; and the theorem will be as follows.

Let there be any number greater than 3 of right lines given by poſition, that are either all parallel to each other, or interſecting each other in one point; and let a, b, c, *&c.* be given magnitudes as many in number as there are right lines given by poſition: 4 right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the 4 lines found, the ſixth power of the perpendicular drawn to one of the right lines given by poſition, together with the power to which the ſixth power of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that a has to b, together with the power to which the ſixth power of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that a has to c, and ſo on, will be to the ſum of the ſixth powers of the perpendiculars drawn to the 4 lines found, as the ſum of *a*, *b*, *c*, &c. to 4*a*.

Again, Let $n = 4$; and the theorem will be as follows.

Let there be any number greater than 4 of right lines given by poſition, that are either all parallel to each other, or interſecting each other in one point; and let *a*, *b*, *c*, &c. be given magnitudes as many in number as there are right lines given by poſition: 5 right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the 5 lines found, the eighth power of the perpendicular drawn to one of the lines given by poſition, together with the power to which the eighth power of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that *a* has to *b*, together with the power to which the eighth power of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that *a* has to *c*, and ſo on, will be to the ſum of the eighth powers of of the perpendiculars drawn to the 5 lines found, as the ſum of *a*, *b*, *c*, &c. to 5*a*. And ſo on.

Cor. Let there be any number of right lines given by poſition, that are either all parallel to each other, or interſecting each other in one point; let *m* be the number of the right lines given by poſition, and let *n* be any number leſs than *m*: there may be found $n+1$ right lines that will be given by poſition, ſuch, that if from any point there be drawn right lines in given angles to all the right lines given by poſition, and likewiſe there be drawn perpendiculars to the right lines found, the ſum of the $2n$ powers of the right lines drawn in given angles to the right lines given by poſition, will be to the ſum of the $2n$ powers of the perpendiculars drawn to the lines found in a given ratio.

PROPOSITION XLVIII.

Theorem XLV.

Let there be any number of right lines given by poſition, that are neither all parallel to each other,

other, nor interſecting each other in one point; and let the number of the right lines given by poſition be m, and let n be any even number leſs than m—1: there may be found n+1 right lines that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to all the right lines given by poſition, and likewiſe there be drawn perpendiculars to the right lines found, the ſum of the n powers of the perpendiculars drawn to the right lines given by poſition, will be to the ſum of the n powers of the perpendiculars drawn to the lines found, as m to n+1.

For example, Let $n = 6$; and the theorem will be as follows.

Let there be any number greater than 7 of right lines given by poſition, that are neither all parallel to each other, nor interſecting each other in one point; and let *m* be the number of the right lines given by poſition: there may be found 7 right lines that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the 7 lines found, the ſum of of the ſixth powers of the perpendiculars drawn to the right lines given by poſition, will be to the ſum of the ſixth powers of the perpendiculars drawn to the 7 lines found, as m to 7.

Again, Let $n = 8$; and the theorem will be as follows.

Let there be any number greater than 9 of right lines given by poſition, that are neither all parallel to each other, nor interſecting each other in one point; and let m be the number of the right lines given by poſition: there may be found 9 right lines that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the 9 lines found, the ſum of the eighth powers of the perpendiculars drawn to the right lines given by poſition, will be to the ſum of the eighth powers of the perpendiculars drawn to the 9 lines found, as m to 9. And ſo on.

PROPOSITION XLIX.

THEOREM XLVI.

Let there be any number of right lines given by position, that are neither all parallel to each other, nor intersecting each other in one point; let m be the number of the right lines given by position, and let a, b, c, &c. be given magnitudes as many in number as there are right lines given by position; let n be any even number less than m: there may be found n + 1 right lines that will be given by position, such, that if from any point there be drawn perpendiculars to the right lines given by position, and likewise there be drawn perpendiculars to the right lines found, the n power of the perpendicular drawn to one of the lines given by position, together with the power to which the n power of the perpendicular drawn to another of the lines given by position has the same ratio that a has to b, together with the power to which the n power of the perpendicular drawn to another of the lines given by position has the same ratio that a has to c, and so on, will be to the

the sum of the n powers of the perpendiculars drawn to the lines found, as the sum of a, b, c, &c. *to* $\overline{n+1} \times a$.

For example, Let $n = 6$; and the theorem will be as follows.

Let there be any number greater than 6 of right lines given by position, that are neither all parallel to each other, nor intersecting each other in one point; and let a, b, c, *&c.* be given magnitudes as many in number as there are right lines given by position: there may be found 7 right lines that will be given by position, such, that if from any point there be drawn perpendiculars to the right lines given by position, and likewise there be drawn perpendiculars to the 7 lines found, the sixth power of the perpendicular drawn to one of the lines given by position, together with the power to which the sixth power of the perpendicular drawn to another of the lines given by position has the same ratio that a has to b, together with the power to which the sixth power of the perpendicular drawn to another of the lines given by position has the same ratio that a has to c, and so on, will be to the sum of the

the ſixth powers of the perpendiculars drawn the lines found, as the ſum of *a*, *b*, *c*, &c. to 7*a*.

Again, Let $n = 8$; and the theorem will be as follows.

Let there be any number greater than 8 of right lines given by poſition, that are neither all parallel to each other, nor interſecting each other in one point; and let *a*, *b*, *c*, &c. be given magnitudes as many in number as there are right lines given by poſition: there may be found 9 right lines that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the lines found, the eighth power of the perpendicular drawn to one of the lines given by poſition, together with the power to which the eighth power of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that *a* has to *b*, together with the power to which the eighth power of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that *a* has to *c*, and ſo on, will be to the ſum of the eighth powers of the perpendicu-

lars

lars drawn to the lines found, as the ſum of *a*, *b*, *c*, *&c.* to 9*a*.

Cor. Let there be any number of right lines given by poſition, that are neither all parallel to each other, nor interſecting each other in one point; let *m* be the number of the lines given by poſition, and let *n* be any even number leſs than *m*: there may be found $n+1$ right lines that will be given by poſition, ſuch, that if from any point there be drawn right lines in given angles to all the right lines given by poſition, and likewiſe there be drawn perpendiculars to the right lines found, the ſum of the *n* powers of the lines drawn in given angles to the right lines given by poſition, will be to the ſum of the *n* powers of the perpendiculars drawn to the right lines found in a given ratio.

PROPOSITION L.

Theorem XLVII.

Let there be any figure given by poſition; let m be the number of the ſides of the figure, and let n be any odd number leſs than $m-1$:

 there

there may be found $n+1$ *right lines that will be given by position, such, that if from any point within the figure there be drawn perpendiculars to the sides of the figure, and likewise there be drawn perpendiculars to the lines found, the sum of the* n *powers of the perpendiculars drawn to the sides of the figure, will be to the sum of the* n *powers of the perpendiculars drawn to the lines found, as* m *to* $n+1$.

For example, Let $n=5$; and the theorem will be as follows.

Let there be any figure given by position of a greater number of sides than 6; and let m be the number of the sides of the figure: there may be found 6 right lines that will be given by position, such, that if from any point within the figure there be drawn perpendiculars to the sides of the figure, and likewise there be drawn perpendiculars to the 6 lines found, the sum of the fifth powers of the perpendiculars drawn to the sides of the figure, will be to the sum of the fifth powers of the perpendiculars drawn to the 6 lines found, as m to 6.

Again,

Again, Let $n=7$; and the theorem will be as follows.

Let there be any figure given by position of a greater number of sides than 8; and let m be the number of the sides of the figure; there may be found 8 right lines that will be given by position, such, that if from any point within the figure there be drawn perpendiculars to the sides of the figure, and likewise there be drawn perpendiculars to the 8 lines found, the sum of the seventh powers of the perpendiculars drawn to the sides of the figure, will be to the sum of the seventh powers of the perpendiculars drawn to the 8 lines found, as m to 8. And so on.

PROPOSITION LI.

THEOREM XLVIII.

Let there be any figure given by position; and let m be the number of the sides of the figure, and let n be any odd number less than m; and let a, b, c, &c. be given magnitudes as many in number as there are sides in the figure: there may be found n+1 right lines *that will be given by position, such, that if from any point within the figure there be drawn perpendiculars to the sides of the figure, and likewise there be drawn perpendiculars to the lines found, the n power of the perpendicular drawn to one of the sides of the figure, together with the power to which the n power of the perpendicular drawn to another of the sides of the figure has the same ratio that a has to b, together with the power to which the n power of the perpendicular drawn to another of the sides of the figure has the same ratio that a has to c, and so on, will be to the sum of the n powers of the perpendiculars drawn to the lines found, as the sum of a, b, c &c. to* $\overline{n+1} \times a$.

For example, Let $n = 5$; and the theorem will be as follows.

Let there be any figure given by position of a greater number of sides than 5; and let a, b, c, &c. be given magnitudes as many in number as there are sides in the figure: there may be found 6 right lines that will be given by position, such, that if from any point within the figure there be drawn perpendiculars

lars to the ſides of the figure, and likewiſe there be drawn perpendiculars to the lines found, the fifth power of the perpendicular drawn to one of the ſides of the figure, together with the power to which the fifth power of the perpendicular drawn to another of the ſides of the figure has the ſame ratio that a has to b, together with the power to which the fifth power of the perpendicular drawn to another of the ſides of the figure has the ſame ratio that a has to c, and ſo on, will be to the ſum of the fifth powers of the perpendiculars drawn to the lines found, as the ſum of a, b, c, &c. to $6a$.

Again, Let $n = 7$; and the theorem will be as follows.

Let there be any figure given by poſition of a greater number of ſides than 7; and let a, b, c, &c. be given magnitudes as many in number as there are ſides in the figure: 8 lines may be found that will be given by poſition, ſuch, that if from any point within the figure there be drawn perpendiculars to the ſides of the figure, and likewiſe there be drawn perpendiculars to the 8 lines found, the ſeventh power of the perpendicular drawn to

one

one of the ſides of the figure, together with the power to which the ſeventh power of the perpendicular drawn to another of the ſides of the figure has the ſame ratio that a has to b, together with the power to which the ſeventh power of the perpendicular drawn to another of the ſides of the figure has the ſame ratio that a has to c, and ſo on, will be to the ſum of the ſeventh powers of the perpendiculars drawn to the lines found, as the ſum of a, b, c, *&c.* to $8a$.

Cor. Let there be any figure given by poſition; and let m be the number of the ſides of the figure, and let n be any odd number leſs than m: there may be found $n+1$ right lines that will be given by poſition, ſuch, that if from any point within the figure there be drawn right lines in given angles to the ſides of the figure, and likewiſe there be drawn perpendiculars to the lines found, the ſum of the n powers of the lines drawn in given angles to the ſides of the figure, will be to the ſum of the n powers of the perpendiculars drawn to the lines found in a given ratio.

N. B. In the following theorems by taking

a

a point always on the ſame ſide the right lines given by poſition, we are to underſtand that the point muſt not be taken on different ſides any one of the right lines given by poſition.

The two laſt theorems may be made more general thus.

PROPOSITION LII.

THEOREM XLIX.

Let there be any number of right lines given by poſition; and let m be the number of the lines, and n any odd number leſs than m—1: there may be found n+1 right lines that will be given by poſition, ſuch, that if from any point always taken on the ſame ſide the right lines given by poſition there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the lines found, the ſum of the n powers of the perpendiculars drawn to the right lines given by poſition, will be to the ſum of the n powers of the perpendiculars drawn to the lines found, as m to n+1.

For example, Let $n = 3$; and the theorem will be as follows.

Let there be any number greater than 4 of right lines given by poſition; and let m be the number of the lines: 4 right lines may be found that will be given by poſition, ſuch, that if from any point always taken on the ſame ſide the right lines given by poſition there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the 4 lines found, the ſum of the cubes of the perpendiculars drawn to the right lines given by poſition, will be to the ſum of the cubes of the perpendiculars drawn to the 4 lines found, as m to 4.

Again, Let $n = 5$; and the theorem will be as follows.

Let there be any number greater than 6 of right lines given by poſition; and let m be the number of the lines: 6 right lines may be found that will be given by poſition, ſuch, that if from any point always taken on the ſame ſide the right lines given by poſition there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the 6 lines found, the ſum

of

of the sixth powers of the perpendiculars drawn to the right lines given by position, will be to the sum of the sixth powers of the perpendiculars drawn to the 6 lines found, as *m* to 6. And so on.

PROPOSITION LIII.

THEOREM L.

Let there be any number of right lines given by position; and let m be the number of the lines, and n any odd number less than m; and let a, b, c, &c. be given magnitudes as many in number as there are right lines given by position: there may be found n+1 right lines that will be given by position, such, that if from any point always taken on the same side the right lines given by position there be drawn perpendiculars to the right lines given by position, and likewise there be drawn perpendiculars to the lines found, the n power of the perpendicular drawn to one of the lines given by position, together with the power to which the n power of the perpendicular drawn to another of the lines given

ven by position has the same ratio that a has to b, together with the power to which the n power of the perpendicular drawn to another of the lines given by position has the same ratio that a has to c, and so on, will be to the sum of the n powers of the perpendiculars drawn to the lines found, as the sum of a, b, c, &c. *to* $\overline{n+1} \times a$.

For example, Let $n = 3$; and the theorem will be as follows.

Let there be any number greater than 4 of right lines given by position; and let *a*, *b*, *c*, *&c.* be given magnitudes as many in number as there are right lines given by position: 4 lines may be found that will be given by position, such, that if from any point always taken on the same side the right lines given by position there be drawn perpendiculars to the right lines given by position, and likewise there be drawn perpendiculars to the lines found, the cube of the perpendicular drawn to one of the lines given by position, together with the solid to which the cube of the perpendicular drawn to another of the lines given by position has the same ratio that *a* has to *b*, to *b*, together with the ſolid to which the cube of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that *a* has to *c*, and ſo on, will be to the ſum of the cubes of the perpendiculars drawn to the lines found, as the ſum of *a*, *b*, *c*, &c. to $4a$.

Again, Let $n = 5$; and the theorem will be as follows.

Let there be any number greater than 6 of right lines given by poſition; and let *a*, *b*, *c*, &c. be given magnitudes as many in number as there are right lines given by poſition: 6 lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the lines found, the fifth power of the perpendicular drawn to one of the lines given by poſition, together with the power to which the fifth power of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that *a* has to *b*, together with the power to which the fifth power of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that

a has

a has to *c*, and ſo on, will be to the ſum of the fifth powers of the perpendiculars drawn to the lines found, as the ſum of *a*, *b*, *c*, &c. to 6*a*. And ſo on.

Cor. Let there be any number of right lines given by poſition; and let *m* be the number of the lines, and *n* any odd number leſs than *m*: there may be found *n* + 1 right lines that will be given by poſition, ſuch, that if from any point always taken on the ſame ſide the right lines given by poſition there be drawn lines in given angles to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the lines found, the ſum of the *n* powers of the lines drawn in given angles to the right lines given by poſition, will be to the ſum of the *n* powers of the perpendiculars drawn to the lines found in a given ratio.

PROPOSITION LIV.

Let there be any number of right lines given by poſition, and parallel to each other; and let n be any given number; and from a point let there be drawn right lines in given angles

to

to all the right lines given by position; and let the sum of the n powers of the lines drawn in given angles be invariable: the point from which the lines are drawn, will be in a right line given by position.

For example, Let $n = 2$; and the proposition will be as follows.

Let there be any number of right lines given by position, and parallel to each other; and from a point let there be drawn right lines in given angles to the right lines given by position; and let the sum of the squares of the lines drawn in given angles be invariable: the point from which the lines are drawn, will be in a right line given by position.

Again, Let $n = 3$; and the proposition will be as follows.

Let there be any number of right lines given by position, and parallel to each other; and from a point let there be drawn right lines in given angles to the right lines given by position; and let the sum of the cubes of the lines drawn in given angles be invariable: the point from which the lines are drawn, will be in a right line given by position.

PRO-

PROPOSITION LV.

Let there be any number of right lines given by position, that are not all parallel to each other; let $n+1$ be the number of the lines; and from a point let there be drawn perpendiculars to the right lines given by position; and let the sum of the n powers of the perpendiculars be invariable: the point from which the perpendiculars are drawn, will be in an oval figure that is a line of the n order, or a line of an inferior order.

This is a locus not only to all the various oval figures of lines of the n order, but likewise to all the various oval figures of lines of any lower order; that is, the various cases of this locus will comprehend all the various oval figures of lines of the n order, and likewise all the various oval figures of lines of any lower order.

For example, Let $n=3$; and the proposition will be as follows.

Let there be four right lines given by position, that are not all parallel to each other; and

and from a point let there be drawn perpendiculars to the right lines given by poſition; and let the ſum of the cubes of the perpendiculars be invariable: the point from which the perpendiculars are drawn, will be in an oval figure that is a line of the third order, or will be in an ellipſe or circle; and the various caſes of this locus will comprehend all the oval figures that are lines of the third order, and likewiſe the ellipſe and circle.

If the right lines given by poſition be the ſides of a ſquare, and the point from which the perpendiculars are drawn be within the ſquare, the point will be in a circle. But

If the right lines given by poſition be the ſides of a parallelogram, and the point from which the perpendiculars are drawn be taken within the parallelogram, the point will be in an ellipſe.

Again, Let $n = 4$; and the propoſition will be as follows.

Let there be five right lines given by poſition, that are not all parallel to each other; and from a point let there be drawn perpendiculars to the right lines given by poſition; and let the ſum of the fourth powers of the perpendiculars be invariable: the point from which the perpendiculars are drawn, will be in an oval figure that is a line of the fourth order, or will be in an oval figure that is a line of an inferior order.

This is a locus not only to all the various oval figures of lines of the fourth order, but likewise to all the oval figures of lines of the third and second order; that is, the various cases of this locus will comprehend all the various oval figures of lines of the fourth order, and likewise all the various oval figures of lines of the third order, and also the ellipse and circle. And so on.

PROPOSITION LVI.

Let there be any number p of right lines given by position, and likewise let there be any number q of right lines given by position; and let all the lines be either parallel to each other, or all intersecting each other in one point; and let n be any number; and from a point let there be drawn right lines in given angles to all the right lines given by position;

ſition; and let the ſum of the n powers of the lines drawn in given angles to the firſt number of right lines given by poſition be to the ſum of the n powers of the lines drawn in given angles to the ſecond number of right lines given by poſition in a given ratio: the point from which the lines are drawn, will be in a right line given by poſition.

For example, Let $n = 2$; and the propoſition will be as follows.

Let there be p number of right lines given by poſition, and likewiſe let there be q number of right lines given by poſition; and let all the right lines be either parallel to each other, or all interſecting each other in one point; and from a point let there be drawn right lines in given angles to all the right lines given by poſition; and let the ſum of the ſquares of the lines drawn in given angles to the firſt number of right lines given by poſition be to the ſum of the ſquares of the lines drawn in given angles to the ſecond number of right lines given by poſition in a given ratio: the point from which the lines are drawn, will be in a right line given by poſition.

 Again,

Again, Let $n = 3$; and the proposition will be as follows.

Let there be any number p of right lines given by position, and likewise let there be any number q of right lines given by position; and let all the right lines be either parallel to each other, or all intersecting each other in one point; and from a point let there be drawn right lines in given angles to all the right lines given by position; and let the sum of the cubes of the lines drawn in given angles to the first number of right lines given by position be to the sum of the cubes of the right lines drawn in given angles to the second number of right lines given by position in a given ratio: the point from which the lines are drawn in given angles, will be in a right line given by position.

PROPOSITION LVII.

Let there be any even number of right lines given by position; let $2n+2$ be the number of the lines; and from a point let there be drawn perpendiculars to the right lines gi-

ven

ven by position; and let the sum of the n powers of the perpendiculars drawn to half the number of right lines given by position be to the sum of the n powers of the perpendiculars drawn to the remaining number of right lines given by position in a given ration: the point from which the perpendiculars are drawn, will be in a line of the n order, or will be in a line of an inferior order.

This is a locus not only to all the various lines of the *n* order, but likewise to all the various lines of any lower order; that is, the various cases of this locus will comprehend all the various lines of the *n* order, and likewise all the various lines of any inferior order.

For example, Let $n=3$; and the proposition will be as follows,

Let there be 8 right lines given by position; and from a point let there be drawn perpendiculars to the right lines given by position; and let the sum of the cubes of the perpendiculars drawn to four of the lines given by position be to the sum of the cubes of the perpendiculars drawn to the other 4 lines given by position in a given ratio; the point from which

which the perpendiculars are drawn, will be in a line of the third order, or will be in a line of the second order, or in a right line.

This is a locus not only to all the various lines of the third order, but likewise to all the lines of the second order, and likewise to a right line; that is, the various cases of this locus will comprehend all the lines of the third order, and all the lines of the second order, and a right line.

Let there be two parallelograms given by position; and from a point within both figures let there be drawn perpendiculars to the sides of the figures; and let the sum of the cubes of the perpendiculars drawn to the sides of the one be to the sum of the cubes of the perpendiculars drawn to the sides of the other in a given ratio: the point from which the perpendiculars are drawn, will be in a conic section; and if the figures be both squares, the point will be in a circle, or in a right line.

In many other cases of this proposition, the point will be in a conic section, and also in a circle, or in a right line.

Again, Let $n = 4$; and the proposition will be as follows.

Let

Let there be ten right lines given by poſi-tion; and from a point let there be drawn per-pendiculars to the right lines given by poſition; and let the ſum of the fourth powers of the perpendiculars drawn to five of the right lines given by poſition be to the ſum of the fourth powers of the perpendiculars drawn to the o-ther five lines given by poſition in a given ratio: the point from which the perpendicu-lars are drawn, will be in a line of the fourth order, or will be in a line of the third or ſe-cond order, or will be in a right line.

This is a locus not only to all the various lines of the fourth order, but likewiſe to all the various lines of the third and ſecond order; that is, the various caſes of this locus will comprehend all the lines of the fourth order, and likewiſe all the lines of the third and ſe-cond order, and alſo a right line.

PROPOSITION LVIII.

Let there be any number of right lines given by poſition, that are not all parallel to each o-ther; and let n be any number; and from a point let there be drawn right lines in given

angles

angles to the right lines given by position; and let the sum of the n powers of the lines drawn in given angles be invariable: the point from which the lines are drawn, will be in an oval figure that is a line of the n order, or a line of an inferior order.

If the number of the right lines given by position be greater than *n*, there may be found [by *Cor.* to 49. & 53.] $n+1$ right lines that will be given by position, such, that if from any point there be drawn right lines in given angles to all the right lines given by position, and likewise there be drawn perpendiculars to the lines found, the sum of the *n* powers of the lines drawn in given angles to the right lines given by position, will be to the sum of the *n* powers of the perpendiculars drawn to the lines found, as the number of the lines given by position to $n+1$; and because the sum of the *n* powers of the lines drawn in given angles to the right lines given by position is invariable, therefore the sum of the *n* powers of the perpendiculars drawn to the $n+1$ lines found will be invariable: Therefore [54.] the point from which the perpendiculars are

are drawn, will be in an oval figure that is a line of the n order, or a line of an inferior order: And therefore the point from which the lines are drawn in given angles to the right lines given by position, will be in an oval figure that is a line of the n order, or a line of an inferior order.

Let $n = 2$; and the proposition will be as follows.

Let there be any number of right lines given by position that are not all parallel to each other; and from a point let there be drawn right lines in given angles to all the right lines given by position; and let the sum of the squares of the lines drawn in given angles be invariable: the point from which the lines are drawn will be in an ellipse.

Again, Let $n = 3$; and the proposition will be as follows.

Let there be any number of right lines given by position that are not all parallel to each other; and from a point let there be drawn right lines in given angles to all the right lines given by position; and let the sum of the cubes of the lines drawn in given angles be invariable: the point from which the lines are

 drawn

drawn in given angles, will be in an oval figure that is a line of the third order, or will be in an ellipse or circle.

PROPOSITION LIX.

Let there be any number p of right lines given by position, and likewise let there be any number q of right lines given by position, and let n be any given number; and from a point let there be drawn right lines in given angles to all the right lines given by position; and let the sum of the n powers of the lines drawn in given angles to the first number of right lines given by position be to the sum of the n powers of the lines drawn in given angles to the second number of right lines given by position in a given ratio: the point from which the lines are drawn will be in a line of the n order, or will be in a line of an inferior order.

Let $n = 2$; and the proposition will be as follows.

Let there be any number p of right lines given by position, and likewise let there be any

any number *q* of right lines given by poſition; and from a point let there be drawn right lines in given angles to all the right lines given by poſition; and let the ſum of the ſquares of the lines drawn in given angles to the firſt number of right lines given by poſition be to the ſum of the ſquares of the lines drawn in given angles to the ſecond number of right lines given by poſition in a given ratio: the point from which the lines are drawn, will be in a line of the ſecond order, or in a right line; that is, the point will be in a conic ſection.

Let there be two circles given by poſition, and about each of the circles let there be a regular figure circumſcribed; and from a point let there be drawn perpendiculars to the ſides of both figures; and let the ſum of the ſquares of the perpendiculars drawn to the ſides of one of the figures be to the ſum of the ſquares of the perpendiculars drawn to the ſides of the other figure in a given ratio: the point from which the perpendiculars are drawn, will be in a circle, or in a right line. If the given ratio be the ſame with that of the number of the ſides of the firſt figure to the number of the ſides of the ſecond, the point from which the perpendiculars are drawn, will be in a right line given by poſition.

In many other caſes of this propoſition, the point will be either in a circle, or in a right line.

Again, Let $n = 3$; and the propoſition will be as follows.

Let there be any number p of right lines given by poſition, and likewiſe let there be any number q of right lines given by poſition; and from a point let there be drawn right lines in given angles to all the right lines given by poſition; and let the ſum of the cubes of the lines drawn in given angles to the firſt number of right lines given by poſition be to the ſum of the cubes of the lines drawn in given angles to the ſecond number of right lines given by poſition in a given ratio: the point from which the lines are drawn, will be in a line of the third order, or in a line of the ſecond order, or in a right line.

For example, Let there be a parallelogram given by poſition, and likewiſe let there be a regular figure of a greater number of ſides than three circumſcribed about a circle given by poſition; and from a point within both fi-

gures

gures let there be drawn perpendiculars to the ſides of both figures; and let the ſum of the cubes of the perpendiculars drawn to the ſides of one of the figures be to the ſum of the cubes of the perpendiculars drawn to the ſides of the other figure in a given ratio: the point from which the perpendiculars are drawn, will be in a line of the ſecond order, or will be in a right line; that is, the point will be in a conic ſection.

Let there be two circles given by poſition, and about each of the circles let there be a regular figure of a greater number of ſides than three circumſcribed; and from a point within both figures let there be drawn perpendiculars to the ſides of both figures; and let the ſum of the cubes of the perpendiculars drawn to the ſides of one of the figures be to the ſum of the cubes of the perpendiculars drawn to the ſides of the other in a given ratio: the point from which the perpendiculars are drawn, will be in a circle, or in a right line.

If the multiple of the diameter of the firſt circle by the number of the ſides of the figure circumſcribed about it be to the multiple of the diameter of the ſecond circle by the number

ber of the ſides of the figure circumſcribed about it in the given ratio, that is, as the ſum of the cubes of the perpendiculars drawn to the ſides of the figure circumſcribed about the firſt circle to the ſum of the cubes of the perpendiculars drawn to the ſides of the figure circumſcribed about the ſecond circle; the point from which the perpendiculars are drawn, will be in a right line.

In many other caſes of this propoſition, the point from which the lines are drawn, will be in a circle, or in a right line.

The following propoſitions are a few properties of the circle that occurred when conſidering ſome of the foregoing propoſitions.

PROPOSITION LX. *Fig.* 19.

Let there be a circle given by poſition, and let A, B *be two given points; a point* C *may be found within the circle, ſuch, that if through the point* C *there be drawn any line meeting the circle in* D, E, *and* AD, BD, AE, BE *be joined, the rectangle* ADB *will be to the rectangle* AEB *as* CD *to* CE.

PRO-

PROPOSITION LXI. *Fig.* 20.

Let there be a circle given by position, and two points A, B *given; two right lines* DE, DF *may be found, such, that if from the points* A, B *there be drawn* AG, BG *to any point* G *in the circumference of the circle, and from the point* G *there be drawn* GH, GK *perpendicular to* DE, DF, *the sum of the squares of* GH, GK *will be to the rectangle* AGB *as the rectangle* AGB *to a certain given space; that is, the rectangle* AGB *will be a mean proportional between the sum of the squares of* GH, GK *and a certain given space.*

PROPOSITION LXII. *Fig.* 21.

Let there be a circle given by position, and let there be two right lines AB, AC *given by position; and let a, b be two given magnitudes: a point* D *may be found, such, that if through the point* D *there be drawn any right line meeting the circle in* E, F, *and from the point* E *there be drawn* EG, EH *perpendicular*

perpendicular to AB, AC, *and from the point* F *there be drawn likewise* FK, FL *perpendicular to* AB, AC, *the square of* EG *together with the space to which the square of* EH *has the same ratio that a has to b, will be to the square of* FK *together with the space to which the square of* FL *has the same ratio that a has to b, as the square of* ED *to the square of* DF.

PROPOSITION LXIII. *Fig.* 22. 23.

Let there be a circle given by position, and let AB, AC *be two right lines given by position, and let the angle* BAC *be equal to two angles of an equilateral triangle; two right lines* DE, DF *may be found that will be given by position, such, that if from any point* G *in the circumference of the circle within the angle* BAC *there be drawn* GH, GK *perpendicular to* AB, AC, *and likewise there be drawn* GL, GM *perpendicular to* DE, DF, *the sum of the cubes of* GH, GK *will be equal to a solid whose base is the sum of the squares of* GL, GM, *and altitude a given line.*

PRO-

PROPOSITION LXIV. *Fig.* 24.

Let there be a circle given by position, and AB, AC *two right lines given by position, and let the angle* BAC *be equal to two angles of an equilateral triangle, and let the circle be contained within the angle* BAC; *a point* D *may be found, such, that if through the point* D *there be drawn any line meeting the circle in* E, F, *and from the point* E *there be drawn* EG, EH *perpendicular to* AB, AC, *and likewise there be drawn* FK, FL *perpendicular to* AB, AC, *the sum of the cubes of* EG, EH, *will be to the sum of the cubes of* FK, FL, *as the square of* DE *to the square of* DF.

There are many properties of the circle and conic sections similar to these, that will naturally occur to such as consider some of the foregoing propositions, and that may be of considerable use in solving several problems, that at first view would seem to require a line of a high order, when the solution may be easily had by a conic section.

For example, *Fig.* 20. Let there be a circle given by poſition, and two points A, B given; and let it be required to draw from the given points A, B right lines to a point G in the circumference of the circle, ſuch, that the rectangle AGB may be equal to a given ſpace.

The ſolution of this problem, at firſt view, would ſeem to require the Caſſinian curve, a line of the fourth order; but it may be ſolved by the interſection of an ellipſe and circle.

Becauſe [61.] two right lines DE, DF are given by poſition, ſuch, that if from the given points A, B there be drawn right lines to any point G in the circle, and from the point G there be drawn GH, GK perpendicular to DE, DF, the ſum of the ſquares of GH, GK, will be to the rectangle AGB as the rectangle AGB to a given ſpace; and becauſe the rectangle AGB is equal to a given ſpace, the ſum of the ſquares of GH, GK, will be equal to a given ſpace: Therefore the point G will be in a given ellipſe. And therefore the point G may be found by the interſection of a given ellipſe and circle.

If the given points A, B be in the diameter, or if they be equally diſtant from the centre, or if the rectangle contained by their diſtances from ſrom the centre be equal to the ſquare of the ſemidiameter of the circle, the point G may be found by the interſection of a right line and circle.

This problem has both a maximum and minimum, if none of the given points be in the circle; but if one of the given points be in the circle, it has only a maximum.

This problem may be ſolved otherwiſe by the ſixth propoſition.

From this it is evident, that a circle can interſect the Caſſinian curve in no more than four points.

F I N I S.

Y,
ni-
rds
D.
Irs.

pag. last.

Fig. 21

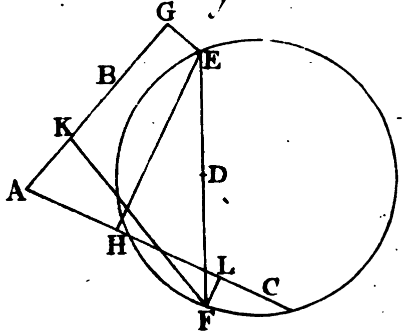
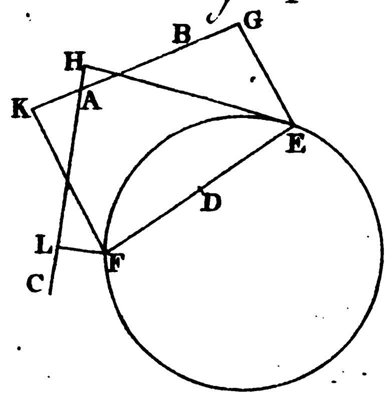


Fig. 24



A TREATISE OF PRACTICAL GEOMETRY. IN THREE PARTS.

BY

The late Dr. *DAVID GREGORY*,
Sometime Professor of Mathematicks in the University of EDINBURGH, and afterwards *Savilian* Professor of Astronomy at OXFORD.

Translated from the Latin; *with Additions.*

EDINBURGH:
Printed by W. and T. RUDDIMANS, for Messrs. HAMILTON and BALFOUR.
M. DCC. XLV.

Entred in Stationers-hall according to Act of Parliament.

ERRATA.

Page 81. Line 14. after *Glaziers* read *use*. Page 143. Line 17. for .0036, read .0034.

ADVERTISEMENT.

THIS Treatise was composed in Latin about sixty Years ago by Dr. David Gregory, *then Professor of Mathematicks in the University of* Edinburgh, *where it has been constantly taught since that time, immediately after* Euclid's *Elements and the plain Trigonometry, as proper for exercising the Students in the Application of Geometry to Practice. The Bookseller having procured an* English *Translation of it, which had been made by an ingenious Gentleman when a Student here, this Translation has been revised; and several Additions have been made to the Treatise itself, in order to render it more useful at this time. The Reader will find these distinguished from the Author's Text.*

Col. Maclaurin

College of EDINB.
May 1. 1745.

A

TREATISE

OF

Practical Geometry.

HAving explained the first books of *Euclid*, with the eleventh and twelfth, which may ſerve for geometrical elements; and having alſo taught the plain Trigonometry; we are now to ſubjoin ſome corollaries which are eaſily deduced from them, that contain practical rules of great use in the affairs of life, concerning the mensuration of lines, angles, surfaces and solids.

This treatise of practical Geometry is divided into three parts. In the first, we treat of the mensuration of lines and angles; to which we have subjoined Surveying. In the second, we treat of surfaces, not of such as are plane only, but of some curve surfaces likewise; as of the surface of the cylinder, cone and sphere; and of those parts of the sphere which we have frequently occasion to consider. It is shewn how to express the area of these in the superficial measures that are now in use amongst us. The third part treats of solid figures and their mensuration. After deducing the rules for finding the solid content of the paralellopipedon, prism, pyramid, cylinder, cone, &c. from *Euclid*, we add from *Archimedes* the mensuration of the sphere and spheroid, and of their segments,

ments, demonſtrated in an eaſy manner; from whence a method is derived for finding the contents of veſſels that are either full, or in part empty, in the wet as well as the dry meaſures that are now in uſe amongſt us.

PART I.

A Line, or length to be meaſured, whether it be diſtance, height or depth, is meaſured by a line leſs than it. With us the leaſt meaſure of length is an inch; not that we meaſure no line leſs than it, but becauſe we do not uſe the name of any meaſure below that of an inch; expreſſing leſſer meaſures by the fractions of an inch; and in this treatiſe, we uſe decimal fractions as the eaſieſt. Twelve inches make a foot, three feet and an inch make the *Scotch* ell, ſix ells make a fall, forty falls make a furlong, eight furlongs make a mile. So that the *Scotch* mile is 1184 paces, accounting every pace to be five feet. Theſe things are according to the ſtatutes of *Scotland*, notwithſtanding which the glaziers uſe a foot of only eight inches; and other artiſts for the moſt part uſe the *Engliſh* foot, on account of the ſeveral Scales marked on the *Engliſh* foot-meaſure for their uſe. But the *Engliſh* foot is ſomewhat leſs than the *Scotch*, ſo as that 185 of theſe make 186 of thoſe.

Lines, to the extremities and any intermediate point of which you have eaſy acceſs, are meaſured, by applying to them the common meaſure a number of times. But lines, to which you cannot have ſuch acceſs, are meaſured by methods taken from Geometry. The chief whereof we ſhall here endeavour to explain. The firſt is by the help of the geometrical ſquare.

" As for the *Engliſh* meaſures, the

" yard

" yard is three feet or thirty ſix in-
" ches. A pole is ſixteen feet and a
" half, or five yards and a half. The
" chain, commonly called *Gunter's*
" chain, is four poles, or twenty
" two yards, that is ſixty ſix feet. An
" *Engliſh* ſtatute mile is fourſcore
" chains, or 1760 yards, that is
" 5280 feet.

" The chain (which is now much
" in uſe, becauſe it is very conveni-
" ent for ſurveying) is divided into
" a hundred links, each of which
" is $7\frac{92}{100}$ of an inch; whence it is
" eaſy to reduce any number of thoſe
" links to feet, or any number of
" feet to links.

" A chain that may have the ſame
" advantages in ſurveying in *Scot-*
" *land* as *Gunter's* chain has in *Eng-*
" *land*, ought to be in length ſeventy
" four feet, or twenty four *Scots* ells,
" if no regard is had to the differ-
" ence of the *Scotch* and *Engliſh*
" foot above-mentioned. But if re-

" gard

" gard is had to that difference, the
" *Scotch* chain ought to consist of
" 74 $\frac{2}{7}$ *English* feet, or 74 feet, 4
" inches, and $\frac{4}{7}$ of an inch. This
" chain being divided into an hun-
" dred links, each of those links is
" 8 inches, and $\frac{925}{1000}$ of an inch."
" In the following table the most
" noted measures are expressed in
" *English* inches and decimals of an
" inch.

	Eng. Inch.	*Dec.*
The *English* foot is -	12	000
The *Paris* foot, -	12	788
The *Rhinland* foot, measured by Mr. *Picart*, -	12	362
The *Scotch* foot, -	12	065
The *Amsterdam* foot by *Snellius* and *Picart*, -	11	172
The *Dantzick* foot by *Hevelius*, - -	11	297
The *Danish* foot by Mr. *Picart*, - - -	12	465
The *Swedish* foot by the same, - -	11	692

The

	Inch.	Dec.
The *Bruſſels* foot by Mr. *Picart*, - - -	10	828
The *Lyons* foot by Mr. *Auzout*, - -	13	458
The *Bononian* foot by Mr. *Caſſini*, - -	14	938
The *Milan* foot by Mr. *Auzout*, - -	15	631
The *Roman* Palm uſed by merchants, according to the ſame, -	9	791
The *Roman* palm uſed by Architects, - -	8	779
The palm of *Naples*, according to Mr. *Auzout*,	10	314
The *Engliſh* yard, -	36	000
The *Engliſh* ell, - -	45	000
The *Scotch* ell, -	37	200
The *Paris* aune uſed by Mercers, according to Mr. *Picard*, -	46	786
The *Paris* aune uſed by Drapers, according to the ſame, - -	46	680

The

	Inch.	Dec.
The *Lyons* aune by Mr. *Auzout*, - -	46	570
The *Geneva* aune, -	44	760
The *Amsterdam* ell, -	26	800
The *Danish* ell. by Mr. *Picart*, - -	24	930
The *Swedish* ell, -	23	380
The *Norway* ell, -	24	510
The *Brabant* or *Antwerp* ell,	27	170
The *Brussels* ell, -	27	260
The *Bruges* ell, -	27	550
The brace of *Bononia* according to *Auzout*, -	25	200
The brace used by architects in *Rome*, -	30	730
The brace used in *Rome* by merchants, -	34	270
The *Florence* brace used by merchants, according to *Picart*. - -	22	910
The *Florence* geographical brace, - -	21	570
The vara of *Seville*, -	33	127
The vara of *Madrid*, -	39	166

The

The vara of *Portugal,*	44	031
The cavedo of *Portugal,*	27	354
The antient *Roman* foot,	11	632
The *Persian* arish, according to Mr. *Greaves,*	38	364
The shorter pike of *Constantinople,* according to the same, -	25	576
Another pike of *Constanstinople,* according to Messrs. *Mallet* and *De la Porte,* - -	27	920

PROPOSITION I.

PROBLEM I.

To describe the structure of the geometrical square.

THE geometrical square is made of any solid matter, as brass or wood, or of any four plain rulers joined together at right angles, (as

 in

in fig. 1.) where A is the centre, from which hangs a thread with a ſmall weight at the end, ſo as to be directed always to the centre. Each of the ſides, BE, and DE, is divided into an hundred equal parts, or (if the ſides be long enough to admit of it) into a thouſand parts; C, and F, are two ſights fixed on the ſide AD. There is moreover an index GH, which, when there is occaſion, is joined to the centre A in ſuch manner as that it can move round, and remain in any given ſituation. On this index are two ſights perpendicular to the right line going from the centre of the inſtrument; theſe are K and L. The ſide DE of the inſtrument is called the upright ſide, BE the reclining ſide.

PROPO-

PROPOSITION II. Fig. 2.

To measure an accessible height, AB, *by the help of a geometrical square, its distance being known.*

LET BR be an horizontal plain, on which there stands perpendicularly any line AB: Let BD, the given distance of the observator from the height, be 96 feet; let the height of the observator's eye be supposed 6 feet; and let the Instrument held by a steady hand, or rather leaning on a support, be directed towards the summit A, so that one eye, (the other being shut,) may see it clearly thro' the sights; the perpendicular or plum-line mean while hanging free, and touching the surface of the instrument: Let now the perpendicular be supposed to cut off on the right side KN 80 equal parts: It is clear that LKN, ACK, are ſimilar triangles; for the angles LKN, ACK, are right angles, and therefore equal; moreover LN and AC are paralell, as being both perpendicular to the horizon; conſequently, by *prop*. 29. 1. *B. of Euclid.* the angles KLN, KAC are equal; wherefore by the ſecond corrolary of the 32. *prop*. 1. *B*. of *Euclid*, the angles LNK, and AKC, are likewiſe equal: So that in the triangles NKL, KAC (by the 4. *prop*. of the 6. *B*. of *Euclid*) as NK : KL : : KC (*i. e*. DB) : CA; that is, as 80 to 100, ſo is 96 feet to CA. Therefore, by the rule of three, CA will be found to be 120 feet; and CB, which is 6 feet, being added, the whole height is 126 feet.

But if the diſtance of the obſervator from the height, as BE, be ſuch that when the inſtrument is directed as formerly toward the ſummit A, the perpendicular fall on the angle P, oppoſite to H, the centre

of

of the inſtrument; and BE or CG be given of 120 feet, CA will alſo be 120 feet. For in the triangles HGP, ACG, æqui-angular, as in the preceeding caſe, as PG : GH : : GC : GA. But PG is equal to GH; therefore GC is likewiſe equal to CA. That is, CA will be 120 feet, and the whole height 126 feet, as before.

Let the diſtance BF be 300 feet, and the perpendicular or plum-line cut off 40 equal parts from the reclining ſide: Now in this caſe the angles QAC, QZI, are equal, by the 29. *prop.* 1. *B.* of *Euclid.* And by the ſame *prop.* the angles QZI, ZIS are equal; therefore the angle ZIS is equal to the angle QAC. But the angles ZSI, QCA are equal, being right angles. Therefore in the æquiangular triangles ACQ, SZI, by the 4. *prop.* of the 6. *B.* of *Euclid*, it will be as ZS : SI : : CQ : CA. That is, as 100 to 40, ſo is 300 to CA. Wherefore, by the rule of of three, CA will be found to be of 120 feet. And by adding the height of the obſervator, the whole BA will be 126 feet. Note, that the height is greater than the diſtance, when the perpendicular cuts the right ſide, and leſs, if it cut the reclined ſide: And that the height and diſtance are equal, if the perpendicular fall on the oppoſite angle.

SCHOLIUM. Fig. 3.

If the height of a tower, to be meaſured as above, end in a point, as in fig. 3. the diſtance of the obſervator oppoſite to it, is not CD, but is to be accounted to the perpendicular from the point A; that is, to CD muſt be added the half of the thickneſs of the tower, viz. BD: Which muſt likewiſe be underſtood in the following propoſitions, when the caſe is ſimilar.

PRO-

PROPOSITION III. PROB. FIG. 4.

From the height of a tower AB given, to find a diſtance on the horizontal plain B C, by the geometrical ſquare.

LET the inſtrument be ſo placed as that the mark C in the oppoſite plain may be ſeen through the ſights, and let it be obſerved how many parts are cut off by the perpendicular. Now by what hath been already demonſtrated, the triangles AEF, ABC, are ſimilar; therefore, by 4th, 6. *Eucl.* it will be as EF to AE, ſo AB (compoſed of the height of the Tower BG, and of the height of the centre of the inſtrument A, above the tower AG) to the diſtance BC. Wherefore, if by the rule of three, you ſay, as EF to AE, ſo is AB to BC, it will be the diſtance ſought.

PROPOSITION IV. FIG. 5.

To measure any distance at land or sea, by the geometrical square.

IN this operation the index is to be applied to the instrument, as was shown in the description; and by the help of a support, the instrument is to be placed horizontally at the point A; then let it be turn'd till the remote point F, whose distance is to be measured, be seen through the fixed sights: and bringing the index to be parallel with the other side of the instrument, observe by the sights upon it any accessible mark B, at a sensible distance: then carrying the instrument to the point B, let the immoveable sights be directed to the first station A, and the sights of the index to the point F. If the index cut the right side of the square, as in K, in the two triangles BRK, and BAF, which

which are æquiangular, it will be (by 4th 6. *Eucl.*) as BR to RK, ſo BA (the diſtance of the ſtations to be meaſured with a chain) to AF; and the diſtance AF ſought, will be found by the rule of three. But if the index cut the reclined ſide of the ſquare in any point L, where the diſtance of a more remote point is ſought; in the triangles BLS, BAG, the ſide LS ſhall be to SB, as BA to AG the diſtance ſought, which accordingly will be found by the rule of three.

PROPOSITION V.

PROB. FIG. 6.

To meaſure an acceſſible height by means of a plain mirror.

LEt AB be the height to be meaſured; let the mirror be placed at C, in the horozontal plain BD at a known diſtance BC: let the obſerver go back to D, till he ſee the image of the ſummit in the mirror, at a certain point of it which he muſt diligently mark, and let DE be the height of the obſervator's eye. The triangles ABC and EDC are æquiangular: For the angles at D and B are right angles; and ACB, ECD, are equal, being the angles of incidence and reflection of the ray AC, as is demonſtrated in optics; wherefore the remaining angles at A, and E, are alſo equal: therefore, by 4th, 6. *Eucl.* it will be, as CD to DE, ſo CB to BA; that is, as the diſtance of the obſervator from the point of the mirror in the right line betwixt the obſervator and the height, is to the height of the obſervator's eye, ſo is the diſtance of the tower from that point of the mirror, to the height of the tower ſought; which therefore will be found by the rule of three.

Note

Note 1st, The obſervator will be more exact if at the point D, a ſtaff be placed in the ground perpendicularly, over the top of which the obſervator may ſee a point of the glaſs exactly in a line betwixt him and the tower.

Note 2d, In place of a mirror may be uſed the ſurface of water contained in a veſſel, which naturally becomes parallel to the horizon.

PROPOSITION VI. FIG. 7.

To meaſure an acceſſible height A B by means of two ſtaves.

LET there be placed perpendicularly in the ground a longer ſtaff DE, likewiſe a ſhorter one FG, ſo as the obſervator may ſee A, the top of the height to be meaſured, over the ends D, F, of the two ſtaves; let FH and DC parallel to the horizon meet DE and AB in H and C, then the triangles FHD, DCA, ſhall be æquiangular; for the angles at C, and H, are right ones, likewiſe the angle A, is equal to the angle FDH, by 29. 1. *Eucl.* wherefore the remaining angles DFH, and ADC, are alſo equal: wherefore by 4. 6. *Eucl.* as FH, the diſtance of the ſtaves, to HD the exceſs of the longer ſtaff above the ſhorter, ſo is DC the diſtance of the longer ſtaff from the tower, to CA the exceſs of the height of the tower above the longer ſtaff, And thence CA will be found by the rule of three.

To which if the length DE be added, you will have the whole height of the tower BA, Q. E. F.

SCHOLIUM Fig. 8.

Many other methods may be occaſionally contrived for meaſuring an acceſſible height. For example, from the given length of the ſhadow BD, BD, I find out the height AB thus; Let there be erected a ſtaff CE perpendicularly, producing the ſhadow EF: the triangles ABD, CEF, are æquiangular, for the angles at B, and E, are right; and the angles ADB, and CFE, are equal, each being equal to the angle of the ſun's elevation above the horizon; therefore, by 4. 6. *Eucl.* as EF the ſhadow of the ſtaff, to EC the ſtaff itſelf, ſo BD the ſhadow of the tower, to BA the height of the tower; tho' the plain on which the ſhadow of the tower falls be not parallel to the horizon, if the ſtaff be erected in the ſame plain, the rule will be the ſame.

PROPO-

PROPOSITION VII.

To meaſure an inacceſſible height by means of two ſtaves.

HItherto we have ſuppoſed the height to be acceſſible, or that we can come at the lower end of it: now if, becauſe of ſome impediment, we cannot get to a tower, or if the point whoſe height is to be found out, be the ſummit of a hill, ſo that the perpendicular be hid within the hill; if, I ſay, for want of better inſtruments, ſuch an inacceſſible height is to be meaſured by means of two ſtaves, let the firſt obſervation be made with the ſtaves DE and FG as in Prop. I. then the obſervator is to go off in a direct line from the the height and firſt ſtation, till he come to the ſecond ſtation, where he is to place the longer ſtaff perpendicularly at RN, and the ſhorter ſtaff

ſtaff at KO, ſo that the ſummit A may be ſeen along their tops, that is, ſo that the points KNA may be in the ſame right line. Through the point N let there be drawn the right line NP parallel to FA: wherefore in the triangles KNP, KAF, the angles KNP, KAF are equal by the 29th 1. *Eucl.* alſo the angle AKF is common to both; conſequently the remaining angle KPN is equal to the remaining angle KFA. And therefore by 4th 6. *Eucl.* PN : FA :: KP : KF. But the triangles PNL, FAS, are ſimilar. Therefore, by 4th 6. *Eucl.* PN : FA :: NL : SA. Therefore by the 11th 5. *Eucl.* KP : KF :: NL : SA. Thence alternately it will be as KP (the Exceſs of the greater diſtance of the ſhort ſtaff from the long one above its leſſer diſtance from it) to NL, the exceſs of the longer Staff above the ſhorter, ſo KF, the diſtance of the two ſtations of the ſhor-

ſhorter ſtaff, to SA the exceſs of the height ſought above the height of the ſhorter ſtaff. Wherefore SA will be found by the rule of three. To which let the height of the ſhorter ſtaff be added, and the ſum will give the whole inacceſſible Height BA, Q. E. F.

Note 1. In the ſame manner may an inacceſſible height be found by a geometrical ſquare, or by a plain ſpeculum. But we ſhall leave the rules to be found out by the ſtudent for his own exerciſe.

Note 2. That by the height of the ſtaff we underſtand its height above the ground in which it is fixed.

Note 3. Hence depends the method of uſing other inſtruments invented by geometricians; for example of the geometrical croſs; and if all things be juſtly weighed, a like rule will ſerve for it as here: but we incline to touch only upon what is moſt material.

PRO-

PROPOSITION VIII. Fig. 9.

To measure the distance AB, to one of whose extremities we have access, by the help of four staves.

LET there be a staff fixed at the Point A, then going back at some sensible distance in the same right Line, let another be fixed in C, (so as that both the points A and B be cover'd and hid by the Staff C.) Likewise going off in a perpendicular from the right line CB at the point A, (the method of doing which shall be shown in the following *Scholium*) let there be placed another Staff at H; and in the right line CKG (perpendicular to the same CB at the point C,) and at the point of it K, such that the points K, H and B may be in the same right line, let there be fixed a fourth staff. Let there be drawn, or let there be supposed to be drawn a light line HG parallel to CA. The triangles KHG, HAB will be equiangular; for the angles HAB, KGH are right angles. Also by 29th 1. *Eucl.* the angles ABH, KHG are equal, wherefore by the 4th 6. *Eucl.* as KG (the excess of CK above AH) to GH or to CA, the distance betwixt the first and second staff, so is AH the distance betwixt the first and third staff to AB the distance sought.

SCHOLIUM. Fig. 10.

To draw on a plain a right line AE perpendicular to CH, from a given point A, take the right lines AB AD, on each side equal, and in the points B and D, let there be fixed stakes, to which let there be tied two equal ropes BE DE (or one having a mark in the middle) and holding in your hand their extremities joined, (or the mark in the middle, if it is

FIG. 2.

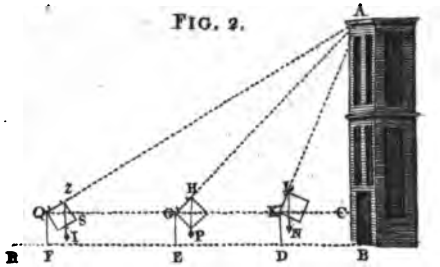


FIG. 4.

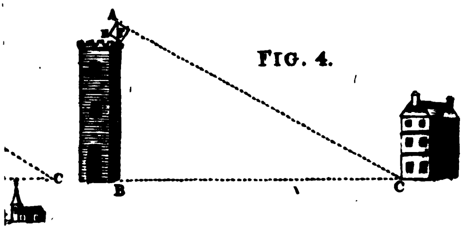


FIG. 6.

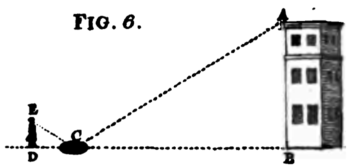
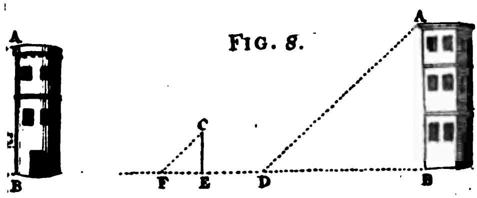
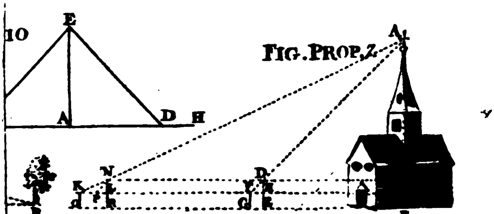


FIG. 8.



10

FIG. PROP. 7.



is but one) draw out the ropes on the ground; and then, where the two ropes meet, or at the mark, when by it the rope is fully ſtretched, let there be placed a third ſtake at E, the right line AE will be perpendicular to CH in the point A, by 11th 1. *Eucl.* In a manner not unlike to this may any problems that are reſolved by the ſquare and compaſſes, be done by ropes and a cord turned round as a radius.

PROPOSITION 9. FIG. 11.

To meaſure the diſtance AB, one of whoſe extremities is acceſſible.

FRom the point A, let the right line AC of a known length be made perpendicular to AB (by the preceeding Scholium,) likewiſe draw the right line CD perpendicular to CB meeting the right line AB in D, then by (the 8th 6.) *Eucl.* as DA :

AC : : AC : AB; wherefore when DA and AC are given, AB will be found by the rule of three, Q. E. F.

SCHOLIUM.

All the preceeding operations depend on the equality of some angles of triangles, and on the similarity of the triangles arising from that equality. And on the same principles depend innumerable other operations which a Geometrician will find out of himself, as is very obvious. However, some of these operations require such exactness in the work, and without it are so liable to errors, that, *cæteris paribus*, the following operations which are performed by a trigonometrical calculation, are to be preferred. Yet could we not omit those above, being most easy in practice, and most clear and evident to those who have only the first elements of Geometry. But if you

you are provided with instruments, the following operations are more to be relied upon. We do not insist on the easiest cases to those who are skilled in plain Trigonometry, which is indeed necessary to any one who would apply himself to practice. It will be easy to the reader to find examples, and we have shown in plain Trigonometry how to find the angle or side of any plain triangle that is required, from the angles or sides that may be given.

PROPOSITION X. FIG. 12.

To describe the construction and use of the geometrical quadrant.

THE geometrical quadrant is the fourth part of a circle, divided into ninety degrees, to which two sights are adapted, with a perpendicular or plum-line hanging from the center. The general use of it is for in-

investigating angles in a vertical plain, comprehended under right lines going from the center of the instrument, one of which is horizontal, and the other is directed to some visible point. This instrument is made of any solid matter, as wood, copper, &c.

PROPOSITION XI. Fig. 13.

To describe the make and use of the graphometer.

THE graphometer is a semicircle made of any hard matter, of wood, for example, or brass, divided into 180 degrees; so fixed on a *fulcrum*, by means of a brass ball and socket, that it easily turns about and retains any situation; two sights are fixed on its diameter. At the center there is commonly a magnetical needle in a box. There is likewise a moveable ruler, which turns round the

the center, and retains any ſituation given it. The uſe of it is to obſerve any angle, whoſe vertex is at the center of the inſtrument in any plain, (though it is moſt commonly horizontal, or nearly ſo) and to find how many degrees it contains.

PROPOSITION XII.

FIG. 14. and 15.

To deſcribe the manner in which angles are meaſured by a quadrant or graphometer.

LET there be an angle in a vertical plain, comprehended between a line parallel to the horizon HK, and the right line RA, coming from any remarkable point of a tower or hill, or from the ſun, moon or a ſtar. Suppoſe that this angle RAH is to be meaſured by the quadrant: let the inſtrument be placed

in

in the vertical plain, so as that th center A may be in the angular point and let the sights be directed to wards the object at R, (by the help of the ray coming from it, if it b the sun or moon, or by the help of the visual ray, if it is any thing else,) the degrees and minutes in the arch BC, cut off by the perpendicular, will measure the angle RAH required. For from the make of the quadrant, BAD is a right angle; therefore BAR is likewise right, being equal to it. But because HK is horizontal, and AC perpendicular, HAC will be a right angle; and therefore equal also to BAR. From those angles substract the part HAB that is common to both; and there will remain the angle BAC equal to the angle RAH. But the arch BC is the measure of the angle BAC; consequently it is likewise the measure of the angle RAH.

Note,

Note, That the remaining arch on the quadrant DC is the meaſure of the angle RAZ, comprehended between the foreſaid right line RA and AZ, which points to the Zenith.

Let it now be required to meaſure the angle ACB (Fig. 15.) in any plain, comprehended between the right lines AC and BC drawn from two points A and B, to the place of ſtation C. Let the graphometer be placed at C, ſupported by its *fulcrum* (as was ſhown above) and let the immoveable ſights on the ſide of the inſtrument DE be directed towards the point A, and likewiſe (while the inſtrument remains immoveable) let the ſights of the ruler FG (which is moveable about the center C) be directed to the point B. It is evident that the moveable ruler cuts off an arch DH, which is the meaſure of the angle ACB ſought. Moreover, by the ſame method, the inclination of DE, or of FG may be obſerved with the meridian line, which is pointed out by the magnetick needle incloſed in the box, and is moveable about the center of the inſtrument, and the meaſure of this inclination or angle found in degrees.

PROPOSITION XIII. Fig. 16.

To meaſure an acceſſible height by the geometrical quadrant.

BY the 12th Prop. of this part, let the angle C be found by means of the quadrant. Then in the triangle ABC, right angled at B (BC being ſuppoſed the horizontal diſtance of the obſervator from the tower) having the angle at C, and the ſide BC, the required height BA will be found by the third caſe of plain Trigonometry.

PROPO-

PROPOSITION XIV. Fig. 17.

To meaſure an inacceſſible height by the geometrical quadrant.

LET the angle ACB be obſerved with the quadrant (by the 12th prop. of this part) then let the obſerver go from C to the ſecond ſtation D, in the right line BCD (providing BCD be a horizontal plain) and after meaſuring this diſtance CD, take the angle ADC likewiſe with the quadrant. Then in the triangle ACD, there is given the angle ADC, with the angle ACD, becauſe ACB was given before: therefore (by 32d 1. *Eucl.*) the remaining angle CAD is given likewiſe. But the Side CD is likewiſe given, being the diſtance of the ſtations C and D; therefore (by the firſt caſe of oblique angled triangles in Trigonometry) the ſide AC will be found. Wherefore in the

the right angled triangle ABC, all the angles and the hypotenuſe AC are given; conſequently, by the 4th caſe of Trigonometry, the height ſought AB will be found; as alſo (if you pleaſe) the diſtance of the ſtation C from AB the perpendicular within the hill or inacceſſible height.

PROPOSITION XV. FIG. 18.

From the top of a given height, to meaſure the diſtance B C.

LET the angle BAC be obſerved by the 12th of this part; wherefore in the triangle ABC right-angled at B, there is given by obſervation the angle at A; whence (by the 32d 1. *Eucl.*) there will alſo be given the angle BCA; moreover the ſide AB (being the height of the tower) is ſuppoſed to be given. Wherefore

by

by the 3d caſe of Trigonometry BC the diſtance ſought will be found.

PROPOSITION XVI. Fig. 19.

To meaſure the diſtance of two places A and B, of which one is acceſſible, by the graphometer.

LET there be erected at two points A and C, ſufficiently diſtant, two viſible ſigns; then (by the 12th of this) let the two angles BAC, BCA be taken by the graphometer. Let the diſtance of the ſtations A and C be meaſured with a chain. Then the third angle B being known, and the ſide AC being likewiſe known; therefore by the firſt caſe of Trigonometry the diſtance required AB will be found.

PRO-

PROPOSITION XVII. Fig. 20.

To measure, by the graphometer, the distance of two places, neither of which is accessible.

LET two stations C and D be chosen, from each of which the places may be seen, whose distance is sought; let the angles ACD, ACB, BCD, and likewise the angles BDC, BDA, CDA, be measured by the graphometer, the distance of the stations C and D be measured by a chain, or (if it be necessary) by the preceeding practice. Now in the triangle ACD, there are given two angles ACD and ADC; therefore the third CAD is likewise given. Moreover the side CD is given; therefore, by the first case of Trigonometry the side AD will be found; after the same manner in the triangle BCD from all the angles, and one side

side CD given, the side BD is found. Wherefore in the triangle ADB, from the given sides DA and DB, and the angle ADB contained by them, the side AB (the distance sought) is found by the 4th case of Trigonometry of oblique angled triangles.

Let it be noted, that it is not necessary that the points A, B, C, and D be in one plain, and that any triangle is in one plain, by 2d prop. 11th of *Eucl.*

PROPOSITION XVIII. Fig. 21.

It is required by the graphometer and quadrant to measure an inaccessible height AB, placed so on a steep, that one can neither go near it in an horizontal plain, nor recede from it, as we supposed in the solution of the 14th prop.

LET there be chosen any situation as C, and another D, where let some

ſome mark be erected; let the angles ACD and ADC be found by the graphometer, then the third angle DAC will be known. Let the ſide CD, the diſtance of the ſtations, be meaſured with a chain, and thence (by Trigon.) the ſide AC will be found. Again in the triangle ACB, right angled at B, having found by the quadrant the angle ACB, the other angle CAB is known likewiſe; but the ſide AC in the triangle ADC is already known; therefore the height required AB will be found by the 4th caſe of right-angled triangles. If the height of the tower is wanted, the angle BCF will be found by the quadrant, which being taken from the angle ACB already known, the angle ACF will remain; but the angle FAC was known before; therefore the remaining angle AFC will be known; but the ſide AC was alſo known before; therefore, in the triangle AFC, all the angles, and one

of the ſides AC being known, the height of the tower above hill will be found by Trigono-:y.

SCHOLIUM.

t were eaſy to add many other hods of meaſuring heights, and di-ces; but if what is above be un-ſtood, it will be eaſy (eſpecially one that is vers'd in the elements) contrive methods for this purpoſe, ording to the occaſion: ſo that re is no need of adding any more this ſort. We ſhall ſubjoin here a thod by which the diameter of earth may be found out.

PROPOSITION XIX. Fig.

To find the diameter of the earth [by] one observation.

LET there be chosen a high [hi]ll AB, near the sea-shore, [and] let the observator on the top of [it] with an exact quadrant divided in[to] minutes and seconds by transver[se] divisions, and fitted with a telesco[pe] in place of the common sights, mea[sure] the angle ABE contained under the right line AB, which goes *to* th[e] center, and the right line BE drawn t[o] the sea, a tangent to the globe at E, [let] there be drawn from A perpendicular to BD, the line AF meeting BE in F. Now in the right angled triangle BAF all the angles are given, also the side AB, the height of the hill, which *is to* be found by some of the foregoing methods, as exactly as possible; and, by Trigonometry, the sides BF and AF are

are found. But by corol. 36. 3d *Eucl.* AF is equal to FE; therefore BE will be known. Moreover, by 36th 3d *Eucl.* the rectangle under BA and BD is equal to the square of BE. And thence by 17th 6. *Eucl.* as AB : BE : : BE : BD. Therefore, since AB and BE are already given, BD will be found by 11th 6. *Eucl.* or by the rule of three, and substracting BA, there will remain AD the diameter of the earth sought.

SCHOLIUM.

Many other methods might be proposed for measuring the diameter of the earth. The most exact in my opinion is that proposed by Mr. *Picart*, of the academy of sciences at *Paris*; but since it does not belong to this place, we refer you to the philosophical transactions, where you will find it described.

" According to Mr. *Picart*, a de-

" gree of the meridian at the lati-
" tude of 49° 21′ was 57060 *French*
" *Toises*, each of which contains six
" feet of the same measure; from
" which it follows, that if the earth
" be an exact sphere, the circumfe-
" rence of a great circle of it will
" be 123,249,600 *Paris* feet, and
" the semidiameter of the earth
" 19,615,800 feet: but the *French*
" Mathematicians who have exami-
" ned Mr. *Picart's* operations of
" late assure us, That the degree
" in that latitude is 57,183 *Toises*.
" They measured a degree in *Lap-*
" *land*, in the latitude of 66° 20′
" and found it of 57,438 *Toises*.
" By comparing these degrees, as
" well as by the observations on
" pendulums, and the theory of
" gravity; it appears that the
" earth is an oblate spheroid; and
" (supposing those degrees to be
" accurately measured) the axis or
" diameter that passes through the
" poles

"poles will be to the diameter of the "equator as 177 to 178, or, the "the earth will be 22 miles high- "er at the equator than at the poles. "A degree has likewife been mea- "fured at the equator, and found to "be confiderably lefs than at the "latitude of *Paris*; which confirms "the oblate figure of the earth; "but an account of this laft men- "furation has not been publifhed as "yet. If the earth was of an u- "niform denfity from the furface "to the center, then, according to "the theory of gravity, the meridi- "an would be an exact ellypfis, "and the axis would be to the di- "ameter of the equator as 230 to "231; and the difference of the "femidiameter of the equator and "femi-axis about 17 miles.

In what follows, a figure is often to be laid down on paper, like to another figure given; and becaufe this likenefs confifts in the equality

of

of their angles, and in the ſides h ving the ſame proportion to each o ther (by the definitions of the 6 of *Eucl.*) we are now to ſhow wh methods practical Geometricians u for making on paper an angle equ to a given angle, and how they con ſtitute the ſides in the ſame propor tion. For this purpoſe they mak uſe of a protractor, (or, when it i wanting, a line of chords) and of a line of equal parts.

PROPOSITION XX.

FIG. 23, 24, 25, 26, and 27.

To deſcribe the conſtruction and uſe of the protractor, of the line of chords, and of the line of equal parts.

THE protractor is a ſmall ſemi-circle of braſs, or ſuch ſolid matter. The ſemi-circumference is divided into 180 degrees. The uſe

uſe of it is to draw angles on any plain, as on paper, or to examine the extent of angles already laid down. For this laſt purpoſe, let the ſmall point in the center of the protractor be placed above the angular point, and let the ſide AB coincide with one of the ſides that contain the angle propoſed; the number of degrees cut off by the other ſide, computing on the protractor from B, will ſhow the quantity of the angle that is to be meaſured.

But if an angle is to be made of a given quantity on a given line, and at a given point of that line, let AB coincide with the given line, and let the center A of the inſtrument be applied to that point. Then let there be a mark made at the given number of degrees; and a right line drawn from that mark to the given point will conſtitute an angle, with the given right line of the

the quantity required; as is manifest.

This is the most natural and easy method, either for examining the extent of an angle on paper, or for describing on paper an angle of a given quantity.

But when there is scarcity of instruments, or because a line of chords is more easily carried about, (being described on a ruler on which there are many other lines besides) practical Geometricians frequently make use of it. It is made thus: let the quadrant of a circle be divided into 90 degrees, as in fig. 24. the right line AB is the chord of 90 degrees; the chord of every arch of the quadrant is transferred to this line AB, which is always marked with the number of degrees in the corresponding arch.

Note, That the chord of 60 degrees is equal to the radius, by corol. 15. 4th *Eucl.* If now a given angle

EDF

EDF is to be measured by the line of chords; from the center D, with the distance DG (the chord of 60 degrees;) describe the arch GH, and let the points G and H be marked where this arch intersects the sides of the angle. Then if their distance GH, applied on the line of chords from A to B, gives (for example) 25 degrees, this shall be the measure of the angle proposed.

When an obtuse angle is to be measured with this line, let its complement to a semicircle be measured; and thence it will be known. It were easy to transfer to the diameter of a circle the chords of all arches to the extent of a semicircle, but such are rarely found marked upon rules.

But now if an angle of a given quantity, suppose of 50 degrees, is to be made at a given point M of the right line KL (fig. 26.) From the center M, and the distance MN equal to the chord of 60 degrees, describe the arch QN. Take off an arch NR, whose chord is equal to that of 50 degrees on the line of chords; join the points M and R, and it is plain that MR shall contain an angle of 50 degrees with the line KL proposed.

But sometimes we cannot produce the sides, till they be of the length of a chord of 60 degrees on our scale; in which case it is fit to work by a circle of proportions (that is a a sector) by which an arch may be made of a given number of degrees to any radius.

The quantities of angles are likewise determined by other lines usually marked upon rules, as the lines of sines, tangents and secants; but as these methods are not so easy or so proper in this place, we omit them.

To delineate figures similar or like to others given, besides the equality of the angles, the same proportion is is to be preſerved among the ſides of the figure that is to be delineated, as is among the ſides of the figure given. For which purpoſe, on the rules uſed by artiſts, there is a line divided into equal parts, more or leſs in number, and greater or leſſer in quantity, according to the pleaſure of the maker.

A Foot is divided into inches, and an inch, by means of tranſverſe lines, into 100 equal parts; ſo that with this ſcale, any number of inches below twelve, with any part of an inch, can be taken by the compaſſes, providing ſuch part be greater than one 100th part of an inch. And this exactneſs is very neceſſary in delineating the plans of houſes, and in other caſes.

PROPOSITION XXI. Fig. 28.

To lay down on paper, by the protractor, or line of chords, and line of equal parts, a right lined figure like to one given, providing the angles and sides of the figure given be known by observation or mensuration.

FOR example, ſuppoſe that it is known that in a quadrangular figure, one ſide is of 235 feet, that the angle contained by it and the ſecond ſide is of 84°, the ſecond ſide of 288 feet, the angle contained by it and the third ſide of 72°, and that the third ſide is 294 feet. Theſe things being given, a figure is to be drawn on paper like to this quadrangular figure. On your paper at a proper point A let a right line be drawn, upon which take 235 equal parts, as AB. The part repreſent-

ing

ing a foot is taken greater or lesser, according as you would have your figure greater or less. In the adjoining figure, the 100th part of an inch is taken for a foot. And accordingly an inch divided into 100 parts, and annexed to the figure is called a scale of 100 feet. Let there be made at the point B, (by the preceeding prop.) an angle ABC of 84°, and let BC be taken of 288 parts like to the former. Then let the angle BCD be made of 72°, and the side CD of 294 equal parts. Then let the side AD be drawn; and it will compleat the figure like to the figure given. The measures of the angles A and D can be known by the protractor or line of chords, and the side AD by the line of equal parts; which will exactly answer to the corresponding angles and to the side of the primary figure.

After the very same manner, from the sides and angles given, which bound

bound any right lined figure, a figure like to it may be drawn, and the rest of its sides and angles be known.

COROLLARY.

Hence any trigonometrical problem in right lined triangles, may be resolved by delineating the triangle from what is given concerning it, as in this proposition. The unknown sides are examined by a line of equal parts, and the angles by a protractor or line of chords.

PROPOSITION XXII. Prob.

The diameter of a circle being given, to find its circumference nearly.

THE periphery of any polygon inscribed in the circle is less than the circumference, and the periphery of any polygon described about a circle is greater than the circumcumference. Whence *Archimedes* first discovered that the diameter was in proportion to the circumference, as 7 to 22 nearly; which serves for common use. But the moderns have computed the proportion of the diameter to the circumference to greater exactness. Supposing the diameter 100, the periphery will be more than 314, but less than 315 *. But *Luctolphus van Ceulen* exceeded the labours of all; for, by immense study he found, that supposing the diameter

100,000,000,000,000,000,000,000,000,000,000

the periphery will be less than

314,159,265,358,979,323,846,264,338,327,951,

but greater than

314,159,265,358,979,323,846,264,338,327,950

whence it will be easy, any part of the circumference being given in degrees

* *The diameter is more nearly to the circumference, as* 113 *to* 355.

grees and minutes, to assign it in parts of the diameter.

Of surveying and measuring of land.

HItherto we have treated of the measuring of angles and sides, whence it is abundantly easy to lay down a field, a plain, or an entire country: for to this nothing is requisite but the protraction of triangles, and of other plain figures, after having measured their sides and angles: but as this is esteemed an important part of practical Geometry, we shall subjoin here an account of it, with all possible brevity, suggesting withall, that a surveyor will improve himself more by one day's practice, than by a great deal of reading.

PROPO-

PROPOSITION XXIII. PROB.

To explain what surveying is, and what instruments surveyors use.

FIrst, it is necessary that the surveyor view the field that is to be measured, and investigate its sides and angles, by means of an iron chain (having a particular mark at each foot of length, or at any number of feet as may be most convenient for reducing lines or surfaces to the received measures*) and the graphometer described above. *Secondly*, It is necessary to delineate the field *in plano*, or to form a map of it; that is, to lay down on paper a figure similar to the field, which is done by the protractor, (or line of chords) and of the line of equal

 parts

* See above p. 5. the account of Gunter's chain, and of the chain that is most convenient for measuring land in Scotland.

parts. *Thirdly*, It is necessary to find out the area of the field so surveyed and represented by a map. Of this last we are to treat below in the second part.

The sides and angles of small fields are surveyed by the help of a plain table, which is generally of an oblong rectangular figure, and supported by a *fulcrum*, so as to turn every way by means of a ball and socket. It has a moveable frame, which surrounds the board, and serves to keep a clean paper put on the board close and tight to it. The sides of the frame facing the paper are divided into equal parts every way. The board hath besides, a box with a magnetick needle, and moreover a large index with two sights. On the edge of the frame of the board are marked degrees and minutes, so as to supply the room of a graphometer.

PRO-

PROPOSITION XXIV.

PROB. FIG. 29.

To delineate a field by the help of a plain table, from one ſtation whence all its angles may be ſeen, and their diſtances meaſured by a chain.

LET the field that is to be laid down be ABCDE. At any convenient place F, let the plain table be erected, cover it with clean paper, in which let ſome point near the middle repreſent the ſtation. Then applying at this place the index with the ſights, direct it ſo as that through the ſights ſome mark may be ſeen at one of the angles, ſuppoſe A, and from the point F, repreſenting the ſtation, draw a faint right line along the ſide of the index, then, by the help of the chain, let FA the diſtance of the ſtation from the foresaid angle be measured. Then taking what part you think convenient for a foot or pace from the line of equal parts, set off on the faint line the parts corresponding to the line FA that was measured, and let there be a mark made representing the angle of the field, A. Keeping the table immoveable, the same is to be done with the rest of the angles; then right lines joining those marks shall include a figure like to the field, as is evident from 5th 6. *Eucl.*

COROLLARY.

The same thing is done in like manner by the graphometer; for having observed in each of the triangles, AFB, BFC, CFD, *&c.* the angle at the station F, and having measured the lines from the station to the angles of the field, let similar triangles be protracted on paper (by the

the 21. of this) having their common vertex in the point of station. All the lines, excepting those which represent the sides of the field, are to be drawn faint or obscure.

Note 1. When a surveyor wants to lay down a field, let him place distinctly in a register all the observations of the angles, and the measures of the sides, until at time and place convenient, he draw out the figure on paper.

Note 2. The observations made by the help of the graphometer are to be examined; for all the angles about the point F ought to be equal to four right ones by 13th 1. *Eucl.*

PRO-

PROPOSITION XXV.

Prob. Fig. 30.

To lay down a field by means of two stations, from each of which all the angles can be seen, by measuring only the distance of the stations.

LET the instrument be placed at the station F; and having chosen a point representing it upon the paper which is laid upon the plain table, let the index be applied at this point, so as to be moveable about it. Then let it be directed successively to the several angles of the field; and when any angle is seen through the sights, draw an obscure line along the side of the index. Let the index, with the sights, be directed after the same manner to the station G; on the obscure line drawn along its side, pointing to A, set off from the

the ſcale of equal parts a line correſponding to the meaſured diſtance of the ſtations, and this will determine the point G. Then remove the inſtrument to the ſtation G, and applying the index to the line repreſenting the diſtance of the ſtations, place the inſtrument ſo that the firſt ſtation may be ſeen through the ſights. Then the inſtrument remaining immoveable, let the index be applied at the point repreſenting the ſecond ſtation G, and be ſucceſſively directed by means of its ſights, to all the angles of the field, drawing (as before) obſcure lines; and the interſection of the two obſcure lines that were drawn to the ſame angle from the two ſtations will always repreſent that angle on the plan. Care muſt be taken that thoſe lines be not miſtaken for one another. Lines joining thoſe interſections will form a figure on the paper like to the field.

SCHO-

SCHOLIUM.

It will not be difficult to do the ſame by the graphometer, if you keep a diſtinct account of your obſervations of the angles made by the line joining the ſtations and the lines drawn from the ſtations to the reſpective angles of the field. And this is the moſt common manner of laying down whole countries. The tops of two mountains are taken for two ſtations, and their diſtance is either meaſured by ſome of the methods mentioned above, or is taken according to common repute. The ſights are ſucceſſively directed towards cities, churches, villages, forts, lakes, turnings of rivers, woods, &c.

Note. The diſtance of the ſtations ought to be great enough, with reſpect to the field that is to be meaſured, ſuch ought to be choſen as are not in a line with any angle of the field. And care ought to be taken likewise that the angles, for example FAG, FDG, &c. be neither very acute, nor very obtuse. Such angles are to be avoided as much as possible; and this admonition is found very useful in practice.

PROPOSITION XXVI.

PROB. FIG. 31.

To lay down any field, however irregular its figure may be, by the help of the graphometer.

LET ABCEDHG be such a field. Let its angles (in going round it) be observed with a graphometer (by the 12th of this) and noted down; let its sides be measured with a chain, and, (by what was said on the 21st of this) let a figure like to the given field be protracted on paper. If any mountain is in the circumference, the horizontal line hid under it is to be taken for a side, which may be found by two or three observations, according to some of the methods described above; and its place on the map is to be distinguished by a shade, that it may be known a mountain is there.

If not only the circumference of the field is to be laid down in the plan, but also its contents, as villages, gardens, churches, publick roads, we must proceed in this manner.

Let there be (for example) a church F, to be laid down on the plan. Let the angles ABF, BAF be observed and protracted on paper in their proper places, the intersection of the two sides BF and AF will give the place of the church on the paper; or more exactly, the lines BF, AF being measured, let circles be described from the centers B and A, with parts from the scale correspond-

ing

ing to the diſtances BF and AF, and the place of the church will be at their interſection.

Note 1. While the angles obſerved by the graphometer are taken down, you muſt be careful to diſtinguiſh the external angles as E and G, that they may be rightly protracted afterwards on paper.

Note 2. Our obſervations of the angles may be examined by computing if all the internal angles make twice as many right angles, four excepted, as there are ſides of the figure: for this is demonſtrated by 32d 1. *Eucl.* But in place of any external angle DEC, its compliment to a circle is to be taken.

PROPOSITION XXVII.

Prob. Fig. 32.

To lay down a plain field without instruments.

IF a ſmall field is to be meaſured, and a map of it to be made, and you are not provided with inſtruments; let it be ſuppoſed to be divided into triangles, by right lines, as in the figure, and after meaſuring the three ſides of any of the triangles, for example of ABC, let its ſides be laid down from a convenient ſcale on paper, by the 22d of this. Again, let the other two ſides BD, CD of the triangle CBD be meaſured and protracted on the paper by the ſame ſcale as before. In the ſame manner proceed with the reſt of the triangles of which the field is compoſed, and the map of the field will be per-

erfected; for the three ſides of a riangle determine the triangle; hence each triangle on the paper ſimilar to its correſpondent triangle in the field, and is ſimilarly ituated; conſequently the whole figure is like to the whole field.

S C H O L I U M.

If the field be ſmall, and all its ngles may be ſeen from one ſtation, may be very well laid down by the lain table by the 24th of this. If e field be larger, and have the reuiſite conditions, and great exactneſs not expected, it likewiſe may be lotted by means of the plain table, r by the graphometer, according the 25th of this; but in fields hat are irregular and mountainous, hen an exact map is required we re to make uſe of the graphometer in the 26th of this, but rarely of e plain table.

Having

Having protracted the bounding lines, the particular parts contained within them may be laid down by the proper operations for this purpose, delivered in the 26th proposition; and the method described in the 27th proposition may be sometimes of service; for we may trust more to the measuring of sides than to the observing of angles. We are not to compute four-sided and many-sided figures till they are resolved into triangles, for the sides do not determine those figures.

In the laying down of cities, or the like, we may make use of any of the methods described above that may be most convenient.

The map being finished, it is transferred on clean paper by putting the first sketch above it, and marking the angles by the point of a small needle. These points being joined by right lines, and the whole illuminated by colours proper to each part,

part, and the figure of the Mariner's compass being added to distinguish the North and South, with a scale on the margin, the map or plan will be finished and neat.

We have thus briefly and plainly treated of Surveying, and shown by what instruments it is performed, having avoided those methods which depend on the magnetick needle, not only because its direction may vary in different places of a field (the contrary of this at least doth not appear) but because the quantity of an angle observed by it cannot be exactly known; for an error of two or three degrees can scarcely be avoided in taking angles by it. As for the remaining part of surveying, whereby the area of a field already laid down on paper is found in acres, roods, or any other superficial measures; this we leave to the following part, which treats of the mensuration of surfaces.

'Besides the instruments descri-
'bed above, a Surveyor ought to be
'provided with an off-set staff, e-
'qual in length to ten links of the
'chain, and divided into ten equal
'parts. He ought likewise to have
'ten arrows, or small streight sticks
'near two feet long, shod with iron
'ferrils. When the chain is first
'opened, it ought to be examined
'by the off-set staff. In measuring
'any line, the leader of the chain
'is to have the ten arrows at first
'setting out. When the chain is
'stretcht in the line, and the near
'end touches the place from which
'you measure, the leader sticks one
'of the ten arrows in the ground,
'at the far end of the chain. Then
'the leader leaving the arrow, pro-
'ceeds with the chain another
'length; and the chain being
'stretcht in the line, so that the
'near end touches the first arrow,
'the leader sticks down another ar-

'row at his end of the chain. The
'line is preſerved ſtreight, if the ar-
'rows be always ſet ſo as to be in
'a right line with the place you
'meaſure from and that to which
'you are going. In this manner
'they proceed till the leader have
'no more arrows. At the eleventh
'chain the arrows are to be carried
'to him again, and he is to ſtick
'one of them into the ground, at the
'end of the chain. And the ſame
'is to be done at the 21, 31, 41,
'*&c.* chains, if there are ſo many
'in the right line to be meaſured.
'In this manner you can hardly
'commit an error in numbring the
'chains, unleſs of ten chains at
'once.

'The off-ſet ſtaff ſerves for mea-
'ſuring readily the diſtances of a-
'ny things proper to be repreſented
'in your plan, from the ſtation-line
'while you go along. Theſe di-
'ſtances ought to be entred into

'your field-book, with the corre-
'sponding distances from the last sta-
'tion, and proper remarks, that you
'may be enabled to plot them just-
'ly, and be in no danger of mistak-
'ing one for another, when you
'extend your plan. The field-book
'may be conveniently divided into
'five pages. In the middle column
'the angles at the several stations
'taken by the Theodolite are to be
'entred, with the distances from
'the stations. The distances taken
'by the off-set staff, on either side
'of the station-line, are to be entred
'into columns on either side of the
'middle-column, according to their
'position with respect to that line.
'The names or characters of the
'objects, with proper remarks may
'be entred in columns on either
'side of these last.

'Because in the place of the gra-
'phometer described by our Author,
'surveyors now make use of the
'Theo-

'Theodolite, we ſhall ſubjoin a
'deſcription of Mr. *Siſſon*'s lateſt
'improved Theodolite from Mr.
'*Gardner*'s *practical Surveying im-*
'*proved.* See a figure of it in plate 4.

'In this inſtrument the three
'ſtaves by braſs ferrils at top ſcrew
'into bell-metal joints, that are
'moveable between braſs pillars fixt
'in a ſtrong braſs plate, in which
'round the center is fixt a ſocket
'with a ball moveable in it, and upon
'which the four ſcrews preſs, that ſet
'the Limb horizontal; next above
'is another ſuch plate, thro' which
'the ſaid ſcrews paſs, and on which
'round the center is fixt a fruſtum
'of a cone of bell-metal, whoſe axis
'(being connected with the center
'of the ball) is always perpendicular
'to the Limb, by means of a coni-
'cal braſs ferril fitted to it, whereon
'is fixt the Compaſs-box, and on
'it the Limb, which is a ſtrong bell-
'metal ring, whereon are moveable

'three brass indexes, in whose 'plate are fixt four brass pillars, 'that, joining at top, hold the cen'ter-pin of the bell-metal double 'Sextant, whole double Index is fixt 'on the center of the same plate; 'within the double Sextant is fixt 'the Spirit-Level, and over it the 'Telescope.

'The Compass-box is graved 'with two diamonds for *North* and '*South*, and with 20 degrees on both 'sides of each, that the needle may 'be set to the variation, and its er'ror also known.

'The Limb has two flower-de'luces against the diamonds in the 'Box, instead of 180 each, and is 'curiously divided into whole de'grees, and number'd to the left 'hand at every ten to twice 180, 'having three Indexes distant 120, '(with *Nonius*'s divisions on each 'for the decimals of a degree) that 'are moved by a pinion fixt below

'one

'one of them without moving the 'Limb, and in another is a ſcrew 'and ſpring under, to fix it to any 'part of the Limb: it has alſo di-'viſions number'd for taking the 'quarter Girt in inches of round '*Timber* at the middle height, when 'ſtanding ten feet horizontally di-'ſtant from its center, which at 20 'muſt be doubled, and at 30 trebled, 'to which a ſhorter index is uſed, 'having *Nonius*'s diviſions for the 'decimals of an inch; but an abate-'ment muſt be made for the bark, 'if not taken off.

'The double Sextant is divided 'on one ſide from under its center '(when the Spirit-Tube and Tele-'ſcope are level) to above 60 de-'grees each way, and numbred at 10, '20, &c. and the double Index '(through which it is moveable) 'ſhews on the ſame ſide the degree 'and decimal of any Altitude or De-'preſſion to that extent by *Nonius*'s

'di-

'divisions; on the other side are
'divisions number'd for taking the
'upright height of *Timber*, &c. in
'feet, when distant ten feet, which
'at 20 must be doubled, and at 30
'trebled; and also the quantities
'for reducing hypothenusal Lines
'to horizontal: it is moveable by a
'pinion fixt in the double index.

'The Telescope is a little shorter
'than the diameter of the Limb,
'that a fall may not hurt it; yet
'it will magnify as much, and shew
'a distant object as perfect, as
'most of treble its length; in its
'focus are very fine cross-wires,
'whose interfection is in the plane
'of the double Sextant, and this was
'a whole Circle, and turned in a
'Lathe to a true Plane, and is fixt
'at right angles to the Limb; so
'that whenever the Limb is set ho-
'rizontal (which is readily done by
'making the Spirit-Tube level o-
'ver two screws, and the like over

'the

'the other two) the double Sexant 'and Telescope are moveable in a 'vertical Plane, and then every An- 'gle taken on the Limb (tho' the 'Telescope be never so much ele- 'vated or deprest) will be an Angle 'in the Plane of the Horizon, and 'this is absolutely necessary in plot- 'ting a horizontal Plane.-----

'If the Lands to be plotted are 'hilly and not in any one Plane, 'the Lines measured cannot be truly 'laid down on paper, without being 'reduced to one Plane, which must 'be the horizontal, because Angles 'are taken in that Plane.

'In viewing my objects, if they 'have much altitude or depression, 'I either write down the degree and 'decimal shewn on the double Sex- 'tant, or the links shewn on the 'back-side, which last subtracted 'from every chain in the station- 'line, leaves the length in the ho- 'rizontal Plane; but if the degree is

'taken,

'taken, the following Table will
' shew the quantity.

A Table of the links to be subtracted out of every Chain in hypothenusal Lines *of several degrees altitude, or depression, for reducing them to* horizontal.

Degrees	Links	Degrees	Links	Degrees	Links
4,05	$\frac{1}{4}$	14,07	3	23,074	8
5,73	$\frac{1}{2}$	16,26	4	24,495	9
7,02	$\frac{3}{4}$	18,195	5	25,84	10
8,11	1	19,95	6	27,13	11
11,48	2	21,565	7	28,36	12

' Let the first station-line really measure 1107 links, and the Angle of altitude or depression be 19°,93; looking in the Table I find against 19°,95 is 6 links, now 6 times 11 is 66, which subtracted from 1107 leaves 1041, the true length to be laid down in the Plan.

It

'It is useful in surveying to take 'the angles which the bounding 'lines form with the magnetick 'needle; in order to check the an-'gles of the figure, and to plot 'them conveniently afterwards.'

PART II.

Of the surfaces of bodies.

THE smallest superficial measure with us is a square inch, 144 of which make a square foot. Wrights make use of these in the measuring of deals and planks; but the square foot which the Glaziers in measuring of glass, consists only of 64 square inches. The other measures are, *first*, the ell square; 2*dly*, The fall, containing 36 square

 ells.

ells. 3*dly*, The rood containing 40 falls. 4*thly*, The acre containing 4 roods. Slaters, Masons and Pavers use the ell square and the fall. Surveyors of land use the square ell, the fall, the rood, and the acre.

The superficial measures of the *English*, are, 1*st*, the square foot; 2*dly*, the square yard, containing 9 square feet; for their yard contains only 3 feet; 3*dly*, the pole, containing 30 ¼ sqaure yards; 4*thly*, the rood, containing 40 poles; 5*thly*, the acre, containing 4 roods. And hence it is easy to reduce our superficial measures to the *English*, or theirs to ours.

' In order to find the content of ' a field, it is most convenient to ' measure the lines by the chains de- ' scribed above, *p.* 5. that of 22 ' yards for computing the *English* ' acres, and that of 24 *Scots* ells for ' the acres of *Scotland*. The chain ' is divided into 100 links, and the

' square

' square of the chain is 10000 square
' links; ten squares of the chain, or
' 100000 square links give an acre.
' Therefore, if the area be expressed
' by square links, divide by 100000
' or cut off five decimal places, and
' the quotient shall give the area
' in acres and decimals of an acre.
' Write the entire acres apart, but
' multiply the decimals of an acre
' by 4, and the product shall give
' the remainder of the area in roods
' and decimals of a rood. Let the
' entire roods be noted apart after
' the acres, then multiply the deci-
' mals of a rood by 40, and the pro-
' duct shall give the remainder of
' the area in falls or poles. Let the
' entire falls or poles be then writ
' after the roods, and multiply the
' decimals of a fall by 36, if the
' area is required in the measures of
' *Scotland*; but multiply the deci-
' mals of a pole by $30\frac{1}{4}$, if the area
' is required in the measures of *Eng-*

'*land*, and the product shall give
'the remainder of the area in square
'ells in the former case, but in
'square yards in the latter. If, in
'the former case, you would reduce
'the decimals of the square ell
'to square feet, multiply them by
'9.50694; but in the latter case,
'the decimals of the *English* square
'yard are reduced to square feet,
'by multiplying them by 9.

'Suppose, for example, that the
'area appears to contain 1265842
'square links of the chain of 24
'ells, and that this area is to be ex-
'pressed in acres, roods, falls, &c.
'of the measures of *Scotland*. Di-
'vide the square links by 100000,
'and the quotient 12.65842 shows
'the area to contain 12 acres and
'$\frac{65842}{100000}$ of an acre. Multiply the de-
'cimal part by 4, and the product
'2.63368 gives the remainder in
'roods and decimals of a rood.
'Those decimals of the rood being
mul-

'multiplied by 40, the product 'gives 25.3472 falls. Multiply 'the decimals of the fall by 36, and 'the product gives 12.4992 square 'ells. The decimals of the square 'ell multiplied by 9.50694 give '4.7458 square feet. Therefore the 'area proposed amounts to 12 acres, '2 roods, 25 falls, 12 square ells, 'and 4 $\frac{7458}{10000}$ square feet.

'But if the area contains the same 'number of square links of *Gunter's* 'chain, and is to be expressed by '*English* measures, the acres and 'roods are computed in the same 'manner as in the former case. 'The poles are computed as the 'falls. But the decimals of the 'pole, *viz.* $\frac{3472}{10000}$ are to be multiplied 'by 30$\frac{1}{4}$ (or 30.25) and the pro-'duct gives 10.5028 square yards. 'The decimals of the square yard 'multiplied by 9, give 4.5252 square 'feet: therefore in this case the a-'rea is in *English* measure 12 acres,

2

' 2 roods, 25 poles, 10 square yards, ' and 4 $\frac{5252}{10000}$ square feet.

' The *Scots* acre is to the *English* ' acre by statute as 100000 to 78694, ' if we have regard to the difference ' betwixt the *Scots* and *English* foot ' above mentioned. But it is custo- ' mary in some parts of *England* to ' have 18, 21, *&c.* feet to a pole, ' and 160 such poles to an acre; ' whereas, by the statute, 16 $\frac{1}{2}$ feet ' make a pole. In such cases the ' acre is greater in the duplicate ' ratio of the number of feet to a ' pole.

' They who measure land in *Scot-* ' *land* by an ell of 37 *English* inches, ' make the acre less than the true ' *Scots* acre by 593 $\frac{6}{10}$ square *English* ' feet, or by about $\frac{1}{93}$ of the acre.

' An husband-land contains 6 acres ' of sock and sythe land, that is, of ' land that may be tilled with a ' plough, and mowen with a sythe; ' 13 acres of arable land make an ox-

' gang

'gang or oxengate; four oxengate
'make a pound land of old extent (by
'a decree of the Exchequer, *March*
'11. 1585) and is called *librata ter-*
'*ræ*. A forty shilling land of old
'extent contains eight oxgang, or
'104 acres.

'The *arpent* about *Paris* contains
'32400 square *Paris* feet, and is
'equal to $2\frac{4}{7}$ *Scots* roods or $3\frac{37}{108}$
'*English* roods.

'The *actus quadratus*, according
'to *Varro*, *Columella*, &c. was a
'square of 120 *Roman* feet. The
'*Jugerum* was the double of this.
''Tis to the *Scots* acre as 10000 to
'20456, and to the *English* acre as
'10000 to 16097. It was divid-
'ed (like the *As*) into 12 *uncias*, and
'the *uncia* into 24 *scrupula*. This
'with the three preceeding para-
'graphs are taken from an ingenious
'manuscript written by Sir *Robert*
'*Stewart* Professor of Natural Philo-
'phy. The greatest part of the table
in

' in *p.* 6. was taken from it likewise.

PROPOSITION I.

PROBLEM FIG. 1.

To find the area of a rectangular parallelogram ABCD.

LET the side AB, for example, be five feet long, and BC (which constitutes with BC a right angle at B) be 17 feet. Let 17 be multiplied by 5, and the product 85 will be the number of square feet in the Area of the figure ABCD. But if the parallelogram proposed is not rectangular as BEFC, its base BC multiplied into its perpendicular height AB (not into its side BE) will give its area. This is evident from 35th 1. *Eucl.*

PRO-

PROPOSITION II.

PROB. FIG. 2.

To find the area of a given triangle.

LET the triangle BAC be given, whose base BC is supposed 9 feet long, let the perpendicular AD be drawn from the angle A opposite to the base, and let us suppose AD to be 4 feet. Let the half of the perpendicular be multiplied into the base, or the half of the base into the perpendicular, or take the half of the product of the whole base into the perpendicular, the product gives 18 square feet for the area of the given triangle.

But if only the sides are given, the perpendicular is found either by protracting the triangle or by 12. and 13. 2d *Eucl.* or by trigonometry; but how the area of a triangle may be found from the given ſides only ſhall be ſhown in the 4th Prop. of this part.

PROPOSITION III.

PROB. FIG. 3.

To find the area of any rectilineal figure.

IF the figure be irregular, let it be reſolved into triangles; and drawing perpendiculars to the baſes in each of them, let the area of each triangle be found by the preceeding Prop. and the ſum of theſe areas will give the area of the figure.

SCHOLIUM 1.

In meaſuring boards, planks and glaſs, their ſides are to be meaſured by a foot rule divided into 100 equal parts; and after multiplying the ſides, the

the decimal fractions are easily reduced to lesser denominations. The mensuration of these is easy, when they are rectangular parallelograms.

SCHOLIUM. 2.

If a field is to be measured, let it first be plotted on paper by some of the methods described in the preceeding part, and let the figure so laid down be divided into triangles, as was shown in the preceeding proposition.

The base of any triangle, or the perpendicular upon the base, or the distance of any two points of the field is measured, by applying it to the scale according to which the map is drawn.

SCHOLIUM 3.

But if the field given be not in a horizontal plain, but uneven and mountainous, the ſcale gives the horizontal line between any two points, but not their diſtance meaſured on the uneven ſurface of the field. And indeed it would appear that the horizontal plain is to be accounted the area of an uneven and hilly country. For if ſuch ground is laid out for building on, or for planting with trees or bearing corn, ſince theſe ſtand perpendicular to the horizon, it is plain that a mountainous country cannot be conſidered as of greater extent for thoſe uſes than the horizontal plain; nay, perhaps, for nouriſhing of plants, the horizontal plain may be preferable.

If however the area of a figure as it lies irregularly on the ſurface of the earth, is to be meaſured, this may be eaſily done by reſolving it into triangles as it lyes. The ſum of their areas will be the area ſought, which exceeds the area of the horizontal figure more or leſs according as the field is more or leſs uneven.

PRO-

PROPOSITION IV.

PROB. FIG. 2.

The sides of a triangle being given, to find the area, without finding the perpendicular.

LET all the sides of the triangle be collected into one sum, from the half of which let the sides be separately substracted, that three differences may be found betwixt the foresaid half sum and each side; then let these three differences, and the half sum be multiplied into one another, and the square-root of the product will give the area of the triangle. For example, let the sides be 10, 17, 21, the half of their sum is 24, the three differences betwixt this half sum and the three sides, are 14, 7 and 3. The first being multiplied by the second, and their product by the third, we we have 294 for the product of the differences, which multiplied by the foresaid half sum 24 gives 7056; the square-root of which 84 is the area of the triangle. The demonstration of this, for the sake of brevity we omit. It is to be found in several treatises, particularly in *Clavius*'s *practical Geometry*.

PROPOSITION V.

Theor. Fig. 4.

The area of the ordinate figure ABEFGH is equal to the product of the half-circumference of the polygon multiplied into the perpendicular drawn from the center of the circumscribed circle to the side of the polygon.

For the ordinate figure can be resolved into as many equal triangles, as there are sides of the figure; and

and ſince each triangle is equal to the product of half the baſe into the perpendicular, it is evident that the ſum of all the triangles together, that is the polygon, is equal to the product of half the ſum of the baſes (that is the half of the circumference of the polygon) into the common perpendicular height of the triangles drawn from the center C to one of the ſides, for example to AB.

PROPOSITION VI.

PROB. FIG. 5.

The area of a circle is found by multiplying the half of the periphery into the radius, or the half of the radius into the periphery.

FOR a circle is not different from an ordinate or regular polygon of an infinite number of ſides, and the common height of the triangles into

into which the polygon (or circle) may be ſuppoſed to be divided, is the radius of the circle.

Were it worth while, it were eaſy to demonſtrate accurately this propoſition by means of the inſcribed and circumſcribed figures, as is done in the 5th prop. of the treatiſe of *Archimedes*, concerning the dimenſions of the circle.

COROLLARY.

Hence alſo it appears that the area of the ſector ABCD is produced by multiplying the half of the arch into the radius; and likewiſe that the area of the ſegment of the circle ADC is found by ſubſtracting from the area of the ſector the area of the triangle ABC.

PRO-

PROPOSITION VII.

THEOR. FIG. 6.

The circle is to the square of the diameter as 11 *to* 14 *nearly.*

FOR if the diameter AB be supposed to be 7, the circumference AHKB will be almost 22 (by the 22d prop. of the first part of this) and the area of the square DC will be 49; and by the preceeding prop. of this, the area of the circle will be 38 $\frac{1}{2}$, therefore the square DC will be to the inscribed circle as 49 to 38$\frac{1}{2}$, or as 98 to 77, that is, as 14 to 11. *Q. E. D.*

If greater exactness is required, you may proceed to any degree of accuracy; for the square DC is to the inscribed circle as 1 to $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13}$ *&c. in infinitum.*

' This series will be of no service

‘ for computing the area of the cir-
‘ cle accurately, without ſome fur-
‘ ther artifice, becauſe it converges
‘ at too ſlow a rate. The area of
‘ the circle will be found exactly e-
‘ nough for moſt purpoſes, by mul-
‘ tiplying the ſquare of the diame-
‘ ter by 7854, and dividing by
‘ 10000, or cutting off four decimal
‘ places from the product; for the
‘ area of the circle is to the circum-
‘ ſcribed ſquare nearly as 7854 to
‘ 10000.

PROPOSITION VIII.

PROB. FIG. 7.

To find the area of a given ellipſe.

LEt ABCD be an ellipſe, whoſe greater diameter is BD and leſſer AC, biſecting the greater perpendicularly in E. Let a mean proportional HF be found (by the 13th

6.

6. *Eucl.*) between AC and BD, and (by the 6th of this) find the area of the circle deſcribed on the diameter HF. I ſay that this area is equal to the area of the ellipſe ABCD. For becauſe, as BD to AC, ſo the ſquare of BD to the ſquare of HF (by 2. *Cor.* 20th 6. *Eucl.*) but (by the 2d 12. *Eucl.*) as the ſquare of BD to the ſquare of HF, ſo is the circle of the diameter BD to the of the diameter HF: therefore as BD to AC, ſo the circle of the diameter BD to the circle of the diameter HF. And (by the 5. prop. of *Archimedes* of ſpheroids) as the greater diameter BD to the leſſer AC, ſo is the circle of the diameter BD to the ellipſe ABCD. Conſequently (by the 11th 5. *Eucl.*) the cirlec of the diameter BD will have the ſame proportion to the circle of the diameter HF, and to the ellipſe ABCD. Therefore, by 9th 5. *Eucl.*

N 2 the

the area of the circle of the diameter HF will be equal to the area of the ellipse ABCD. *Q. E. D.*

SCHOLIUM.

From this and the two preceeding propositions, a method is derived of finding the area of an ellipse. There are two ways; 1st, say, as 1 is to the lesser diameter, so is the greater diameter to a fourth number (which is found by the rule of three.) Then again say, as 14 to 11, so is the 4th number found to the area sought. But the second way is shorter. Multiply the lesser diameter into the greater, and the product by 11; then divide the whole product by 14, and the quotient will be the area sought of the ellipse. For example, let the greater diameter be 10, and the lesser 7, by multiplying 10 by 7, the product is 70, and multiplying that by 11, it is 770, and dividing 770 by by 14, the quotient will be 55, which is the area of the ellipsis sought.

'The area of the ellipse will be 'found more accurately, by multi-'plying the product of the two di-'ameters by .7854.

We shall add no more about other plain surfaces, whether rectilinear or curvilinear, which seldom occur in practice; but shall subjoin some propositions about measuring the surfaces of solids.

PROPOSITION IX. PROB.

To measure the surface of any prism.

BY the 14th definition of the 11. *Eucl.* a prism is contained by plains, of which two opposite sides (commonly called the bases) are plain rectilinear figures; which are either regular and ordinate, and measured by prop. 5. of this part, or however irregular, and then they

are

are measured by the 3d prop. of this book. The other sides are parallelograms which are measured by the 1. prop. of this second part, and the whole superficies of the prism, consists of the sum of those taken altogether.

PROPOSITION X. PROB.

To measure the superficies of any pyramid.

SInce its basis is a rectilinear figure, and the rest of the plains terminating in the top of the pyramid are triangles, these measured separately, and added together, give the surface of the pyramid required.

PRO-

PROPOSITION XI. PROB.

To measure the superficies of any regular body.

THese bodies are called regular, which are bounded by equilateral and equiangular figures. The superficies of the tetraedron consists of four equal and equiangular triangles; the superficies of a hexaedron or cube, of six equal squares; an octaedron of eight equal equilateral triangles; a dodecaedron of twelve equal and ordinate pentagons; And the superficies of an icosiaedron of twenty equal and equilateral triangles. Therefore it will be easy to measure these surfaces from what has been already shown.

In the same manner we may measure the superficies of a solid contained by any plains.

PRO-

PROPOSITION XII.

PROB. FIG. 8.

To measure the superficies of a cylinder.

BEcause a cylinder differs very little from a prism whose opposite plains (or bases) are ordinate figures of an infinite number of sides, it appears that the superficies of a cylinder, without the bases, is equal to an infinite number of parallelograms, the common altitude of all which is, with the height of the cylinder, and the bases of them all differ very little from the periphery of the circle which is the base of the cylinder. Therefore this periphery multiplied into the common height, gives the superficies of the cylinder, excluding the bases; which are to be measured separately by the help

help of the 6th prop. of this part.

This proposition concerning the measure of the surface of the cylinder (excluding its basis) is evident from this, That when it is conceived to be spread out, it becomes a parallelogram, whose base is the periphery of the circle of the base of the cylinder stretched into a right line, and whose height is the same with the height of the cylinder.

PROPOSITION XIII.

PROB. FIG. 9.

To measure the surface of a right cone.

THE surface of a right cone is very little different from the surface of a right pyramid, having an ordinate polygon for its base of an infinite number of sides; the surface of which (excluding the base) is equal to the sum of the triangles.

The sum of the bases of these triangles is equal to the periphery of the circle of the base, and the common height of the triangles is the side of the cone AB; wherefore the sum of these triangles is equal to the product of the sum of the bases (*i. e.* the periphery of the base of the cone) multiplied into the half of the common height, or it is equal to the product of the side of the cone multiplied into the half of the periphery of the base.

If the area of the base is likewise wanted, it is to be found separately by the 6th prop. of this part. If the surface of a cone is supposed to be spread out on a plane, it will become a sector of a circle, whose radius is the side of the cone, and the arch terminating the sector is made from the periphery of the base. Whence by corol. 6. prop. of this, its dimension may be found.

CO-

COROLLARY.

Hence it will be easy to measure the surface of a *frustum* of a cone cut by a plain parallel to the base. As to what relates to the measuring of the surface of the scalenous cone, because it is not very useful in practice, we shall not describe the method, which would carry us beyond the limits of this treatise.

PROPOSITION XIV.

PROB. FIG. 10.

To measure the surface of a given sphere.

LEt there be a sphere, whose center is A, and let the area of its convex surface be required. *Archimedes* demonstrates (37. prop. 1. book of the sphere and cylinder) that its surface is equal to the area of four great circles of the ſphere; that is, let the area of the great circle be multiplied by 4, and the product will give the area of the ſphere, or by 20th 6, and 2d 12. of *Eucl.* the area of the ſphere given is equal to the area of a circle whoſe radius is the right line BC the diameter of the ſphere. Therefore having meaſured (by 6. prop. of this part) the circle deſcribed with the radius BC, this will give the ſurface of the ſphere.

PROPOSITION XV.

PROB. FIG. 10.

To meaſure the ſurface of a ſegment of a ſphere.

LET there be a ſegment cut off by the plain ED. *Archimedes* demonſtrates (49. and 50. 1. *de ſphæra*) that the ſurface of this ſegment, exclucluding the circular base, is equal to the area of a circle whose radius is the right line BE drawn from the vertex B of the segment to the periphery of the circle DE. Therefore by the 6. prop. of this part, it is easily measured.

COROLLARY 1.

Hence that part of the surface of a sphere that lieth between two parallel plains is easily measured, by substracting the surface of the lesser segment from the surface of the greater segment.

COROLLARY 2.

Hence likewise it follows that the surface of a cylinder described about a sphere (excluding the basis) is equal to the surface of the sphere, and the parts of the one to the parts of the other intercepted between plains

pa-

parallel to the basis of the cylinder.

PART III.

Of solid figures and their mensuration.

AS in the preceeding parts we took an inch for the smallest measure in length, and an inch square for the smallest superficial measure; so now, in treating of the mensuration of solids, we take a cubical inch for the smallest solid measure. Of these 109 make a *Scots* pint; other liquid measures depend on this, as is generally known.

In dry measures, the firlot by statute, contains 19 ½ pints, and on this depend the other dry measures: there-

therefore, if the content of any solid be given in cubical inches, it will be easy to reduce the same to the common liquid or dry measures, and conversely to reduce these to solid inches. The liquid and dry measures in use among other nations are known from their writers.

' As to the *English* liquid mea-
' sures, By act of Parliament 1706,
' any round vessel, commonly cal-
' led a cylinder, having an even
' bottom, being seven inches in di-
' ameter throughout, and six inches
' deep from the top of the inside
' to the bottom; (which vessel will
' be found by computation to con-
' tain 230 $\frac{907}{1000}$ cubical inches) or a-
' ny vessel containing 231 cubical
' inches, and no more, is deemed to
' to be a lawful wine gallon. An
' *English* pint therefore contains
' 28 $\frac{7}{8}$ cubical inches; two pints make
' a quart, four quarts a gallon, 18
' gallons a roundlet, three round-

' lets

'lets and an half, or 63 gallons
'make a hogshead; the half of
'a hogshead is a barrel; one hog-
'shead, and a third, or 84 gal-
'lons make a puncheon; one pun-
'cheon and a half, or two hog-
'sheads, or 126 gallons, make a
'pipe or butt; the third part of a
'pipe, or 42 gallons make a tierce;
'two pipes, or three puncheons, or
'four hogsheads make a tun of
'wine. Tho' the *English* wine-gal-
'lon is now fixed at 231 cubical in-
'ches, the standard kept in *Guildhall*
'being measured before many per-
'sons of distinction, *May* 25, 1688,
'it was found to contain only 224
'such inches.

'In the *English* Beer-measure, a
'gallon contains 282 cubical inch-
'es; consequently 35 ¼ cubical inch-
'es make a pint, two pints make a
'quart, four quarts make a gallon,
'nine gallons a firkin, four firkins
'a barrel. In ale, eight gallons make
'a

'a firkin, and 32 gallons make a
'a barrel. By an Act of the first
'of *William* and *Mary*, 34 gallons
'is the barrel, both for beer and
'ale, in all places, except within
'the weekly bills of mortality.

'In *Scotland* it is known that four
'gills make a mutchkin, two mutch-
'kins make a chopin, a pint is two
'chopins, a quart is two pints, and
'a gallon is four quarts or eight
'pints. The accounts of the cubi-
'cal inches contained in the *Scots*
'pint vary considerably from each
'other. According to our Author
'it contains 109 cubical inches.
'But the standard jugs kept by the
'Dean of Gild of *Edinburgh* (one
'of which has the year 1555, with
'the arms of *Scotland*, and of the
'town of *Edinburgh* marked upon it)
'having been carefully measured se-
'veral times, and by different per-
'sons, the *Scots* pint, according
'to those standards, was found to

'contain about 103 $\frac{4}{11}$ cubic inches. 'The Pewterers jugs (by which the 'vessels in common use are made) 'are said to contain sometimes be-'twixt 105 and 106 cubic inch-'es. A cask that was measured 'by the Brewers of *Edinburgh*, be-'fore the Commissioners of Excise 'in 1707, was found to contain '46 $\frac{7}{8}$ *Scots* pints; the same vessel 'contained 18 $\frac{1}{16}$ *English* ale gal-'lons. Supposing this mensuration 'to be just, the *Scots* pint will be to 'the *English* ale gallon as 289 to '750; and if the *English* ale-gallon 'be supposed to contain 282 cubi-'cal inches, the *Scots* pint will con-'tain 108.664 cubical inches. But it 'is suspected on several grounds that 'this experiment was not made 'with sufficient care and exactness.

'The Commissioners appointed 'by authority of Parliament to settle 'the measures and weights in their 'Act of *February* 19. 1618, relate,

That

'That having caused fill the *Lin-* '*lithgow* firlot with water, they found 'that it contained 21 ¼ pints of the 'just *Stirling* jug and measure. They 'likewise ordain that this shall be 'the just and only firlot, and add, '*That the wideness and breadness of* '*the which firlot, under and above,* '*even over within the buirds, shall* '*contain nineteen inches, and the* '*sixth part of an inch, and the deip-* '*ness seven inches, and a third part* '*of an inch.* According to this Act '(supposing their experiment and 'computation to have been accurate) 'the pint contained only 99.56 cu- 'bical inches; for the content of 'such a vessel as is described in the 'act is 2115.85, and this divided 'by 21 ¼, gives 99.56. But by the 'weight of water said to fill this fir- 'lot in the same Act, the measure of 'the pint agrees nearly with the *E-* '*dinburgh* standard above mentioned.

'As for the *English* measures of

 'corn

'corn the *Winchester* gallon contains
'272 ¼ cubical inches, two gallons
'make a peck, four pecks or eight
'gallons (that is 2178 cubical inch-
'es) make a bushel, and a quarter is
'eight bushels.

'Our Author says, that 19 ½ *Scots*
'pints, make a firlot; but this does
'not appear to be agreeable to the
'statute above mentioned, nor to the
'standard jugs. It may be conjectu-
'red that the proportion assigned by
'him has been deduced from some
'experiment of how many pints, ac-
'cording to common use, were con-
'tained in the firlot. For if we sup-
'pose those pints to have been each
'of 108.664 cubical inches, accord-
'ing to the experiment made in the
'1707, before the Commissioners
'of Excise described above; then
'19 ½ such pints will amount to
'2118.94 cubical inches which a-
'grees nearly with 2115.85 the
'measure of the firlot by the statute

'above

‘ above mentioned. But it is pro-
‘ bable, that in this he followed the
‘ Act 1587, where it is ordained
‘ that the wheat firlot ſhall contain
‘ 19 pints and twa joucattes. A
‘ wheat firlot marked with the *Lin-*
‘ *lithgow* ſtamps being meaſured, was
‘ found to contain about 2211 cubi-
‘ cal inches. By the ſtatute of 1618,
‘ the barley-firlot was to contain 31
‘ pints of the juſt *Stirling* jug.

‘ A *Paris* pint is 48 cubical *Paris*
‘ inches, and is nearly equal to an
‘ *Engliſh* wine quart. The *Boiſſeau*
‘ contains 644.68099 *Paris* cubical
‘ inches, or 780.36 *Engliſh* cubical
‘ inches.

‘ The *Roman Amphora* was a cu-
‘ bical *Roman* foot, the *Congius* was
‘ the 8th part of the *Amphora*, the
‘ *Sextarius* was one ſixth of the *Con-*
‘ *gius*. They divided the *Sextarius*
‘ like the *As* or *Libra*. Of dry
‘ meaſures the *Medimnus* was equal
‘ to two *Amphoras*, that is about $1\frac{11}{27}$
Eng-

'*English* legal bushels; and the *Me-*
'*dius* was the third part of the *Am-*
'*phora.*

PROPOSITION I. Prob.

To find the solid content of a given prism.

BY the 2. prop. of the 2d part of this, let the area of the base of the prism be measured, and be multiplied by the height of the prism, the product will give the solid content of the prism.

PROPOSITION II. Prob.

To find the solid content of a given pyramid.

THE area of the base being found (by the 3d prop. of the 2d part) let it be multiplied by the third part of the height of the pyramid, or the third

third part of the baſe by the height, the product will give the ſolid content by 7th 12. *Eucl.*

CORROLARY.

If the ſolid content of a *fruſtum* of a pyramid is required, firſt let the ſolid content of the entire pyramid be found, from which ſubſtract the ſolid content of the part that is wanting, and the ſolid content of the broken pyramid will remain.

PROPOSITION III. PROB.

To find the content of a given cylinder.

THE area of the baſe being found (by prop. 6. of 2. part) if it be a circle, and by prop. 8. if it be an ellipſe for in both caſes it is a cylinder) multiply it by the height of the cylinder, and the ſolid content of the cylinder will be produced.

CO-

COROLLARY, FIG. 1.

And in this manner may be measured the solid content of vessels and casks not much different from a cylinder as ABCD. If towards the middle EF it be somewhat grosser, the area of the circle of the base being found (by 6th prop. of 2. part) and added to the area of the middle circle EF, and the half of their sum (that is an arithmetical mean between the area of the base, and the area of the middle circle) taken for the base of the vessel, and multiplied into its height, the solid content of the given vessel will be produced.

Note, That the length of the vessel as well as the diameters of the base and of the circle EF ought to be taken within the staves; for it is the solid content within the staves that is sought.

PRO-

PROPOSITION IV. Prob.

To find the ſolid content of a given cone.

LET the area of the baſe (found by prop. 6th, 2. part) be multiplied into $\frac{1}{3}$ of the height, the product ſhall give the ſolid content of the cone; for (by 10th, 12. *Eucl.*) a cone is the third part of a cylinder that has the ſame baſe and height.

PROPOSITION V.

Prob. Fig. 2. and 3.

To find the ſolid content of a fruſtum *of a cone cut by a plane parallel to the plain of the baſe.*

FIrſt, let the height of the entire cone be found, and thence, (by the preceeding prop.) its ſolid content; from which ſubſtract the ſolid content of the cone cut off at the top, there will remain the solid content of the *frustum* of the cone.

How the content of the entire cone may be found, appears thus, let ABCD be the *frustum* of the cone (either right or scalenous, as in the figures 2. and 3.) let the cone ECD be supposed to be compleated; let AG be drawn parallel to DE, and let AH and EF be perpendicular on CD. It will be (by 2d 6. *Eucl.*) as CG : CA : : CD : CE; but (by the 4th prop. of the same book) as CA : AH : : CE : EF; consequently (by 22d 5. *Eucl.*) as CG : AH : : CD : EF, that is, as the excess of the diameter of the greater base of the *frustum* above the lesser base, is to the height of the *frustum*, so is the diameter of the greater base to the height of the entire cone.

CO-

COROLLARY, Fig. 4.

Some casks whose staves are remarkably bended about the middle, and streight towards the ends, may be taken for two portions of cones, without any considerable error. Thus ABEF is a *frustum* of a right cone, to whose base EF, on the other side, there is another similar *frustum* of a cone joined EDCF. The vertices of these cones, if they be supposed to be completed, will be found at G and H. Whence, by the preceeding prop. the solid content of such vessels may be found.

PROPOSITION VI.

Theor. Fig. 5.

A Cylinder circumscribed about a sphere, that is, having its base equal to a great circle of the ſphere, and its height equal to the diameter of the ſphere, is to the ſphere as 3 to 2.

Let ABEC be the quadrant of a circle, and ABDC the circumſcribed ſquare; and likewiſe the triangle ADC; by the revolution of the figure about the right line AC as axis, a hemiſphere will be generated by the quadrant, a cylinder of the ſame baſe and height by the ſquare, and a cone by the triangle. Let theſe three be cut any how by the plain HF parallel to the baſe AB, the ſection in the cylinder will be a a circle whoſe radius is FH; in the hemiſphere a circle of the radius EF; and in the cone, a circle of the radius GF.

By the 47th 1. *Eucl.* EAq, or HFq = EFq and FAq taken together, (but AFq = FGq, becauſe AC = CD) therefore the circle of the radius HF is equal to a circle of the radius EF together with a circle of the radius GF, GF, and ſince this is true every where, all the circles together deſcribed by the reſpective *radii* HF, (that is the cylinder,) are equal to all the circles deſcribed by the reſpective *radii* EF and FG (that is to the hemiſphere and the cone taken together;) but by 10th 12. *Eucl.* the cone generated by the triangle DAC is one third part of the cylinder generated by the ſquare BC. Whence it follows that the hemiſphere generated by the rotation of the quadrant ABEC, is equal to the remaining two third parts of the cylinder, and that the whole ſphere is $\frac{2}{3}$ of the double cylinder circumſcribed about it.

This is that celebrated 39th prop. 1. book of *Archimedes* of the ſphere and cylinder, in which he determines the proportion of the cylinder to the ſphere inſcribed to be that of 3 to 2.

COROLLARY.

Hence it follows that the ſphere is equal to a cone having a circle equal to the ſuperficies of the ſphere, or to four great circles of the ſphere, or to a circle whoſe radius is equal to the diameter of the ſphere (by 14th prop. 2. part of this) for its baſe, and its height equal to the ſemidiameter of the ſphere. And indeed a ſphere differs very little from the ſum of an infinite number of cones, that have their baſes in the ſurface of the ſphere, and their common vertex in the center of the ſphere: ſo that the ſuperficies of the ſphere, (of whoſe dimenſion ſee 14th prop. 2. part of this) multiplied into the third part of the ſemidiameter, gives the ſolid content of the ſphere.

PRO-

PROPOSITION VII.

PROB. FIG. 6.

To find the ſolid content of a ſector of the ſphere.

A Spherical ſector ABC (as appears by the corol. of the preceeding prop.) is very little different from an infinite number of cones, having their baſes in the ſuperficies of the ſphere BEC, and their common vertex in the center. Wherefore the ſpherical ſuperficies BEC being found (by 15. prop. 2. part) and multiplied into the third part of AB the radius of the ſphere, the product will give the ſolid content of the ſector ABC.

COROLLARY.

It is evident how to find the ſolidity

dity of a ſpherical ſegment leſs than a hemiſphere, by ſubſtracting the cone ABC from the ſector already found. But if the ſpherical ſegment be greater than a hemiſphere, the cone correſponding muſt be added to the ſector, to make the ſegment.

PROPOSITION VIII.

PROB. FIG. 7.

To find the ſolidity of the ſpheroid, and of its ſegments cut by plains perpendicular to the axis.

IN the ſecond prop. of this part, it is ſhown that every where EH : EG : : CF : CD ; but circles are as the ſquares deſcribed upon their rays, that is, the circle of the radius EH, is to the circle of the radius EG as CF*q* to CD*q*. And ſince it is ſo every where, all the circles deſcribed with the reſpective rays EH

(that

(that is the ſpheroid made by the rotation of the ſemiellipſis AFB around the axis AB) will be to all the circles deſcribed by the reſpective *radii* EG (that is the ſphere deſcribed by the rotation of the ſemicircle ADB on the axis AB) as FG*q* to CD*q*; that is, as the ſpheroid to the ſphere on the ſame axis, ſo is the ſquare of the other axis of the generating ellipſe to the ſquare of the axis of the ſphere.

And this holds, whether the ſpheroid be found by a revolution around the greater or leſſer axis.

COROLLARY I.

Hence it appears that the half of the ſpheroid, formed by the rotation of the ſpace AHFC around the axis AC, is double of the cone generated by the triangle AFC about the ſame axis; which is the 32d prop. of *Archimedes*, of conoids and ſpheroids.

COROLLARY 2.

Hence likewise is evident the measure of segments of the spheroid cut by plains perpendicular to the axis. For the segment of the spheroid made by the rotation of the space ANHE round the axis AE, is to the segment of the sphere having the same axis AC, and made by the rotation of the segment of the circle AMGE, as CF*q* to CD*q*.

But if the measure of this solid be wanted with less labour, by the 34th prop. of *Archimedes*, of conoids and spheroids, it will be as BE to AC + EB, so is the cone of the same base and height to the segment of the sphere made by the rotation of the space ANHE around the axis EA; which could easily be demonstrated, (was this a proper place for it) by the method of indivisibles.

CO-

COROLLARY 3.

Hence it is easy to find the solid content of the segment of a sphere or spheroid intercepted between two parallel plains, perpendicular to the axis. This agrees as well to the oblate as to the oblong spheroid, as is obvious.

COROLLARY 4. Fig. 8.

If a cask is to be valued as the middle piece of an oblong spheroid, cut by the two plains DC and FG, at right angles to the axis. First let the solid content of the half spheroid ABCED be measured by the preceeding prop. from which let the solidity of the segment DEC be substracted, and there will remain the segment ABCD, and this doubled, will give the capacity of the cask required.

The following method is generally made use of for finding the solid content of such vessels. The double area of the greatest circle, that is of that which is described by the diameter AB at the middle of the cask, is added to the area of the circle at the end, that is, of the circle DC or FG (for they are usually equal) and the third part of this sum is taken for a mean base of the cask, which therefore multiplied into the length of the cask OP gives the content of the vessel required.

Sometimes vessels have other figures, different from those we have mentioned; the easy methods of measuring which may be learned from those who practise this art. What hath already been delivered is sufficient for our purpose.

PROPOSITION IX.

PROB. FIG. 9. and 10.

To find how much is contained in a vessel that is in part empty, whose axis is parallel to the horizon.

LET AGBH be the great circle in the middle of the cask, whose segment GBH is filled with liquor, the segment GAH being empty; the segment GBH is known, if the depth EB be known, and EH a mean proportional between the segments of the diameter AE and EB, which are found by a rod or ruler put into the vessel at the orifice. Let the basis of the cask, at a medium, be found, which suppose to be the circle CKDL, and let the segment KCL be similar to the segment GAH (which is either found by the rule of three, because as the circle AGBH

is

is to the circle CKDL, ſo is the ſegment GAH to the ſegment KCL; or is found from the tables of ſegments made by authors) and the product of this ſegment multiplied by the length of the cask, will give the liquid content remaining in the cask.

PROPOSITION X. Prob.

To find the ſolid content of a regular and ordinate body.

A Tetraedron being a pyramid, the ſolid content is found by the ſecond prop. of this part. The hexaedron or cube, being a kind of priſm, it is meaſured by the firſt prop. of this part. An octaedron conſiſts of two pyramids of the ſame ſquare baſe, and of equal heights, conſequently its meaſure is found from the ſecond prop. of this part. A dodecaedron conſiſts of twelve pyramids

ramids having equal equilateral and equiangular pentagonal bases; and so one of these being measured (by 2d prop. of this) and multiplied by 12, the product will be equal to the solid content of the dodecaedron. The icosiaedron consists of 20 equal pyramids having triangular bases, the solid content of one of which being found (by the 2d prop. of this) and multiplied by 20, gives the whole solid. The bases and heights of these pyramids, if you want to proceed more exactly, may be found by Trigonometry.

PROPOSITION XI. PROB.

To find the solid content of a body however irregular.

LET the given body be immersed in a vessel of water, having the figure of a parallellopipedon or prism, and let it be noted how much the

the water is raiſed upon the immerſion of the body. For it is plain that the ſpace which the water fills, after the immerſion of the body, exceeds the ſpace filled before its immerſion, by a ſpace equal to the ſolid content of the body however irregular. But when this exceſs is of the figure of a parallellopipedon or priſm, it is eaſily meaſured by the firſt prop. of this part, to wit, by multiplying the area of the baſe, or mouth of the veſſel, into the difference of the elevations of the water before and after immerſion. Whence is found the ſolid content of the body given. *Q. E. I.*

In the ſame way the ſolid content of a part of a body may be found, by immerſing that part only in water.

There is no neceſſity to inſiſt here on diminiſhing or enlarging ſolid bodies in a given proportion. It will be eaſy to deduce theſe things from the 11th and 12th B. of *Euclid.*

'THE following rules are subjoined for the ready computation of the contents of vessels, and of any solids in the measures in use in *Great Britain*.

'I. To find the content of a cylindric vessel in *English* wine gallons, the diameter of the base and altitude of the vessel being given in inches and decimals of an inch.

'Square the number of inches in the diameter of the vessel; multiply this square by the number of inches in the height; then multiply the product by the decimal fraction .0034; and this last product shall give the content in wine gallons and decimals of such a gallon. To express the rule arithmetically, let D represent the number of inches and decimals of an inch in the diameter of the vessel, and H the inches and decimals of an inch in the height of the vessel; then the content in

‘ wine gallons ſhall be DDH × $\frac{34}{10000}$,
‘ or DDH × .0034. *Ex.* Let the di-
‘ ameter D = 51.2 inches, the height
‘ H = 62.3 inches, then the content
‘ ſhall be 51.2 × 51.2 × 62.3 ×.0034
‘ = 555.27342 wine gallons. This
‘ rule follows from prop. 7. of the
‘ ſecond part, and prop. 3. of the
‘ third part; for by the former, the
‘ area of the baſe of the veſſel is in
‘ ſquare inches DD × .7854; and
‘ by the latter, the content of the
‘ veſſel in ſolid inches is DDH ×
‘ .7854; which divided by 231 (the
‘ number of cubical inches in a wine
‘ gallon) gives DDH × .0034 the
‘ content in wine gallons. But tho'
‘ the charges in the exciſe are
‘ made (by ſtatute) on the ſuppoſi-
‘ tion that the wine-gallon contains
‘ 231 cubical inches, yet it is ſaid
‘ that in ſale 224 cubical inches
‘ (the content of the ſtandard mea-
‘ ſured in *Guildhall*, as was mentioned
above)

'above) are allowed to be a wine
'gallon.

'II. Supposing the *English* ale
'gallon to contain 282 cubical inch-
'es, the content of a cylindric vef-
'sel is computed in such gallons,
'by multiplying the square of the
'diameter of the vessel by its height
'as formerly, and their product by
'the decimal fraction .0027851.
'That is, the solid content in ale
'gallons is DDH × .0027851.

'III. Supposing the *Scots* pint to
'contain about 103.4 cubical inch-
'es (which is the measure given by
'the standards at *Edinburgh*, accor-
'ding to experiments mentioned
'above) the content of a cylindric
'vessel is computed in *Scots* pints,
'by multiplying the square of the
'diameter of the vessel by its height,
'and the product of these by the
'decimal fraction .0076. Or the
'content of such a vessel in *Scots*
'pints is DDH × .0076.

'IV. Supposing the *Winchester* 'bushel to contain 2178 cubical 'inches, the content of a cylindric 'vessel is computed in those bushels 'by multiplying the square of the 'diameter of the vessel by the 'height, and the product by the 'decimal fraction .0003606. But 'the standard bushel having been 'measured by Mr. *Everard* and o-'thers in 1696, it was found to 'contain only 2145.6 solid inches; 'and therefore it was enacted in the 'Act for laying a duty upon malt; '*That every round bushel, with a plain* '*and even bottom, being* 18½ *inches* '*diameter throughout, and* 8 *inches* '*deep, should be esteemed a legal* Win-'chester *bushel*. According to this 'Act (ratified in the first Year of 'Queen *Anne*) the legal *Winchester* 'bushel contains only 2150.42 so-'lid inches. And the content of a 'cylindric vessel is computed in such 'bushels, by multiplying the square 'of

‘ of the diameter by the height, and ‘ their product by the decimal fra‑ ‘ ction .0003652. Or the content ‘ of the vessel in those bushels is ‘ DDH × .0003652.

‘ V. Supposing the *Scots* wheat ‘ firlot to contain 21 ¼ *Scots* pints, ‘ (as is appointed by the Statute ‘ 1618) and the pint to be conform ‘ to the *Edinburgh* standards above ‘ mentioned, the content of a cylin‑ ‘ dric vessel in such firlots is com‑ ‘ puted by multiplying the square ‘ of the diameter by the height, and ‘ their product by the decimal fra‑ ‘ ction .00358. This firlot, in 1426 ‘ is appointed to contain 17 pints, ‘ in 1457 it was appointed to con‑ ‘ tain 18 pints, in 1587 it is 19 ¼ ‘ pints, in 1618 it is 21 ¼ pints; ‘ and tho' this last Statute appears to ‘ have been founded on wrong com‑ ‘ putations in several respects, yet ‘ this part of the Act that relates to ‘ the number of pints in the firlot

‘ seems

'seems to be the least exceptionable; 'and therefore we suppose the firlot to contain 21 ¼ pints of the '*Edinburgh* standard, or about 2197 'cubical inches; which a little exceeds the *Winchester* bushel, from 'which it may have been originally copied.

'Supposing the bear firlot to contain 31 *Scots* pints (according to the 'Statute 1618) and the pint conform to the *Edinburgh* standards, 'the content of a cylindric vessel in 'such firlots is found by multiplying the square of the diameter by 'the height, and this product by '.000245.

'When the section of the vessel is 'not a circle but an ellipsis, the product of the greatest diameter by 'the least is to be substituted in 'those rules for the square of the 'diameter.

'VII. To compute the content 'of a vessel that may be considered

'as

‘ as a *frustum* of a cone in any of ‘ those measures.

‘ Let A represent the number of inches in the diameter of the greater base, B the number of inches in the diameter of the lesser base. Compute the square of A, the product of A multiplied by B, and the square of B, and collect these into a sum. Then find the third part of this sum, and substitute it in the preceding rules in the place of the square of the diameter; and proceed in all other respects as before. Thus, for example, the content in wine gallons is $\overline{AA + AB + BB} \times \frac{1}{3} \times H \times .0034$.

‘ Or to the square of half the sum of the diameters A and B, add one third part of the square of half their difference; and substitute this sum in the preceding rules for the square of the diameter of the vessel; for the square of $\frac{1}{2}A + \frac{1}{2}B$ added to $\frac{1}{3}$ of the square of $\frac{1}{2}A -$

‘ $\frac{1}{2}B$

'$\frac{1}{2}$ B, gives $\frac{1}{3}$ AA + $\frac{1}{3}$ AB + $\frac{1}{3}$ BB.

'. VIII. When a veſſel is a *fruſtum* ' of a parabolic conoid, meaſure the ' diameter of the ſection at the ' middle of the height of the *fru-* ' *ſtum*; and the content will be pre- ' ciſely the ſame as of a cylinder ' of this diameter, of the ſame height ' with the veſſel.

' IX. When a veſſel is a *fruſtum* ' of a ſphere, if you meaſure the ' diameter of the ſection at the ' middle of the height of the *fru-* ' *ſtum*, then compute the content of ' a cylinder of this diameter of the ' ſame height with the veſſel, and ' from this ſubſtract $\frac{1}{3}$ of the con- ' tent of a cylinder of the ſame ' height on a baſe whoſe diameter is ' equal to its height; the remain- ' der will give the content of the ' veſſel. That is, if D repreſent the ' diameter of the middle ſection, and ' H the height of the *fruſtum*, you ' are to ſubſtitute DD — $\frac{1}{3}$ HH for

' the

'the square of the diameter of the 'cylindric vessel in the first six rules.

'X. When the vessel is a *frustum* 'of a spheroid, if the bases are e-'qual, the content is readily found 'by the rule in *p.* 132. In other 'cases, let the axis of the solid be to 'the conjugate axis, as n to 1; let 'D be the diameter of the middle 'section of the *frustum*, H the 'height or length of the *frustum*; 'and substitute in the first six rules '$DD - \frac{HH}{3nn}$ for the square of the di-'ameter of the vessel.

'XI. When the vessel is an hyper-'bolic conoid, let the axis of the solid 'be to the conjugate axis, as n to 1, 'D the diameter of the section at the 'middle of the *frustum*, H the height 'or length, compute $DD + \frac{1}{3nn} \times HH$, 'and substitute this sum for the 'square of the diameter of the cy-'lindric vessel in the first six rules.

'XII. In general, it is usual to

'measure any round vessel, by di'stinguishing it into several *frustums*, 'and taking the diameter of the se'ction at the middle of each *fru'stum*; thence to compute the con'tent of each, as if it was a cylin'der of that mean diameter; and to 'give their sum as the content of the 'vessel. From the total content 'computed in this manner they sub'stract successively the numbers 'which express the circular areas 'that correspond to those mean di'meters, each as often as there are 'inches in the altitude of the *fru'stum* to which it belongs, begin'ning with the uppermost; and in 'this manner calculate a table for 'the vessel, by which it readily ap'pears how much liquor is at any 'time contained in it, by taking ei'ther the dry or wet inches; having 'regard to the inclination or drip of 'the vessel when it has any.

'This method of computing the content

'content of a *frustum* from the diameter of the section at the middle of its height is exact in that case only when it is a portion of a parabolic conoid; but in such vessels as are in common use, the error is not considerable. When the vessel is a portion of a cone or hyperbolic conoid, the content by this method is found less than the truth; but when it is a portion of a sphere or spheroid, the content computed in this manner exceeds the truth. The difference or error is always the same in the different parts of the same or of similar vessels when the altitude of the *frustum* is given. And when the altitudes are different, the error is in the triplicate *ratio* of the altitude. If exactness be required, the error in measuring the *frustum* of a conical vessel, in this manner, is $\frac{1}{4}$ of the content of a cone, similar to the vessel, of an altitude equal to the

 'height

'height of the *frustum*. In a sphere, 'it is $\frac{1}{3}$ of a cylinder of a diameter 'and height equal to the *frustum*. 'In the spheroid and hyperbolic 'conoid, it is the same as in a cone 'generated by the right-angled tri- 'angle contained by the two semi- 'axes of the figure revolving about 'that side which is the semi-axis of 'the *frustum*. These are demon- 'strated in a treatise of fluxions by 'Mr. *Mac Laurin*, *p.* 25. and 715. 'where those theorems are extended 'to *frustums* that are bounded by 'planes oblique to the axis in all the 'solids that are generated by any 'conic section revolving about ei- 'ther axis.

'In the usual method of comput- 'ing a table for a vessel, by sub- 'ducting from the whole content 'the number that expresses the up- 'permost area as often as there are 'inches in the uppermost *frustum*, 'and afterwards the numbers for the

'other

'other areas successively, it is obvious that the contents assigned by the table, when a few of the uppermost inches are dry, are stated a little too high, if the vessel stands on its lesser base, but too low when it stands on its greater base; because, when one inch is dry, for example, it is not the area at the middle of the uppermost *frustum*, but rather the area at the middle of the uppermost inch, that ought to be subducted from the total content, in order to find the content in this case.

'XIII. To measure round timber, let the mean circumference be found in feet and decimals of a foot; square it, multiply this square by the decimal .079577, and the product by the length. *Ex.* Let the mean circumference of a tree be 10.3 feet, and the length 24 feet. Then 10.3 × 10.3 × .079577 × 24 = 202.615 is the

'num-

'number of cubical feet in the tree.
'The foundation of this rule is, that
'when the circumference of a crcle
'is 1, the area is .079577471 5, and
'that the areas of circles are as the
'squares of their circumferences.

'But the common way used by
'Artificers for measuring round tim-
'ber, differs much from this rule.
'They call one fourth part of the
'circumference the *girt*, which is
'by them reckoned the side of a
'square, whose area is equal to the
'area of the section of the tree;
'therefore they square the *girt*, and
'then multiply by the length of the
'tree. According to their method
'the tree of the last example would
'be computed at 159.13 cubical
'feet only. How square timber is
'measured, will be easily under-
'stood from the preceeding propo-
'sitions. Fifty solid feet of hewn
'timber, and forty of rough timber
'make a load.

' XIV.

'XIV. To find the burden of a
'ship, or the number of tons it will
'carry, the following rule is com-
'monly given. Multiply the length
'of the keel taken within board, by
'the breadth of the ship within
'board, taken from the mid-ship
'beam from plank to plank, and
'the product by the depth of the
'hold, taken from the plank below
'the keelson to the under part of
'the upper deck plank, and divide
'the product by 94, the quotient
'is the content of the tonnage re-
'quired. This rule however can-
'not be accurate, nor can one rule
'be supposed to serve for the
'measuring exactly the burden of
'ships of all sorts. Of this the
'reader will find more in the me-
'moires of the Royal Academy of
'Sciences at *Paris* for the year
'1721.

'Our Author having said nothing
'of weights, it may be of use to
'add

‘ add briefly that the *English* Troy-
‘ pound contains 12 ounces, the
‘ ounce twenty penny weight, and
‘ the penny weight 24 grains; that
‘ the *Averdupois* pound contains 16
‘ ounces, the ounce 16 drams, and
‘ that 112 pounds is usually cal-
‘ led the hundred weight. It is
‘ commonly supposed that 14 pounds
‘ *Averdupois* are equal to 17 pounds
‘ Troy. According to Mr. *Ever-*
‘ *ard*'s experiments, one pound *A-*
‘ *verdupois* is equal to 14 ounces,
‘ 11 penny weight and 16 grains
‘ Troy, that is, to 7000 grains;
‘ and an *Averdupois* ounce is 437 $\frac{1}{2}$
‘ grains. The *Scots* Troy pound,
‘ (which by the Statute 1718 was
‘ to be the same with the *French*)
‘ is commonly supposed equal to
‘ 15 $\frac{3}{4}$ ounces *English* Troy, or 7560
‘ grains. By a mean of the stan-
‘ dards kept by the Dean of Gild of
‘ *Edinburgh*, it is 7599 $\frac{3}{32}$, or 7600
‘ grains. They who have measur-
‘ ed

'ed the weights which were ſent
'from *London*, after the Union of
'the Kingdoms, to be the ſtandards
'by which the weights in *Scotland*
'ſhould be made, have found the
'*Engliſh Averdupois* pound (from a
'medium of the ſeveral weights)
'to weigh 7000 grains, the ſame as
'Mr. *Everard*; according to which
'the *Scots*, *Paris* or *Amſterdam*
'pound will be to the pound *Aver-*
'*dupois* as 38 to 35. The *Scots*
'Troy ſtone contains 16 pounds,
'the pound two marks or 16 ounces,
'an ounce 16 drops, a drop 36
'grains. Twenty *Scots* ounces make
'a Trone pound; but becauſe it is
'uſual to allow one to the ſcore, the
'Trone pound is commonly 21
'ounces. Sir *John Skene* however
'makes the Trone ſtone to con-
'tain only 19½ pounds.

FINIS.