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ADVERTISEMENT.

The Committee appointed by the Royal Society to direct the publication of the Philosophical Transactions take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former Transactions, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the Transactions had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future Transactions; which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body,
upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.
PHILOSOPHICAL TRANSACTIONS.

1. On the Refraction and Dispersion of the Halogens, Halogen Acids, Ozone, Steam, Oxides of Nitrogen and Ammonia.

By Clive Cuthbertson, Fellow of University College, London University, and Maude Cuthbertson.

Communicated by A. W. Porter, F.R.S.

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It has long been well known that the refractive indices of simple gaseous compounds do not obey the additive law so closely as those of solids or liquids. From the study of these last Gladstone and Dale, and their followers, succeeded in obtaining refraction equivalents for a large number of the elements which were fairly constant for the same class of compound. But in gases the discrepancies were found to be

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much wider, and this appeared the more surprising since, in other fields of research, the gaseous state has proved peculiarly favourable for the discovery of simple relations.

Accumulation of Data.—The investigation, of which the present paper forms part, was designed to throw light on the cause of these anomalies.

The first step was to enlarge the field of the enquiry by the accumulation of data, and with this object we have, either together or in collaboration with others, determined and redetermined the refraction and dispersion in the gaseous state of fourteen elements and ten compounds within the limits of the visible spectrum, and the refraction of six elements and four compounds for a single wave-length.

The number is still far too small. Many interesting compounds remain to be investigated. But, as the present instalment of work has occupied nearly two years and has led to certain definite conclusions, it seems better to publish it rather than to await the addition of more difficult and, perhaps, less instructive examples.

Choice of Compounds.—In selecting the compounds to be examined we have been guided by the principle that the molecule should consist of as few atoms as possible, and that the refraction and dispersion of each constituent should be measurable in the gaseous state. It is unfortunate that the list of compounds which comply with these requirements is very short. All compounds of carbon are excluded. Of those substances which are dealt with in the present paper, the most important are the halogen acids, which form a regular series of simple diatomic molecules. Steam, SO₂, and H₂S form an interesting cycle, and the two oxides of nitrogen are very instructive since the constituents are the same in both cases.* Ozone is remarkable as an example of the effect of polymerization on the refractivity.

Expression of Results.—The choice of a formula for the expression of results is of fundamental importance. Previous workers on the subject of gaseous refractivities have almost invariably used that of Cauchy, with two terms or three, according to the degree of accuracy of their figures. But this formula is not based on modern physical theory. Moreover, we have shown in a previous paper† that, when only two constants are used, it is inadequate to express the experimental results, even in the visible spectrum; while if a third constant and term involving 1/λ is introduced, the shape of the dispersion curve cannot be easily grasped on inspection of the figures. For these reasons we have abandoned this formula, and have used, tentatively, a formula of Sellmeyer's type, \( \frac{\mu^2 - 1}{2} = \mu - 1 \) (approximately) = \( \frac{N}{n_0^2 - n^2} \).

It is unnecessary to defend the adoption for gaseous indices of this expression, which has been widely used for solids, and is in general outline established on theory

* The examination of NO₂ and N₂O₄ has been postponed on account of its difficulty, but promises to be no less interesting and important owing to the association which characterises it.
and confirmed by experiment. But it is desirable to emphasise the fact that the
calculation of the constants given in this paper from a formula containing only one
term on the right-hand side is only provisional, since the main conclusion of the
authors is that, both for elements and compounds, a single term is inadequate, except
in the case of monatomic gases. The simple form of the formula is, however, useful
for indicating at a glance the direction and magnitude of the changes in refractive
and dispersive power which take place when elements combine to form a compound.

If \( \mu_1, \mu_2 \) are the refractive indices of a substance for two wave-lengths for which the
frequencies are \( n_1, n_2 \), we have

\[
\frac{\mu_1 - \mu_2}{\mu_1 - 1} = \frac{n_1^2 - n_2^2}{n_0^2 - n_2^2} = \frac{n_1^2 - n_2^2}{n_0^2} \text{ approximately,}
\]

since \( n_2^2 \) is usually small compared with \( n_0^2 \). The left side of the equation expresses
the dispersive power of the substance, which is thus seen to be inversely proportional
to \( n_0^2 \).

\textbf{Changes of Refractive Power.} — Let \( \mu_A - 1 = \frac{N_A}{n_0^2 A - n_2^2} \); \( \mu_B - 1 = \frac{N_B}{n_0^2 B - n_2^2} \) be the
formulae which express the refractivity of two gaseous elements in the region of the
visible spectrum, and let \( \mu_{AB} - 1 = \frac{N_{AB}}{n_0^2 AB - n_2^2} \) express the refractivity of the compound
which they form. If the molecules of the two elements are diatomic, and one
molecule of the compound is formed of one atom of each, then the change of refractive
power on combination is

\[
\frac{1}{2} \left( \frac{N_A}{n_0^2 A - n_2^2} + \frac{N_B}{n_0^2 B - n_2^2} \right) - \frac{N_{AB}}{n_0^2 AB - n_2^2}.
\]

\textbf{Changes of Dispersive Power.} — If the additive law were strictly followed the
dispersive power of a compound, measured in a region remote from free frequencies,
would lie between the dispersive powers of its constituents, \( i.e. \), \( n_2^2 AB \) would lie
between \( n_0^2 A \) and \( n_0^2 B \). For it can be shown that, for a short region of the spectrum,
remote from free frequencies, so that \( n_2^2 \) is large compared with \( n_2^2 \) and \( n_2^2/n_0^2 \) can be
neglected,

\[
n_0^2 AB = \left( \frac{N_A}{n_0^2 A} + \frac{N_B}{n_0^2 B} \right) \left( \frac{N_A}{n_0^2 A} + \frac{N_B}{n_0^2 B} \right),
\]

and this expression lies between these limits.

Hence, if the experimental value of \( n_0^2 AB \) differs from this value, the variation must
be due to the changes in one or more of the four quantities \( N_A, N_B, n_0^2 A, n_0^2 B \)
consequent on combination. It is evident that all four unknown quantities cannot
be determined from a knowledge of \( N_{AB}, n_0^2 AB \), which is all we obtain from a deter-
mination of the dispersion of the compound. But two cases should be distinguished.
The value of \( n_0^2 AB \) may vary owing to changes in \( n_0^2 A \) and \( n_0^2 B \) consequent on
combination, \( i.e. \), to real modifications of the free periods of the vibrators; or it may
be due to the introduction of a new free period, or the elimination of an old one, previously wrapped up in \( n_0^2 \) and \( n_0^3 \).

**Summary of Results.**—Before proceeding to give the experimental work it will be convenient to summarise the results obtained:

1. In hydrochloric, hydrobromic, hydriodic acids, sulphur dioxide* and sulphur-retted hydrogen* the refractivity of the compound is less than the sum of the refractivities of the elements, and the dispersive power of the compound lies between those of its constituents;

2. In nitrous oxide, nitric oxide, ammonia, and ozone the refractivity of the compound is greater than the sum of those of its constituents, and the dispersive power is greater than that of either constituent;

3. In steam the refractivity of the compound is less, and the dispersive power greater, than those of its constituents;

4. In all cases the change in dispersive power is great relatively to the change in refractive power.

We have framed a hypothesis which, in our opinion, would account for these changes in a qualitative manner, and we hope to publish it elsewhere.

**Chlorine.**

**Previous Work.**—The refractive index of gaseous chlorine has only been measured twice. Dulong† found \( \mu - 1 = 0.00772 \) for white light with gas, prepared from MnO₂, whose density was 2.47 (air 1).

Mascart‡ found \( \mu - 1 = 0.00768 \) for white light. He worked at low pressures at the temperature 12° C. and compared the refractivity with that of air under the same conditions.

The dispersion of the gas has not previously been measured.

**Preparation.**—The gas we used was prepared by dropping strong hydrochloric acid on potassium permanganate. After washing in water and drying by sulphuric acid it was condensed in a bath of acetone cooled with solid CO₂ and then allowed to boil off till all the air in the connecting tubes had been displaced. The refractometer tube, previously evacuated, was then placed in connection with the chlorine by a capillary tube so fine that the gas entered the tube sufficiently slowly for the interference bands to be counted. When the pressure of gas in the refractometer reached that of the atmosphere the bands came to rest, and the pressure and temperature were read.

‡ 'C. R.,' p. 321, 1878.
After each experiment was over the gas was absorbed over soda lime and only experiments in which the impurities were negligible were used in determining the refractive index.

The light used was that of the green mercury line, \( \lambda = 5461 \).

*Refractivity.*—The following figures were obtained in five experiments, the experimental values being reduced to 0° C. and 760 mm. by the formula:

\[
\mu - 1 = (\nu - 1) \frac{T}{273} \frac{760}{P}.
\]

\((\mu - 1)10^7 \ldots 7976, 7985, 7966, 7986, 7981.\) Mean 7980.

**Standard Conditions.**—The practice of reducing observations of refractivity to the standard temperature of 0° C. and the pressure of 760 mm., dates from a time when deviations from the laws of BOYLE and GAY-LUSSAC were alike unknown. As accuracy improved and the field of research was extended to vapours, these criteria became insufficient and sometimes meaningless. MASCART, the volume of whose work entitles him to be considered the leading authority on the subject, at first adopted the old conditions, and even in the case of \( \text{SO}_2 \), expressed the refractivity as it would be at 0° and 760 mm. But, in his later work, when dealing with chlorine and bromine and with some organic compounds for which the coefficients of thermal expansion and compressibility were unknown, he contented himself with determining the refractivity at pressures as low as possible and comparing it with that of air at the same temperature (12° C.).

Le Roux, in his experiments on sulphur, mercury, phosphorus and arsenic, expressed the ratio of the refractivity to the density, and LORENZ and PRYTZ adopted the same system. It is evident that this principle is the most convenient for those who wish to compare the refractivity of equal numbers of atoms of different elements, or of molecules of different compounds. Accordingly, in the present work, we have reduced all refractivities to the values which they would have had if the gas or vapour had the density of hydrogen at 0° C. and 76 cm., \((000089849)\) gr./cm.) cubed multiplied by the ratio of the theoretical molecular weight of the substance in question to that of hydrogen. But, in order to avoid confusion, we shall denote this value by the symbol \((\mu - 1) \frac{D}{(d_{96})}\), where \(D\) denotes the standard density as here defined, and \((d_{96})\) the density at 0° C. and 76 cm.

The density of chlorine at the temperature and pressure of the atmosphere has been determined recently by TREADWELL and CHRISTIE. They found that at 20° C., the molecular volume was 22039 and 220300 at 10° C.

The average temperature of our experiments was 19°.4 C. and at that temperature the molecular volume of chlorine would be 22038'9 c.c. That of hydrogen is 22428'8 c.c. Hence the refractivity observed must be multiplied by the ratio of these numbers. We thus arrive at the number '00078412; and since the accuracy of the experiments is not greater than 1 part in 1000 we may accept '000784 as the refractivity of chlorine for the green mercury line.

Dispersion.—Assuming this value, the dispersion was measured at seven other points of the visible spectrum by the method described in previous papers.*

The following table shows the results:—

<table>
<thead>
<tr>
<th>(\lambda \times 10^6)</th>
<th>((\mu - 1) 10^8) (\frac{D}{(d_{00})}) Observed</th>
<th>Calculated</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>6707.85</td>
<td>77563</td>
<td>77556</td>
<td>-7</td>
</tr>
<tr>
<td>6438.5</td>
<td>77703</td>
<td>77697</td>
<td>-6</td>
</tr>
<tr>
<td>5779.5</td>
<td>78121</td>
<td>78123</td>
<td>+2</td>
</tr>
<tr>
<td>5769.5</td>
<td>78135</td>
<td>78139</td>
<td>+4</td>
</tr>
<tr>
<td>5460.7</td>
<td>78400</td>
<td>78402</td>
<td>+2</td>
</tr>
<tr>
<td>5209.1</td>
<td>78651</td>
<td>78655</td>
<td>+4</td>
</tr>
<tr>
<td>5085.8</td>
<td>78791</td>
<td>78792</td>
<td>+1</td>
</tr>
<tr>
<td>4799.9</td>
<td>79166</td>
<td>79156</td>
<td>-10</td>
</tr>
</tbody>
</table>

The numbers shown in the column headed "calculated" are derived from the formula

\[
(\mu - 1) \frac{D}{(d_{00})} = \frac{7.3131 \times 10^{27}}{9629.4 \times 10^{27} - \lambda^2},
\]

where \(n\) is the frequency of the light, i.e., \(\frac{V}{\lambda} = \frac{3 \times 10^6}{\lambda} \text{ (in cm.).}\)

Relation of Dispersion to Absorption.—The absorption spectrum of chlorine has been fully investigated by Mrs. Laird.† She describes the spectrum as consisting of (1) a region of general absorption which extends, in a column of gas 60 cm. long, from \(\lambda 4650\) to \(\lambda 2599\), and lengthens in both directions with increasing pressure, but chiefly towards the less refrangible end, reaching \(\lambda 4990\) at 2\(\frac{1}{2}\) atmospheres; and (2) a line spectrum lying between \(\lambda 4799\) and \(\lambda 5350\) which consists of groups of dark lines, giving a fluted appearance to the eye. With increased pressure these lines became more intense but did not increase in numbers where they had been visible before. In addition there may be patches of general absorption in the region of line spectrum.

The region of which the refractivity was measured by us extended from \( \lambda 6708 \) to \( \lambda 4799\cdot9 \) and thus covers the whole range of the line spectrum and 200 Å.U. which are affected by the general absorption.

It is, therefore, of interest to find that the observed values of the refractivity lie on a smooth curve. But it is significant that the calculated curve cuts across the experimental, in which the curvature is greater. This appears to indicate that a single term formula is inadequate to express the results, and that a second term is required, in which both \( N \) and \( n_0^2 \) are small, to represent the influence of the absorption band.

In the region of the line spectrum it was to be expected that if any variations of refractive index accompanied the variations of absorption they would be found either in the immediate neighbourhood of each dark line or possibly affecting the whole breadth of each group of lines forming a fluting. In order to investigate this point the following test was made:

The paths of the two interfering rays of light were equalised, so that when the wave-length of the light employed was continuously changed from red to violet no change was observed in the position of the interference bands in the field of view. Chlorine was then admitted into one tube till the path of that beam had been retarded by 450 bands (\( \lambda = 5461 \)). Next, by means of the compensator,* the same beam was accelerated by an equal amount. If, now, the wave-length of the light be changed from red to violet any movement of the bands would be due to the difference of refractivity of glass and chlorine for the particular wave-length which is in the field of view. The bands can easily be read to 1/10, so that when 450 bands have passed a difference of refractivity of 1/4500 can be detected.

The slit was then narrowed till the interference systems due to \( \lambda 5790 \) and \( \lambda 5769 \) were clearly separated in the field of view: \( i.e. \), till the light composing any particular part of the image varied by less than 20 Å.U. On changing the wave-length continuously from red to violet no sudden change in the bands could be detected. It may therefore be concluded that between \( \lambda 6708 \) and \( \lambda 4799 \) any sudden change of refractivity exceeding \( '000784/4500 = '0000017 \) must be confined to a breadth of less than 20 Å.U. and probably to less than half that amount. It is not possible to detect small changes in the refractivity in a narrower section of the spectrum than this, since, if the light is sufficiently dispersed, it becomes too feeble to read tenths of a band.

**Bromine.**

Previous Work.—DUFKET records MASCART's value \( \mu = 1\cdot001125 \) for the D line. The dispersion of the gas has, apparently, not been attempted.

Preparation.—The purest bromine obtainable from Kahlbaum was used. Before every experiment the bulb containing the liquid was cooled to \( -80^\circ \) C. and exhausted,

* This compensator, of special construction, retards all wave-lengths equally except in so far as dispersion affects them.
so as to get rid of traces of HBr which might have formed since the previous experiment. Since it was necessary for the gas to pass through greased taps more elaborate precautions for purification would have been useless. The grease used was specially prepared from pure paraffin wax and oil. After each experiment the vapour which entered the refractometer tube was admitted to contact with the mercury and absorbed. Any admixture of HBr would have been measurable as hydrogen. In the experiments on which we rely to obtain the index the residue was negligible.

Procedure.—In order to reduce the observed refractivity to standard conditions it was necessary to measure the density of vapour employed. For this purpose the bulb containing the liquid was connected with a density bulb in parallel with the refractometer tube and a determination of the density of the vapour accompanied each experiment. The atomic weight of bromine was taken as 79.97 (O = 16). Owing to the great absorption of the vapour in the green it was necessary to use red light for the determination of the absolute index. With $\lambda = 6438$ as many as 80 or 90 bands could be read, whereas at $\lambda 5461$ the band system was no longer readable after 25 bands had passed.

The absolute index was determined from the following five experiments which were well corroborated by several others not quite so trustworthy:

| Table II.—Refractive Index of gaseous Bromine, $\lambda = 6438$. |
|---|---|---|
| Experiment. | $(\mu - 1) \times 10^8 \frac{D}{d_0\nu^2}$ | Number of bands counted. |
| 1 | 1158 | 55 |
| 2 | 1156 | 65 |
| 3 | 1154 | 89 |
| 4 | 1159 | 50 |
| 5 | 1167 | 80 |
| Mean | 1157 | |

Dispersion.—Owing to the strong absorption of the vapour in the visible region measurements of the dispersion were difficult, and the accuracy attained was much inferior to that shown in the case of chlorine. The following figures show approximately the number of bands which were readable at different points of the spectrum:

<table>
<thead>
<tr>
<th>$\lambda \times 10^8$</th>
<th>Number of bands readable.</th>
<th>$\lambda \times 10^8$</th>
<th>Number of bands readable.</th>
</tr>
</thead>
<tbody>
<tr>
<td>6708</td>
<td>More than 120.</td>
<td>5600</td>
<td>Less than 30.</td>
</tr>
<tr>
<td>6438</td>
<td>&quot; 120.</td>
<td>5461</td>
<td>&quot; 25.</td>
</tr>
<tr>
<td>6000</td>
<td>&quot; 115.</td>
<td>5209</td>
<td>&quot; 20.</td>
</tr>
<tr>
<td>5700</td>
<td>Less 90.</td>
<td>5085</td>
<td>&quot; 10.</td>
</tr>
<tr>
<td>5650</td>
<td>&quot; 60.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Beyond 5461 the number of bands read was not sufficient to ensure trustworthy values.

The following table gives the experimental values in column 2:

**Table III.—Dispersion of gaseous Bromine.**

<table>
<thead>
<tr>
<th>$\lambda \times 10^8$</th>
<th>$(\mu - 1) \times 10^7 \frac{D}{(d_076)}$</th>
<th>Observed.</th>
<th>Calculated.</th>
<th>Difference.</th>
</tr>
</thead>
<tbody>
<tr>
<td>6708</td>
<td>11525</td>
<td>11518</td>
<td>-7</td>
<td></td>
</tr>
<tr>
<td>6438</td>
<td>11570</td>
<td>11571</td>
<td>+1</td>
<td></td>
</tr>
<tr>
<td>6000</td>
<td>11662</td>
<td>11675</td>
<td>+13</td>
<td></td>
</tr>
<tr>
<td>5800</td>
<td>11735</td>
<td>11731</td>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>5750</td>
<td>11741</td>
<td>11746</td>
<td>+5</td>
<td></td>
</tr>
<tr>
<td>5700</td>
<td>11762</td>
<td>11762</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5600</td>
<td>11796</td>
<td>11767</td>
<td>-29</td>
<td></td>
</tr>
<tr>
<td>5461</td>
<td>11849</td>
<td>11842</td>
<td>-7</td>
<td></td>
</tr>
</tbody>
</table>

The experimental numbers fall approximately on a smooth curve which is given by

$$(\mu - 1) \frac{D}{(d_076)} = \frac{4.2838 \times 10^7}{3919.2 \times 10^7 - \nu^2}.$$ 

The figures calculated from this equation are given in column 3.

*Relation of Dispersion to Absorption.*—As in the case of chlorine a test was made for a rapid change of refractivity affecting a narrow section of the spectrum, but none was detected.

In this case also the change of refractivity is small compared with the increase of the absorption as we pass from the red to the green.

**Iodine.**

*Previous Measurements.*—The only determination of the refractive index of iodine on record is that of Hurion,* who gives $\mu = 1.00205$ for the red and 1.00192 for the violet. He employed a prism and heated the iodine to 700° C.

It was hoped that with a refractometer a higher degree of accuracy could be obtained, but the results of experiment were disappointing. The absorption band which has its maximum at $\lambda 5000$ extends so far into the red that, with the faint light available in a Jamin apparatus, the band system was very soon obliterated.

In the red ($\lambda = 6438$) as many as 21 bands could be observed with difficulty, but

* Journal de Physique,' I., VII., p. 181.
at \( \lambda = 5600 \) it was not possible to read more than three, and on the violet side of the region of absorption no measurements were possible. In attempting to measure the dispersion the experimenter has to choose between a small number of bands read over a slightly wider range and a larger number read over a small range. In either case the errors of observation are relatively large.

*Procedure.*—A weighed quantity of iodine was introduced into the refractometer tube which was evacuated and sealed off.*

The tube was then heated till the solid had all sublimed and the bands observed.

*Refraction.*—The wave-length selected for the absolute determination was 6438, and this was obtained from white light of a Nernst lamp by means of a fixed deviation spectroscope. The volume of the refractometer tube was 49'1 c.c. and the weight of iodine which it contained was '00473.

The best experiments gave for \( \mu - 1 \) the value '00210, and this is probably correct to 1 or 2 per cent. It agrees well with Hurion's value, which was probably for a longer wave-length than 6438.

*Dispersion.*—Assuming this value the following numbers were obtained for the refractivity in the red-orange, the number of bands read being 9'7 for \( \lambda = 6438 \).

| Table IV.—Dispersion of gaseous Iodine. |
|-----------------|------------------|
| \( \lambda \times 10^8 \) | \((\mu - 1) 10^8 \frac{D}{d_76}\) |
| 6438            | 2100             |
| 6280            | 2100             |
| 6150            | 2150             |
| 6100            | 2180, 2170, 2140 |

In another set of readings the number of bands read was only 2'1 in the red, and the following readings were taken:—

\( \lambda \times 10^8 \) . . 6708, 6438, 6215, 6180, 5600, 5250, 5100, 5005, 5000.

\( (\mu - 1) 10^8 \) . . 1970, 2100, 2130, 2130, 2170, 2250, 2210, 2160, 2120.

These numbers are, of course, of little individual value, but they show at least the order of magnitude of the variation of refractivity in passing through an absorption band. The last set is interesting in showing the fall of index which seems to occur a little on the red side of the region of greatest absorption.

* The method is more fully described in a previous paper by one of us (see 'Phil. Trans.', A, vol. 204, p. 323, 1909).
REFRACTION AND DISPERSION OF THE HALOGENS, HALOGEN ACIDS, ETC. 11

It was not considered worth while to spend further time in multiplying observations which could never command great confidence, owing to the fewness of the bands read. We hope to return to this element, using the method of crossed prisms, which is more suitable than that of the interferometer.

HYDROCHLORIC ACID.

Previous Determinations.—Dulong obtained 1.000447 for white light, and Mascart 1.000444 for the D line. The dispersion has never before been attempted.

Preparation.—The gas was prepared by dropping sulphuric acid on pure sodium chloride. After passing through two drying bulbs filled with sulphuric acid it was condensed in liquid air and allowed to boil off. When the gas had flowed through the connecting tubes for 15 minutes so as to displace the air, it was admitted to the refractometer and allowed to flow till the pressure was atmospheric. The following table gives the experimental values found, reduced to 0° C. and 760 mm. by the formula

\[ \mu - 1 = (\nu - 1) \frac{T \times 760}{273 \times p} \]

<table>
<thead>
<tr>
<th>Experiment</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\mu - 1) (10^7))</td>
<td>4514</td>
<td>4513</td>
<td>4508</td>
<td>4512</td>
<td>4510</td>
<td>4509</td>
<td>4510</td>
</tr>
</tbody>
</table>

Mean 4511.

This value requires correction for the density of the gas. Gray and Burt\(^*\) found that the volume of hydrogen from two volumes of hydrochloric acid is 1.0079.

Leduc gives the mean coefficient of expansion of the gas at constant pressure as 0.003736.

The average temperature of our experiment was 16° C.

Hence the experimental value must be multiplied by

\[ \frac{1 \times (1 + 16 \times 0.003736)}{1.0079 \times (1 + 16 \times 0.00366)} \]

whence we obtain

\[ (\mu_{451} - 1) \frac{D}{(d_{76})} = 0.000448. \]

Dispersion.—Assuming this value the dispersion was determined from six experiments. The following table shows the results:

\[ \text{ dispersion } \]

\[ \text{ * 'Trans. Chemical Society,' 95, II. of 1909, p. 1654. } \]
TABLE V.—Dispersion of Hydrochloric Acid.

<table>
<thead>
<tr>
<th>( \lambda \times 10^6 )</th>
<th>((\mu - 1) \times 10^8 \frac{D}{(d_0^76)})</th>
<th>((\mu - 1) \times 10^8 \frac{D}{(d_0^76)})</th>
<th>Difference, 3 - 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed.</td>
<td>Calculated.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>6707.85</td>
<td>44375</td>
<td>44367</td>
<td>-8</td>
</tr>
<tr>
<td>6438.5</td>
<td>44444</td>
<td>44437</td>
<td>-7</td>
</tr>
<tr>
<td>5790.5</td>
<td>44656</td>
<td>44661</td>
<td>+5</td>
</tr>
<tr>
<td>5769.5</td>
<td>44666</td>
<td>44670</td>
<td>+4</td>
</tr>
<tr>
<td>5460.7</td>
<td>44800</td>
<td>44803</td>
<td>+3</td>
</tr>
<tr>
<td>5209.1</td>
<td>44930</td>
<td>44933</td>
<td>+3</td>
</tr>
<tr>
<td>5085.8</td>
<td>45007</td>
<td>44994</td>
<td>-13</td>
</tr>
<tr>
<td>4799.9</td>
<td>45187</td>
<td>45191</td>
<td>+4</td>
</tr>
</tbody>
</table>

Using the Sellmyer form of equation the refractivity is expressed by

\[
(\mu - 1) \frac{D}{(d_0^76)} = \frac{4.6425 \times 10^{27}}{10664 \times 10^{27} - n^2}
\]

The values calculated from this expression, in which the constants are calculated from the observations by the method of least squares, are shown in column 3 above, and the differences between columns 3 and 2 are given in column 4.

HYDROBROMIC ACID.

Previous Determinations.—Mascart obtained \(\mu - 1 = 0.00570\) for the D line. The dispersion has not been attempted.

Preparation.—The gas was prepared by dropping the purest aqueous solution of the acid on phosphorus pentoxide. After passing through tubes containing red phosphorus and phosphorus pentoxide, it was condensed in liquid air, sometimes twice and sometimes once only. In successful experiments the acid was obtained as a pure white solid and a colourless liquid. After an experiment the gas was absorbed over a tube containing soda lime in vacuo. Only those experiments in which the impurity was negligible were used for the determination of the index.

Procedure.—The procedure was similar to that adopted for bromine, for, strange to say, the coefficients of expansion and compressibility have not been determined. It was, therefore, necessary to supplement the readings of refractivity by measurements of density. The following values were obtained in six experiments:

Refractivity of Hydrobromic Acid. \(\lambda = 5461\).

<table>
<thead>
<tr>
<th>Experiment</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\mu - 1) \times 10^7 \frac{D}{d_0^76})</td>
<td>6167, 6158, 6151, 6141, 6139, 6141</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>6149</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Dispersion.—Assuming this value the following values of the dispersion were obtained from eight experiments:

**Table VI.—Dispersion of Hydrobromic Acid.**

<table>
<thead>
<tr>
<th>( \lambda \times 10^8 )</th>
<th>( (\mu - 1) \times 10^8 \frac{D}{(d_0 T 6)} )</th>
<th>Observed</th>
<th>Calculated</th>
<th>Difference, 3 - 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>6707·85</td>
<td>60752</td>
<td>60751</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>6438·5</td>
<td>60878</td>
<td>60873</td>
<td>-</td>
<td>5</td>
</tr>
<tr>
<td>5790·5</td>
<td>61245</td>
<td>61245</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5769·5</td>
<td>61266</td>
<td>61260</td>
<td>+</td>
<td>4</td>
</tr>
<tr>
<td>5460·7</td>
<td>61490</td>
<td>61490</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5209·1</td>
<td>61704</td>
<td>61710</td>
<td>+</td>
<td>6</td>
</tr>
<tr>
<td>5085·8</td>
<td>61824</td>
<td>61830</td>
<td>+</td>
<td>6</td>
</tr>
<tr>
<td>4799·9</td>
<td>62160</td>
<td>62149</td>
<td>-11</td>
<td></td>
</tr>
</tbody>
</table>

Using Sellmeier’s formula the refractivity can be expressed by the equation

\[
(\mu - 1) \frac{D}{(d_0 T 6)} = \frac{5'1446 \times 10^{22}}{8668'4 \times 10^{22} - n^2}.
\]

Calculated values are shown in column 3 and differences in column 4.

Density of Hydrobromic Acid.—As the density of the gas at temperatures higher than 0° C. does not appear to have been previously measured the following values are perhaps worth recording. The degree of accuracy was not carried beyond one part in a thousand, since errors in reading the refractivity were not less than this amount.

The gas was weighed at atmospheric pressure and the temperature of the room, which averaged 19° C., and the values were reduced to 0° C. and 760 by the formula

\[
D_0 = D_p \times \frac{T}{273} \times \frac{76}{P}.
\]

Three experiments gave, for the weight of a litre, on these assumptions, 3'648, 3'647, and 3'650 gr., the mean of which is 3'6484.

The theoretical weight is 3'61633.

**Hydriodic Acid.**

Previous Determinations.—Mascart found \( \mu - 1 = 0'000906 \) for the D line. The dispersion has not been attempted.

Preparation.—The gas was prepared by slowly dropping pure aqueous solution of
the acid on phosphorus pentoxide and proceeding as in the case of hydrobromic acid. The solid obtained on freezing was colourless, but the liquid was usually a pale pink, owing to a trace of dissolved iodine. As the boiling point of HI is far below that of iodine the quantity of iodine subliming, at the boiling point of HI, from this mixture was negligible.

Tests for impurity, similar to those in the case of HBr, were equally satisfactory.

Refractivity, \( \lambda = 5461 \).—In this case also measurements of refractivity had to be supplemented by those of density as this has not previously been determined carefully.

In three trustworthy experiments the following figures were obtained for the refractivity at the green mercury line:

\[
(\mu - 1) \times 10^8 \frac{D}{d_{076}} \cdot \cdot \cdot 9237, 9277, 9260. \quad \text{Mean 9258.}
\]

The mean is taken as the best value.

Dispersion.—From seven experiments the following values were obtained for the dispersion:

<table>
<thead>
<tr>
<th>Table VII.—Dispersion of Hydriodic Acid.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda \times 10^6 )</td>
</tr>
<tr>
<td>--------------------------</td>
</tr>
<tr>
<td>6707.5</td>
</tr>
<tr>
<td>6438.5</td>
</tr>
<tr>
<td>5790.5</td>
</tr>
<tr>
<td>5769.5</td>
</tr>
<tr>
<td>5460.7</td>
</tr>
<tr>
<td>5209.1</td>
</tr>
<tr>
<td>5089.8</td>
</tr>
<tr>
<td>4799.9</td>
</tr>
</tbody>
</table>

Using Sellmeyer's equation the refractivity can be expressed by the formula

\[
(\mu - 1) \frac{D}{d_{076}} = \frac{57900 \times 10^{27}}{6556.4 \times 10^{27} - \eta^2}.
\]

The calculated values are shown in column 3 and the differences in column 4.

Density of Hydriodic Acid.—The density of this gas also has not been accurately measured. Calculated in the same way as in the case of hydrobromic acid three experiments gave for the weight of a litre 5789, 5791, 5793, mean = 5791 gr. The theoretical weight is 57151, taking \( H = 1.008 \) and \( I = 126.97 \), and the weight of a litre of oxygen as 1.4290 gr.
Previous determinations on the refractivity of water vapour are given by Dufet as follows:—

<table>
<thead>
<tr>
<th>Observer</th>
<th>Light</th>
<th>$(\mu - 1) \times 10^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fizeau</td>
<td>D</td>
<td>254</td>
</tr>
<tr>
<td>Jamin</td>
<td>D</td>
<td>257.9</td>
</tr>
<tr>
<td>Mascart</td>
<td>D</td>
<td>257</td>
</tr>
<tr>
<td>Lorenz</td>
<td>D</td>
<td>250</td>
</tr>
</tbody>
</table>

No one appears to have attempted the dispersion.

**Procedure.**—A weighed quantity of distilled water, sealed up in a thin capillary tube, was introduced into the refractometer tube, which was then evacuated and sealed off. On breaking the capillary by a jerk the tube was filled with vapour. After adjusting the tubes between the mirrors of the interferometer the centre of the tube containing the water was first cooled to a known temperature and then the tube was heated till the whole of the water present had evaporated. To the number of the bands read was added a proportionate number for the vapour present at the initial temperature.

In order to eliminate the errors of drift other experiments were made in which the ends of the tube were kept near the maximum temperature required (about $140^\circ$ C.) and the centre of the tube gradually cooled to the temperature of ice.

**Refractivity.**—Experiments were made with four charges of water. The results are given below:—

**Refractivity of Steam.**

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$(\mu - 1) \times 10^7 \frac{D}{d_0^2 T}$</th>
<th>Approximate number of bands read.</th>
<th>Remarks.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>2523</td>
<td>178</td>
<td>Mean of 3 experiments,</td>
</tr>
<tr>
<td>(2)</td>
<td>2491</td>
<td>379</td>
<td>&quot;</td>
</tr>
<tr>
<td>(3)</td>
<td>2534</td>
<td>130</td>
<td>&quot;</td>
</tr>
<tr>
<td>(4)</td>
<td>2524</td>
<td>300</td>
<td>&quot;</td>
</tr>
<tr>
<td>Mean of 1, 3, and 4</td>
<td>2527</td>
<td></td>
<td>&quot;</td>
</tr>
</tbody>
</table>

It will be seen that the second charge yielded results considerably lower than the other three. The cause of the discrepancy was found to be the unequal distribution of vapour between the main portion of the refractometer tube and the small "appendix" left when the side tube is sealed off. When the temperature of the ends is markedly higher than that of the middle (as it was in this series), the error becomes considerable. Neglecting this experiment we take the mean of the other
three as the value for the green mercury line. The variations of these experiments indicate that this result is probably correct to 1 part in 500.

*Dispersion.*—Assuming this value the following table shows the values obtained from seven experiments with the largest charge of water, the number of bands read being about 380 for \( \lambda = 5461 \):

**Table VIII.—Dispersion of Steam.**

<table>
<thead>
<tr>
<th>( \lambda \times 10^6 )</th>
<th>((\mu - 1) \times 10^8 \frac{D}{d_{076}})</th>
<th>Difference, 3 - 2.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed</td>
<td>Calculated</td>
<td></td>
</tr>
<tr>
<td>6707.85</td>
<td>25028</td>
<td>-1</td>
</tr>
<tr>
<td>6438.5</td>
<td>25069</td>
<td>-1</td>
</tr>
<tr>
<td>5790.5</td>
<td>25191</td>
<td>0</td>
</tr>
<tr>
<td>5769.5</td>
<td>25195</td>
<td>+1</td>
</tr>
<tr>
<td>5460.7</td>
<td>25270</td>
<td>+2</td>
</tr>
<tr>
<td>5509.1</td>
<td>25345</td>
<td>0</td>
</tr>
<tr>
<td>5085.8</td>
<td>25380</td>
<td>+4</td>
</tr>
<tr>
<td>4799.9</td>
<td>25495</td>
<td>-5</td>
</tr>
</tbody>
</table>

Using the Sellmeyer equation the refractivity can be expressed by

\[
(\mu - 1) \frac{D}{d_{076}} = \frac{2.62707 \times 10^{27}}{10697 \times 10^{27} - n^2}.
\]

The numbers calculated from this expression are shown in the third column and the differences between column 3 and 2 are given in column 4.

**Ozone.**

*Previous Work.*—No previous work on the refractivity of ozone is recorded in the usual books of reference. The difficulties are considerable. It is impossible to prepare ozone even approximately pure, and if it were possible it would be inadvisable to do so, since the decomposition of the molecules during the time necessary to measure the refraction and dispersion would introduce fruitful sources of error.

*Procedure.*—Of the two best methods of preparing the gas, electrolysis of a solution of sulphuric acid has produced the highest percentages of ozone, Fischer and Massenez* having obtained over 28 per cent. by weight. But the objections to this method seemed to us to outweigh its advantages. It was necessary that the gas used should be absolutely pure oxygen, for the smallest trace of moisture, air or hydrogen would introduce large errors; and in the electrolytic process the gas is produced wet and is mixed with air in the connections. For these reasons the method selected was that of ozonising by means of the silent discharge in a vessel of the type

used by Berthelot. The average yield was 6 per cent. by volume, but on one or two occasions it reached 10 per cent. We failed to identify the causes which produced these higher yields, and were unable to repeat them, but succeeded in obtaining between 5 and 7 per cent. with fair regularity.

Methods.—As in the case of other gases, the work was divided into two parts (1) the determination of the refractivity for a single wave-length (the green mercury line), and (2) the measurement of the dispersion in the visible spectrum relatively to this value.

For the measurement of the refractivity two methods were employed. In the first of these the quantity of ozone present was estimated by destroying the ozone by heat, and measuring the increase of the gas in volume. In the second, the ozone was estimated chemically by bubbling the mixture of gases through a solution of potassium iodide, and titrating with thiosulphate of soda.

As the results of the enquiry were remarkable the following details may be of interest:—

Dry oxygen, prepared by heating permanganate of potash, and stored in a gas holder over mercury, was led through an ozoniser into the interferometer tube, which was previously evacuated. The interference bands which crossed the field were counted till atmospheric pressure was reached. The pressure was then read by connecting the apparatus with a mercury manometer filled with oxygen and separated from the ozonised gas by a long capillary tube. The temperature of the water bath was observed and the tap which led to the refractometer tube turned off. Having again evacuated the connections the gas in the refractometer tube was allowed to flow slowly into the pump, passing through a spiral of fused silica heated to redness, which effectually destroyed the ozone. From the pump it was transferred to another gas holder over mercury and thence again allowed to flow into the refractometer tube, where its temperature and pressure were again measured. If \( V_1 \) is the volume of the ozonised, and \( V_2 \) that of the deozonised oxygen, the percentage of ozone is given by \( V_2 - V_1 = xV_1/200 \). In the present case \( V_1 \) was about 150 c.c., so that if \( x = 8 \) the total increase of volume is 6 c.c. In order to determine the value of the refractivity to 1 per cent. it is therefore necessary that the total error in pumping the gas round the cycle should not exceed '06 c.c. In practice this accuracy was not quite attained. It was necessary to grease stopcocks with a mixture of pure paraffin and vaseline, which will not hold a vacuum indefinitely; while, in order to destroy the ozone, the gas had to be pumped through a spiral 12 inches long of fine capillary bore, which made it difficult to evacuate the last traces from the connections. It was also necessary to know the refractivity of the oxygen very accurately, since an error in this figure is multiplied in the ratio of 100/x. After a sufficient number of trials had been made to prove that our oxygen was approximately pure, its refractivity was assumed to be that previously determined by us, * viz., \( \mu - 1 = 0002717 \). \( \lambda 5461 \).


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The following are the details of a typical experiment by this method, in which the refractivity of the deozonised gas was separately determined:—

Part I. Refractivity of the ozonised oxygen—

Bands ($\lambda = 5460\text{~}7$) 489'9. Length of tube 99'786 cm. Pressure difference 742'2 mm. (corrected). Temperature 16°'25 C.

$$\mu - 1 = \frac{489'9 \times 5460'7 \times 289'25 \times 760 \times 10^{-8}}{99'786 \times 273 \times 742'2} = .0002909.$$

Part II. Refractivity of the deozonised oxygen—

Bands 474'2. Pressure difference 764'33 mm. Temperature 14°'7 C.; whence

$$\mu - 1 = .00027193.$$

Part III. Percentage of ozone—

$$\frac{V_2}{V_1} = \frac{764'33 \times 289'25}{742'2 \times 287'7} = 1'0354.$$

Thus percentage of ozone = 3'54 x 2 = 7'08.

Part IV. Refractivity of pure ozone—

The refractivity of the mixture is the sum of the refractivities of its components. Let $\mu_0 - 1$ denote that of pure ozone, then

$$7'08 \times (\mu_0 - 1) + 92'92 \times .00027193 = 100 \times .0002909,$$

whence $\mu_0 - 1 = .000539.$

By this method the following results were obtained:—

**Table IX.**—Refractivity of pure Ozone. (First Method.)

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Percentage of ozone by volume</th>
<th>Refractivity, $(\mu - 1)10^6 \frac{1}{(d_0 \text{~}76)}$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9'5</td>
<td>508</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>7'68</td>
<td>543</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7'08</td>
<td>539</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>6'24</td>
<td>511</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6'24</td>
<td>545</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6'5</td>
<td>560</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>3'5</td>
<td>[585]</td>
<td>Not reliable, percentage of ozone too small.</td>
</tr>
<tr>
<td>8</td>
<td>8'72</td>
<td>502</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>7'14</td>
<td>497</td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td></td>
<td>525</td>
<td></td>
</tr>
</tbody>
</table>
Second method. To check these results a second set of experiments was made, in which the quantity of ozone was estimated by chemical tests. This method was found to give more concordant figures.

**Table X.—Refractivity of pure Ozone. (Second Method.)**

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Percentage of ozone by volume</th>
<th>Refractivity, ( \frac{(\mu - 1) \times 10^6}{d_0/76} )</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.09</td>
<td>515</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5.47</td>
<td>521</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4.86</td>
<td>[495]</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4.40</td>
<td>522</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6.36</td>
<td>530</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>7.08</td>
<td>516</td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td></td>
<td>516.5</td>
<td></td>
</tr>
</tbody>
</table>

The results obtained by the two methods are tolerably concordant, and would be even better if the third experiment were omitted.

Their mean is 520.7, but having regard to the smallness of the proportion of ozone present it would be unsafe to rely on this number beyond the second significant figure, and we therefore conclude that the refractive index of pure ozone for the mercury green line is

\[ \mu = 1.00052. \]

**Comparison with the Refractive Index of Oxygen.**—It will be observed that this result is remarkable.

The refractivity of oxygen is 0.002717, and if the third atom of oxygen on joining the molecule had the same refractive effect as in the normal gas we should expect a refractivity \( (\mu - 1) \times 10^6 \) of \( \frac{3}{3} \times 2717 = 407.5 \).

The experimental value 520 is very largely in excess of this, and indicates the existence of interesting peculiarities in the molecule which may probably be ascribed to the linkage.

**Dispersion of Ozone.**—Nine experiments were made on the dispersion of mixtures of ozone and oxygen. In each of these the refractive index of the mixture for the green mercury line was separately determined, and the other seven refractivities were calculated with reference to it from the observations as previously described.

The first experiment, being a trial, is omitted, and the refractivity of ozone calculated from the remaining eight as follows:—
Table XI.—Dispersion of Mixtures of Ozone and Oxygen.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>( (\mu - 1) \times 10^8 \frac{D}{d_0^{76}} )</th>
<th>Bands read.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6708</td>
<td>6438</td>
</tr>
<tr>
<td>1</td>
<td>28364</td>
<td>—</td>
</tr>
<tr>
<td>2</td>
<td>28143</td>
<td>28194</td>
</tr>
<tr>
<td>3</td>
<td>28509</td>
<td>28554</td>
</tr>
<tr>
<td>4</td>
<td>28500</td>
<td>28555</td>
</tr>
<tr>
<td>5</td>
<td>28499</td>
<td>28543</td>
</tr>
<tr>
<td>6</td>
<td>28374</td>
<td>28416</td>
</tr>
<tr>
<td>7</td>
<td>28365</td>
<td>28410</td>
</tr>
<tr>
<td>8</td>
<td>28368</td>
<td>28412</td>
</tr>
<tr>
<td>9</td>
<td>28624</td>
<td>28667</td>
</tr>
</tbody>
</table>

Mean. . . . . | 28423 | 28469 | 28535 | 28616 | 28703 | 28790 | 28838 | 28967 | 28423 |

Refractivities of \( \text{O}_2 \). . | 26952 | 26988 | 27098 | 27102 | 27170 | 27237 | 27272 | 27366 |

Refractivities of \( \text{O}_3 \). . | 50764 | 50968 | 51514 | 51624 | 52000 | 52375 | 52621 | 53290 |

Adding together all the values of the refractivities for each wave-length separately, and dividing by the number of experiments, we obtain the refractivities for the average mixture of ozone and oxygen, which are given as “means.” Assuming \( [\mu_{5461} - 1]_n = 0.00520 \) the percentage of ozone in this mixture is found as follows:—

\[
520x + (100-x)271.7 = 100 \times 287.03, \quad \text{whence} \quad x = 6.1764.
\]

To find the refractivities for the other seven wave-lengths we have only to use this value and the refractivity for the corresponding wave-length of oxygen which we take from our previous determinations, vide loc. cit. supra p. 2. Thus

\[
[\mu_\lambda - 1]_n \times 6.1764 + 93.8236 \times [\mu_\lambda - 1]_n = [\mu_\lambda - 1]_{\text{mixture}}.
\]

The numbers obtained in this way are shown in the next line.

It is at once noticeable that the dispersive power of ozone is much greater than that of oxygen. And here again, as in the case of chlorine, we find that the curvature of the experimental curve is greater than that calculated. Using \( \mu_{5700} - 1 \) and \( \mu_{4880} - 1 \) we obtain the formula \( \mu - 1 = \frac{2.0414 \times 10^{-7}}{4221.3 \times 10^{-7} - \nu^2} \), whence we find \( \mu_{5461} - 1 = 52082 \), whereas the experimental value is 52000.

As in the case of chlorine, the inference is that a second term is required.
Ammonia.

Previous Determinations.—Previous determinations of the refractivity of ammonia are as follows:—

<table>
<thead>
<tr>
<th>Observer</th>
<th>Light</th>
<th>$(\mu - 1) \times 10^4$</th>
<th>Corrected for density</th>
</tr>
</thead>
<tbody>
<tr>
<td>Biot and Arago</td>
<td>White</td>
<td>381</td>
<td></td>
</tr>
<tr>
<td>Dulong</td>
<td>D</td>
<td>377</td>
<td>376.1</td>
</tr>
<tr>
<td>Mascart</td>
<td>Li</td>
<td>373</td>
<td>373</td>
</tr>
<tr>
<td>Lorenz</td>
<td>D</td>
<td>373</td>
<td>374.3</td>
</tr>
<tr>
<td>W. Walker</td>
<td></td>
<td>379.3 ± 5</td>
<td></td>
</tr>
</tbody>
</table>

It would occupy too much space to analyse the causes of these discrepancies, which are chiefly due to differences in the standard conditions assumed and in the coefficients of thermal expansion and compressibility adopted. But the figures in the last column give approximately the figures corrected for the theoretical density.

Preparation.—Our gas was prepared by warming a mixture of ammonium chloride and calcium oxide in a flask. After passing over red hot lime and cold dry lime it was condensed at $-80^\circ$ C. and allowed to boil off, the middle fraction being collected. Three samples were used.

Calculation of Results.—In reducing the results the figures given below were used, following Guye*:

Coefficient of thermal expansion $(1 + 0.003914t)$.
Coefficient of compressibility

$$1 - \frac{p_1}{p_0} = A (p_1 - p_0), \quad A = 0.002 (1 - 0.000003t).$$

Weight of a litre of ammonia at $0^\circ$ C. and 760 mm., 7708 gr. Theoretical density, 7605 gr.

Thus the equation for reduction is

$$(\mu - 1) \frac{D}{(d_0^4 76)} = \frac{N\lambda 7605}{L 7708} \times 76 \left/ \frac{p_1 [1 - A (76 - p_0)]}{1 + 0.003914t_1} - \frac{p_2 [1 - A (76 - p_2)]}{1 + 0.003914t_2} \right.$$

where $N$ is the number of bands observed, $\lambda$ the wave-length, $L$ the length of the tube, and $p_1, p_2, t_1, t_2$ the initial and final pressures and temperatures.

Refraction.—The determinations for $\lambda$ 5461 were, as usual, made by pairs of experiments, with pressure rising and falling.

The mean of nine such experiments, whose extremes were 1'0003782 and 1'0003790,

was 1'0003786. Seven of these were at room temperature and two at 0° C. We adopt 0003786 as the refractivity for the green mercury line.

*Dispersion.*—Five experiments were made to determine the dispersion.

The following table gives the mean results and compares the observed values with those calculated from the formula

$$\mu - 1 = \frac{D}{d_76} = \frac{2.9658 \times 10^{27}}{8135.3 \times 10^{27} - n^2},$$

which was, as usual, calculated from the observations by the method of least squares:

**Table XII.** Dispersion of Ammonia.

<table>
<thead>
<tr>
<th>$\lambda \times 10^6$</th>
<th>$(\mu - 1) \times 10^8 \frac{D}{(d_76)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6707.85</td>
<td>37376</td>
</tr>
<tr>
<td>6435.5</td>
<td>37455</td>
</tr>
<tr>
<td>5790.5</td>
<td>37701</td>
</tr>
<tr>
<td>5769.5</td>
<td>37707</td>
</tr>
<tr>
<td>5460.7</td>
<td>37860</td>
</tr>
<tr>
<td>5209.1</td>
<td>38002</td>
</tr>
<tr>
<td>5085.8</td>
<td>38083</td>
</tr>
<tr>
<td>4799.9</td>
<td>38300</td>
</tr>
</tbody>
</table>

**Nitric Oxide.** (NO.)

*Previous Work.*—Dufet gives the following:

<table>
<thead>
<tr>
<th>Light.</th>
<th>$(\mu - 1) \times 10^6$</th>
<th>Observer.</th>
</tr>
</thead>
<tbody>
<tr>
<td>White</td>
<td>302</td>
<td>Dulon.</td>
</tr>
<tr>
<td>D</td>
<td>297.1</td>
<td>Mascart.</td>
</tr>
</tbody>
</table>

Mr. E. P. Metcalfe, in collaboration with one of us,* obtained 293'9 for $\lambda = 5893$. The gas used by Mascart had 10 per cent. of impurity.

*Preparation.*—Following the third method described by Guye† we prepared the gas by the action of of dilute sulphuric acid (10 per cent.) on dilute nitrite of soda (6 per cent.) in a vacuum. After bubbling through concentrated sulphuric acid and passing over P₂O₅, it was condensed in liquid air and fractionally distilled. The gas employed, tested with ferrous sulphate, showed less than 1 part in a 1000 of impurity.

---

probably nitrogen. As the refractivity of nitrogen is almost identical with that of nitric oxide the results were not modified by the impurity. The observations were reduced by the ordinary formula \( \mu - 1 = (\nu - 1) \frac{T}{273} \times \frac{76}{P} \).

**Refraction.**—Six careful double experiments (i.e., pressure rising and falling) gave Experiment . . . 1, 2, 3, 4, 5, 6. 

\[
(\mu - 1) 10^7 \frac{D}{(d_{76})^2} = 2959, 2957, 2952, 2955, 2951, 2956. \text{ Mean } 2955.
\]

We adopt this mean '0002955 as the value for the green mercury line.

Calculating the value for the D line from this value and the dispersion formula obtained below we find '0002944, which agrees well with 2939 found in 1908.

**Dispersion.**—From five observations the following values for the dispersion were obtained :

<table>
<thead>
<tr>
<th>Table XIII.—Dispersion of Nitric Oxide.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda \times 10^8 )</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>6707.85</td>
</tr>
<tr>
<td>6438.5</td>
</tr>
<tr>
<td>5790.5</td>
</tr>
<tr>
<td>5769.5</td>
</tr>
<tr>
<td>5460.7</td>
</tr>
<tr>
<td>5209.1</td>
</tr>
<tr>
<td>5085.8</td>
</tr>
<tr>
<td>4799.9</td>
</tr>
</tbody>
</table>

Using Sellmeyer's formula the results are expressed by

\[
(\mu - 1) \frac{D}{(d_{76})^2} = 3.5210 \times 10^{27} \frac{12216 \times 10^{27} - n^2}{n^2}.
\]

**Nitrous Oxide.** (\( \text{N}_2\text{O} \))

**Previous Work.**—Dufet gives

<table>
<thead>
<tr>
<th>Light.</th>
<th>( \mu )</th>
<th>Observer.</th>
</tr>
</thead>
<tbody>
<tr>
<td>White</td>
<td>1.000507</td>
<td>Dulong.</td>
</tr>
<tr>
<td>Red</td>
<td>1.000507</td>
<td>Jamin.</td>
</tr>
<tr>
<td>6439.2</td>
<td>1.0005132</td>
<td>Mascart.</td>
</tr>
<tr>
<td>5378.9</td>
<td>1.0005152</td>
<td></td>
</tr>
<tr>
<td>5086.1</td>
<td>1.0005207</td>
<td></td>
</tr>
<tr>
<td>4800.2</td>
<td>1.0005230</td>
<td></td>
</tr>
</tbody>
</table>
Mascart's gas was prepared from ammonium nitrate and contained 10 per cent. of impurity.

The gas we used was obtained from two sources: (1) The commercial gas, obtained in cylinders, condensed and fractionated at the temperature of liquid air, and (2) gas prepared by the action of ammonium nitrite on hydroxylamine hydrosulphate. It was bubbled through strong potash and dried with sulphuric acid and phosphorus pentoxide.

Refractive Index.—Three sets of experiments on different samples gave

<table>
<thead>
<tr>
<th>Series</th>
<th>((\mu - 1) \times 10^7 \frac{D}{(d_0 76)})</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5092, 5102, 5097, 5098, 5099</td>
<td>Commercial</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>&quot;</td>
</tr>
<tr>
<td>3</td>
<td>5087, 5091</td>
<td>From hydroxylamine.</td>
</tr>
<tr>
<td>Mean</td>
<td>5096</td>
<td></td>
</tr>
</tbody>
</table>

In reducing these experiments the coefficient of thermal expansion used was 0.00371.

The purity of the gas was tested by absorption in an excess of water boiled in vacuo. The bubble of gas left unabsorbed was not so great as 1/2000 of the whole; and even this was probably due to the error of the test experiment, which is not very easy. But as traces of air were probably present we think 5100 a more trustworthy value than the exact experimental mean, and probably correct to 1/500 at least.

Dispersion.—From five experiments the following values were obtained for the dispersion:

<table>
<thead>
<tr>
<th>(\lambda \times 10^8)</th>
<th>((\mu - 1) \times 10^8 \frac{D}{(d_0 76)})</th>
<th>Observed</th>
<th>Calculated</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>6707·85</td>
<td>50544</td>
<td>50540</td>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>6438·5</td>
<td>50616</td>
<td>50616</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5790·5</td>
<td>50848</td>
<td>50848</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5769·5</td>
<td>50857</td>
<td>50857</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5460·7</td>
<td>51000</td>
<td>51003</td>
<td>+3</td>
<td></td>
</tr>
<tr>
<td>5209·1</td>
<td>51145</td>
<td>51142</td>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>5086·8</td>
<td>51208</td>
<td>51215</td>
<td>+7</td>
<td></td>
</tr>
<tr>
<td>4799·9</td>
<td>51415</td>
<td>51420</td>
<td>+5</td>
<td></td>
</tr>
</tbody>
</table>

Table XIV.
The refractivity can be expressed by the formula

\[ (\mu - 1) \frac{D}{(n_0^2 + 6)} = 5'6685 \times 10^{12} \frac{1}{11416 \times 10^{-8} - n^3}. \]

The calculated values are shown in column 3 and the differences in column 4.

**Errors of Experiment.**

*Refraction.*—In the determination of the refractivities for the green, mercury line the principal source of error is the impurity of the gas, and, in the case of vapours which absorb light, such as the halogens and sulphur, the limitation of the number of bands which can be read before the light fails.

It will be seen that experiments of a series generally agreed to 1 part in 500, and the mean is probably within 1 in a 1000 of the truth. In iodine and ozone, however, the errors may amount to 1 or 2 per cent.

*Dispersion.*—It will be observed that the values of the refractivities for the other seven wave-lengths are relative to that found for the green mercury line.

The degree of accuracy attainable depends on the number of bands read and the dispersive power of the gas. As an example take nitric oxide.

Here \( [\mu - 1]_{\lambda = 6706} = '00029306, \quad [\mu - 1]_{\lambda = 4800} = '00029776. \)

The dispersive power is \( \frac{29776 - 29396}{29776} = \frac{470}{29776} ; \) and if 400 green bands are counted, the number which represents the effect of dispersion is \( \frac{188000}{29776} = 6'3 \) bands. We consider that 1/15 of a band can be read; so that the value of the dispersive power should be correct to 1 part in 95. It may be assumed that by determining the constants from eight independent values of the refraction instead of two the accuracy is at least doubled, and the error should not exceed 1/200 of the effect itself.

This claim is supported by the experimental results. Thus, in the six experiments from which the dispersion of HCl was determined, the values of \( (\mu_{6706} - \mu_{4800}) \times 10^8 \) were 816, 815, 813, 818, 805.

In eight experiments on HBr they were 1370, 1388, 1393, 1356, 1372, 1376, 1368, 1363.

**Calculation of the Constants.**

The calculation of the constants \( N \) and \( n_0^2 \) of the formula \( \mu - 1 = N/(n_0^2 - n^2) \) by the method of least squares is very laborious if carried out in the ordinary manner. The following modification was, therefore, adopted. Using subscripts to denote the eight refractive indices and their frequencies we have eight equations of the form

\[ \frac{1}{\mu_i - 1} = \frac{n_0^2 - n_i^2}{N}. \]
REFRACTION AND DISPERSION OF THE HALOGENS, HALOGEN ACIDS, ETC.

Subtracting the \((n+4)^{th}\) from the \(n^{th}\) equation, we obtain four equations similar to

\[
\frac{1}{\mu_5-1} - \frac{1}{\mu_1-1} = \frac{1}{N} \left( n_5^2 - n_1^2 \right).
\]

Let \(\frac{1}{\mu_5-1} - \frac{1}{\mu_1-1}\) be expressed by \(x\), and \((n_5^2 - n_1^2)\) by \(y\), and similarly for the other three equations. Then it can be shown that, applying the method of least squares,

\[
N = \Sigma (x, y)/\Sigma (x^2), \quad \text{and hence} \quad n_0^2 = \frac{1}{4} N \left\{ \Sigma \left( \frac{1}{\mu - 1} \right) \right\} + \Sigma (n^2).
\]

We have much pleasure in recording our deep obligations to many friends. To Prof. Trouton and Prof. A. W. Porter we owe most grateful thanks for their unwearied patience in assisting, guiding, and encouraging us. To Prof. N. Wilsmore and Dr. Whytlaw-Gray we are indebted for instruction and invaluable help in the whole of the chemical side of the work. To the Royal Society we owe our grateful acknowledgment for the assistance of pecuniary grants.
II. On a Cassegrain Reflector with Corrected Field.

By Dr. R. A. Sampson, F.R.S.

Received December 28, 1912,—Read February 13, 1913.

The great advantage enjoyed by the reflecting telescope is its equal treatment of rays of all colours, and the geometrical defects or aberrations of its field are less than those of many of the older refractors. The most serious of these defects is coma, owing to which different zones of the objective do not place the light which they receive from the same object-point symmetrically around any common centre in the image area, but arrange it in a radial fan or flare, the light from the outer zones being most diffused; besides spoiling the image this tends to neutralize, for any except narrow fields, the value of extended aperture in the objective as a light-collector. In the refractor this can be and is now always met by adjusting the curves of the two lenses, for when achromatism, as far as possible, and spherical aberration are allowed for, there still remains one unused datum; in old forms this was often used to make the inner curves contact curves that might be cemented together if it was convenient to do so, but it is properly employed to extinguish coma. But with the reflector the case is different. In the Newtonian form there is only one available surface, and when this is made a paraboloid to cure spherical aberration, nothing is left to adjust. In the Gregorian or Cassegrain forms there are two curved surfaces and, theoretically, these would offer means to correct two faults. An illuminating study of the possibilities of a system of two mirrors has been made by Schwarzschild in his 'Untersuchungen zur Geometrischen Optik';* I shall deal with its outcome below. Its general tenor is comprehensive and exploratory rather than detailed, and it remains doubtful whether any of the forms which he indicates for the reflector, at the point at which his research stops, could actually be made successfully upon a scale that would show their advantages. My own purpose in the present paper is essentially a practical one. I have in mind throughout a telescope of large aperture and considerable focal length, and seek to devise a correction for the faults of its field which shall leave its achromatism unimpaired, which can really be made and which shall effect its purpose without employing any curves and angles outside those that are already known to work well. It has been


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said that "an object-glass cannot be made on paper," but the possibilities of new and somewhat complicated constructions must in all cases first be demonstrated on paper, since practice can never conveniently vary more than a single factor at a time. Study is directed to the Cassegrain because of the great advantage which this design possesses in shortening the tube of the instrument for given focal length, and in placing the observer at the lower, in place of at the upper, end of it.

The best introduction to the subsequent work will be in the form of a few remarks upon Schwarzscld's results. These are not meant as a complete criticism or estimation of it but are merely such as arise naturally in relation to the points with which I deal afterwards. The traditional form of Cassegrain telescope consists of a great concave mirror faced by a small convex one, which is placed between the great mirror and its principal focus, and throws the image out through a hole cut centrally in the great mirror. The small mirror increases the effective focal length in the ratio of its distances respectively from the final principal focus and from the principal focus of the great mirror. This ratio for example is 5·4 in the great Melbourne telescope, 3½ to 4½ in the Mount Wilson 60-inch when used as a Cassegrain, and it can hardly fall much below 2½ unless the small mirror is to cut off a disproportionate amount of the area of the great mirror. The Cassegrain is, therefore, generally speaking, a long focus instrument. From all these features Schwarzscld's forms differ widely, except that they place the small mirror between the great mirror and its principal focus. His small mirror is concave in place of convex, and shortens the effective focal length, bringing the beam to a focus between itself and the great mirror. The effect of this change in design is to render possible a flat field. Spherical aberration and coma are removed from the image by modifying the spherical figures of the two mirrors into definite hyperboloidal and ellipsoidal forms. To confine reference to the case which he considers generally the best (loc. cit., II., §11), the necessary deformations are given respectively by \(b_1 = -13·5\), \(b_2 = +1·97\), where \(b = -1\) would deform a sphere into a paraboloid. The image-surface for this case would be very nearly flat, and the images of points would be very nearly circles, which reached a diameter of 8 seconds at an angular distance of about 1 degree from the centre of the field. This may seem somewhat large but it is a quantity proportional to the aperture-ratio, which in this case is large also, namely 1 : 3·5. The result is in brief a very rapid instrument of short focus and of field about comparable to that of a good long-focus refractor. The chief objection to it is found in the curves that it requires. Until some one turns such curves out, it must remain problematic whether it is feasible at all to make the construction a practical success.

A feature of Schwarzscld's analysis is the use of a concave small mirror. This is not necessary to destroy coma, which may equally be removed in the Cassegrain form by deformations of the mirrors, and those indeed of less pronounced degree than Schwarzscld finds necessary. But as will be shown below there then remains a somewhat severe and irremovable curvature of the field.
The general conclusion which I draw from Schwarzschild's investigation is that modification of the two mirrors is in itself not enough to give a practical solution of the problem. We have to deal with spherical aberration, coma, curvature of the field, and astigmatism. Distortion may be set aside, because in itself it does not vitiate the image of a point, and errors which it introduces into relative distances may be computed and allowed for. We have at our disposal the figures of the two mirrors and their separation and curvatures. The last are so locked up with the kind of telescope which we wish to produce that they are hardly available for adjustment—if we want a short-focus instrument we have to take Schwarzschild's choice, and for a long-focus one the Cassegrain form. It turns out that the former of these may have a flat field and the latter must have a curved field and we have to rest content with that. And with respect to the figures of the mirrors it is not within our control to say whether they shall offer themselves in our equations in a favourable form for removing undesired terms; it appears from the research that they appear somewhat unfavourably entailing the use of surfaces decidedly far from the sphere. It is my object to obtain a workable solution and not merely a theoretical one, and therefore I have recourse to a more complicated system, by passing the beam through a definite set of lenses, the curvatures of which are more or less completely at our disposal. It might, at first sight, appear that this would impair the achromatism of the reflector, but if a system of not less than three separated lenses be made of the same glass, the two conditions for achromatism at a given plane may be completely satisfied, equally for all colours. With such a system we can produce deviation in a beam, but more emphatically we can produce aberrations. The details at which I arrive are given on p. 66, and need not be repeated here, but generally the plan is to replace the convex mirror by a weak convexo-concave lens silvered at the back, and about two-thirds of the way between this and the surface of the great mirror to place a system which I call the Corrector, being a pair of lenses of nearly equal but opposite focal lengths, of which the first is double concave with the lesser curvature first, and the latter nearly plano-convex.

Choosing the curvatures properly a telescope is thus produced which gives, strictly in the focal plane, an image free from chromatic faults, except for minute chromatic residues of aberration, from spherical aberration and from coma, and in which points of the object are represented in the image by spots strictly circular which reach a diameter of 2.2 seconds at a distance of 1 degree from the centre of the field. The greatest angle of incidence upon any of the surfaces is 11 degrees, or not more than about two-thirds of what is customary upon the anterior surface of the flint lens of the object glass of a refractor; all the surfaces are spherical except that of the great mirror which is intermediate between the sphere and paraboloid, and I cannot see that anywhere any serious constructional difficulty is introduced. The effective aperture-ratio is 1 : 14.05, or, say, about 1 : 15, allowing that 12 per cent. more light will be lost in this construction than in other possible ones.
The methods which I employ are those of a memoir recently published.* SCHWARZSCHILD used the Characteristic Function. Our methods thus differ, but since aberrations of the third or any other order are the same things, no matter how they are obtained, where we occasionally touch the same matter the differences are at most those of notation, and occasionally these are slight ones. I have not attempted to remove them because it seems to me that an investigation is easiest to read if expressed in notation that grows naturally out of its own processes. I shall therefore adhere strictly to the notation of my Memoir, amplifying its results as occasion requires.

We may take for reference the following specifications of the faults of an optical field at its principal focus in terms of the coefficients \( \delta_1 G \), &c.:

\[
\begin{align*}
\alpha &= \text{semi-aperture.} \\
\beta &= \text{tangent of inclination of original ray to axis.} \\
&f' &= \text{effective focal length.} \\
&f' &= +\frac{3}{8} f' \alpha \beta \delta_1 G.
\end{align*}
\]

Position of least circle of spherical aberration \( \delta'' f = \frac{25783''}{f} \times \alpha^2 \delta_1 G. \)

Angular radius of this circle \( \delta f' = \frac{103133''}{f} \times \alpha \beta \delta_2 G. \)

Comatic radius \( \delta f' = \frac{103133''}{f} \times \alpha \beta \delta_2 G. \)

Secondary focal line after principal focus \( \delta f' = \frac{1}{2} f' \beta \delta_3 G. \)

Primary focal line after secondary \( \delta f' = f' \beta \delta_3 H. \)

Radius of focal circle \( \delta f' = \frac{103133''}{f} \times \beta \delta_3 H. \)

Curvature of field (convex to ray if positive) \( \delta f' = \frac{1}{f''} (\delta_1 G + \delta_3 H). \)

Distortional displacement \( \delta f' = \frac{103133''}{f''} (\delta_1 G + \delta_3 H). \)

With respect to these it may be explained that the Comatic Radius is the radius of the circle around which rays from a zone of radius \( \alpha \) are distributed, the centre of the comatic circle being displaced from the normal image-point by an amount equal to its diameter; the “secondary” focal line is the line in the plane of the axis; the word “after” means after, in the order in which light reaches the points; the focal circle is the circle half way between the two focal lines, through which, in the absence of coma, all rays of the zone would pass; the curvature of the field refers to the field containing the focal circles of all object-points.

Now, if we secure a field for which

\[
\begin{align*}
\delta_1 G &= 0, \quad \delta_2 G &= 0, \quad \delta_3 G + \delta_3 H &= 0. 
\end{align*}
\]

it will be free from spherical aberration and from coma, and the images of points will be circles in the plane through the principal focus, the radii of which are given by \(103133'' \times (\alpha/f') \times \beta^2 \delta H\). If \(\delta G\), which by (2) is made equal to \(-\delta G\), is not zero, the instrument will be successful for such values of the angular radius of the field as keep this down below desired limits. These conditions give the objects which I aim at attaining. Given the general design of the instrument as regards apertures and focal lengths, it will be found that the lens which is used as a mirror, or the Reverser as I shall call it, is completely determined in its curvatures by the conditions for achromatization, and the quantities available for adjustment are the figure of the great mirror and the curvatures of the two lenses of the corrector. These are used to satisfy rigorously equations (2), and the essential difficulty of the problem is to find a case among the great number of those that are open for trial, the solution of which shall prove to be of a practical kind, not involving excessive curvatures. Once an approximate solution is obtained, to refine it only requires patience, but to arrive in the neighbourhood of a solution is a problem in which trial needs some guide. In this connection I would draw attention to the theory given below of the Thin Corrector. This is an optical system of two or more thin lenses in contact, null as far as deviation and colour are concerned, and introducing aberrations only which are available for correcting existing aberrations. Thus simplified, it is manageable algebraically, and its indications will show the possibility or otherwise of any projected arrangement.

If we denote by \(\mathfrak{B}\) the curvature of the field and by \(\mathfrak{P}\) Petzval’s expression

\[
\mu_1 \sum (1/\mu_{2r+1} - 1/\mu_{2r-1}) B_{2r}
\]

\(B_{2r}\) being the curvature of the surface \((2r)\), as in the Memoir, p. 162, we have

\[
\delta G + \delta H = f'\mathfrak{B}, \quad \delta G - \delta H = H\mathfrak{P} = f'\mathfrak{P}
\]

at the principal focus; hence \(\delta H\) which gives the amount of astigmatism is determined by

\[
\delta H = \frac{1}{2} (\mathfrak{B} - \mathfrak{P}) f' \quad \ldots \quad (3)
\]

a result which can also be deduced at sight from known expressions for astigmatism and curvature of field according to Seidel’s theory. In the special case of a flat field, or \(\mathfrak{B} = 0\), it becomes

\[
\delta G = -\delta H = \frac{1}{2} \mathfrak{P} f' \quad \ldots \quad (3a)
\]

and this may be taken in place of the third of equations (2) as one of our necessary conditions. We notice that it is only possible to control the astigmatism through the value of \(\mathfrak{B}\), and the value of \(\mathfrak{B}\) depends only in small degree upon the distribution of curvatures between the two faces of a lens. It is a matter then of the general design of the instrument to keep \(\delta G\) down to a suitable magnitude. This presents no difficulty. I have been content to keep it small enough for my purpose. If a field of
radius greater than 1 degree were desired, it could be made even smaller, but it would seem to involve the sacrifice of some other conveniences.

The values of the quantities \( \delta_i, G, \&c. \), for the combined system are built up step by step by proceeding from surface to surface or from lens to lens by the sequence equations (17), p. 160, of the Memoir referred to above. For making these steps it is not convenient to lay down any one procedure as being the best for all cases, but two methods may be mentioned, one or other of which is frequently suitable. First we can proceed from conjugate focus to conjugate focus, the first focus being the principal focus of the first or great mirror, and each successive conjugate focus being the principal focus of the whole combination which precedes it. That is to say, at each stage we have

\[
g = 0, \quad h k = -1, \quad h' = 0,
\]

so that the equations we require to consider are

\[
\begin{align*}
\delta_i G &= g' \delta_i g + k^2 \delta_i h', \\
\delta_j G &= g' \delta_j g - k^3 \delta_j h' + k^2 \delta_j h', \\
\delta_k G &= g' \delta_k g + k^{-1} \delta_k h' - 2k \delta_k h' - k^2 \delta_k h'. 
\end{align*}
\]

(4)

In these \( g', \ldots \) refers to the new or added element, \( g, \ldots \) to the combination from the beginning up to this element, and \( G, \ldots \) to the resulting combination including this element. We thus notice that \( \delta g \) contributes to \( \delta G \) simply by multiplying by \( g' \), which is the magnification of the new element between its conjugate foci under consideration. We notice, too, that so long as we confine ourselves to \( \delta G \), the only coefficients which it is necessary to find for each added element are \( \delta_j h' \), calculated between the same conjugate foci. If the aberrations of the second element are given, referred to some other origins, they must be transferred to the conjugate foci in question by means of the equations for change of origin (22), p. 164. A case will present itself that requires a modification of this process, namely, when one of the conjugate foci belonging to an element introduced by one of the steps described is at a great distance; to meet this case we may take this element together with the next following one and combine them into one before adding them to the combination, or we may take a second completely different method as follows:

Let \( O_n, O_n \) be the initial and final origins; \( O_n, O_n \) the origins to which the known aberrations of a part of the system are referred. Calling \( \{ g', h'; k', l' \} \) the subsequent normal system \( O_n \) to \( O_n \) transfer the aberrations to origins \( O_n \ldots O_n \) by use of the first part of equations (17), p. 160, viz.,

\[
\delta_i G = g' \delta_i g + h' \delta_i k, \quad \ldots
\]

Then calling \( \{ g, h; k, l \} \) the preceding normal scheme \( O_n \) to \( O_n \) transfer the so-found coefficients from \( O_n \ldots O_n \) to \( O_n \ldots O_n \) by using the forms of the second part of the same equations. An example of this method will be found on p. 55.

We now study the formulae for thin lenses. It will be pointed out later how to make use of these when the lenses are thick.
Thin Lenses.

The aberration coefficients for a single surface are given in the Memoir, p. 161;
\[
\begin{align*}
\delta_g &= (1-n)B^2, & \delta_k &= 0, & \delta_k &= 0, & \delta_k &= (1-n)B, & \delta_k &= 0, & \delta_k &= 0, \\
\delta_k &= (1-n)(-1+n-n^2)B^2, & \delta_k &= -n^2(1-n)B^2, & \delta_k &= -n(1-n^2)B, \\
\delta_k &= (1-n)(-1+n-n^2)B^2, & \delta_k &= -n^2(1-n)B, & \delta_k &= -n(1-n^2),  \\
\end{align*}
\]

where I have written \(e = 1 - \epsilon\), so that \(e = 0\) for a spherical surface, and \(e = 1\) for a paraboloid.

Both origins are at the surface, and
\[
g = 1, \quad h = 0, \quad k = (n-1)B, \quad l = n, \quad p = k, \quad n = \mu_{-1}/\mu_{+1}.
\]

The case of the thin lens, with origins at its surface, is derived from this by an application of equations (17), p. 160.

Write
\[
k = \left(1 - \frac{1}{n}\right)(B-B'), \quad p = kn, \quad q = \left(1 + \frac{1}{n}\right)(B+B'),
\]
then
\[
g = 1, \quad h = 0, \quad k = \left(1 - \frac{1}{n}\right)(B-B'), \quad l = 1,
\]
\[
\begin{align*}
\delta_g &= -\frac{1}{2}kn(k+q) = -\frac{1}{2}p(k+q), & \delta_k &= 0, & \delta_k &= 0, \\
\delta_h &= -kn = -p, & \delta_k &= 0, & \delta_k &= 0, \\
\delta_k &= \left\{1 + \frac{1}{4} \frac{(2-n)}{(1-n)^2}\right\} k^2 + \frac{1}{2}nk^2g + \frac{1}{4} \frac{n^2(1+2n)}{(1+n)^2} kq^2 + (1 - \frac{1}{n})(cB^2 - c'B^2), \\
\delta_k &= k^2 - \delta_k g, \\
\delta_k &= k(1+n) = k + p, \\
\delta_l &= \delta_k k - kp = k^2 - kp - \delta_k g, \\
\delta_l &= l, \\
\delta_l &= 0.
\end{align*}
\]

It may be mentioned that B, the curvature, is positive when the convex face is presented to the ray.

It seems unnecessary to give the algebra leading to these expressions in all cases. It is quite straightforward, and that for \(\delta_k\), which is relatively long, may be taken as a model. From the Memoir, p. 160, we have, taking \(\delta_kK\) to refer to the joint effect of the two surfaces
\[
\delta_kK = k'\delta_k g + l'\delta_k k + \{\delta_k k' + 2k\delta_k k' + k^2\delta_k k'\} + k\{\delta_l + 2k\delta_l + k^2\delta_l\}.
\]
It is clear that the terms in $e$, $e'$ come to the values given. Leaving these aside

$$
\delta l'^2 + 2k_0 \delta l' + k_0^2 \delta l' = B'^{-1} \{ \delta l' + 2k_0 \delta l' + k_0^2 \delta l' \}
$$

$$
= \left(1 - \frac{1}{n^2}\right) \left(-1 + \frac{1}{n^2}\right) B^2 - 2 (1 - n) B \cdot (-n^2) \left(1 - \frac{1}{n^2}\right) B' + (1 - n)^2 B^2 \cdot \frac{1}{n} \left(-1 + \frac{1}{n^2}\right)
$$

$$
= n^{-3} [(1 - n)^2 (1 + n) B^2 - 2 (1 - n)^2 B B' + (1 - n) (1 - n + n^2) B^2].
$$

This appears, multiplied by $-(1 - n) B + B'$, and added to $k' \delta g + l' \delta l$ which is

$$
- \left(1 - \frac{1}{n^2}\right) B' \cdot (1 - n) B^2 + \frac{1}{n} (1 - n) (-1 + n - n^2) B^3;
$$

the whole is

$$
B^3 \times \frac{1}{n^3} (1 - n)(1 - n + n^2) - \frac{1}{n^3} (1 - n)^2 (1 - n^2) = - \frac{1}{n^3} (1 - n) (1 - 2n + n^2 + n^3)
$$

$$
+ B B' \times \frac{1}{n^3} (1 - n)^2 + \frac{2}{n^3} (1 - n)^2 + \frac{1}{n^3} (1 - n)^2 (1 - n^2) = \frac{1}{n^3} (1 - n) (3 - 5n + 2n^2)
$$

$$
+ B^2 \times \frac{1}{n^3} (1 - n)^2 (1 - n + n^2) - \frac{2}{n^3} (1 - n)^2
$$

$$
= \frac{1}{n^3} (1 - n) (3 - 4n + 2n^2 - n^3)
$$

$$
+ B^3 \times \frac{1}{n^3} (1 - n)(1 - n + n^2).
$$

This may be written

$$
- n^{-3} (1 - n) (B - B') [(1 - 2n + n^2 + n^3) B^2 + (-2 + 3n - n^2 + n^3) B B' + (1 - n + n^2) B^2]
$$

$$
= - n^{-3} (1 - n) (B - B') [(1 - n)^2 (B - B')^2 + n^2 B^2 + (-n + n^2 + n^3) B B' + n B^2]
$$

$$
= K + KX
$$

where

$$
X = nB^2 + (-n^{-1} + 1 + n) BB' + n^{-1} B^2,
$$

$$
= \frac{1}{4} \frac{n(2-n)}{(1-n)^2} K^2 + \frac{1}{2} nKQ + \frac{1}{4} \frac{n^2 (2n+1)}{(1+n)^2} Q^2.
$$

This is the given expression if finally we write small letters for capitals.

It will be noticed that $g$, which contains the reference to the distribution of curvatures, apart from their effect upon focal length only presents itself in the forms in which it is introduced by $\delta k$, $\delta g$. It is somewhat remarkable that the same is true when we have any number of thin lenses in contact; thus, if we have a system of thin lenses in contact, giving a set of coefficients $\delta g$, ..., and add a single thin lens to it for which we have $\delta g'$, ..., then, noticing that

$$
g = l = 1, \quad h = 0, \quad g' = l' = 1, \quad h' = 0,$$
we have
\[ \begin{align*}
\delta G &= \delta g + \delta g' + k \delta h' = \delta g + \delta g' - k \psi', \\
\delta H &= \delta h + \delta h' = -\psi - \psi' = -2 \Psi, \\
\delta K &= \delta_3 K + k' \delta_3 g + \{ \delta_3 k' + 2 k \delta_3 k' + k^2 \delta_3 \psi' \} + k \{ \delta_3 l' + 2 k \delta_3 l' + k^2 \delta_3 \psi' \}, \\
&= \delta_3 k + \delta_3 k' + k' \delta_3 g + 2 k (k^2 - \delta_3 g') + k^2 (k' + \psi'), \\
&+ k (k^2 - k \psi' - \delta_3 g') + 2 k k' + E, \\
&= (k + k')^2 + kX + k'X' + k \delta g - 3 k \delta g' + k (k - k') \psi' + E,
\end{align*} \]

where \( E \) is the sum of terms in \( e, e' \) for each of the lenses;
\[ \begin{align*}
\delta_3 G &= \delta_3 g + \delta_3 g' + k \delta_3 h' = \delta_3 g + \delta_3 g' - k \psi', \\
\delta_3 H &= \delta_3 h + \delta_3 h' = -\psi - \psi' = -2 \Psi, \\
\delta_3 K &= \delta_3 k + k' \delta_3 g + \{ \delta_3 k' + 2 k \delta_3 k' + k^2 \delta_3 \psi' \} + k \{ \delta_3 l' + 2 k \delta_3 l' + k^2 \delta_3 \psi' \}, \\
&= \delta_3 k + \delta_3 k' + k' \delta_3 g + 2 k (k^2 - \delta_3 g') + k^2 (k' + \psi'), \\
&+ k (k^2 - k \psi' - \delta_3 g') + 2 k k' + E,
\end{align*} \]

Thus, to form the coefficients \( \delta G, \ldots \) for any system of thin lenses in contact, we require to know only the forms for \( \delta G \) and \( \delta K \). I add the forms of these for three lenses,
\[ \begin{align*}
\delta G &= \delta g + \delta g' + \delta g'' - k (\psi + \psi') - k \psi'', \\
&= \frac{1}{2} (\psi + \psi' + \psi'' - k \psi - \frac{1}{2} \psi' (2 k + k') - k \psi'' (2 k + 2 k' + k''), \\
\delta K &= -\frac{1}{2} kX + kX' + k'X'' + E, \\
&+ (k' + k'') \delta_3 g + (-3 k + k') \delta_3 g' + (-3 k - 3 k') \delta_3 g'', \\
&+ k (k - k' - k'') \psi' + (k + k') (k + k' - k'') \psi'', \\
&= K^3 + kX + kX' + k''X'' + E, \\
&- \frac{1}{2} (k' + k'') \psi + \frac{1}{2} (3 k - k') \psi' + \frac{1}{2} (k + k') \psi''.
\end{align*} \]

From these, if necessary, the general case may be written down by analogy without much difficulty, e.g., in \( \delta K \) the coefficient of \( \frac{1}{2} \psi' \) is three times the \( k \) of the preceding system minus the \( k \) of the following system; but I shall not require more than three.

We may employ these equations where we require to obtain algebraically rough but reliable indications of the properties of a given actual system. Thus, consider the aberrations of any set of thin lenses in contact, at their principal focus, that is, at a distance \(-K^{-1}\) beyond their common surfaces. We must form \( \delta_3 l' = \delta_3 G - K^{-1} \delta_3 K, \ldots \) where \( \delta G, \ldots \) are the quantities just found which refer to the surfaces of the lenses as origins. Hence for example, referring to p. 30, we see that the radius of the focal
circle, and the separation of the focal lines is constant in such a system, the former being equal to \(103139^\circ \times (a f') \times \beta^2\), and the latter to \(f \beta^2\). The curvature of the field is \(2K + \beta\), or the radius of curvature is always about two-fifths of the focal length.

The condition for absence of coma, which is usually given as ABBE's Sine Condition, may be put

\[
0 = K\delta_2 \Gamma = K\delta_2 G - \delta_2 K = \delta_2 G - K^2;
\]

in this the right-hand member, apart from the focal lengths, is a linear function of the quantities \(q\).

The condition for absence of spherical aberration is

\[
0 = K\delta_1 \Gamma = K\delta_1 G - \delta_1 K,
\]

which is a quadratic function of \(q\), ....

A numerical example of the use of such approximations will be given later.

It is necessary to deal with express care with the case of the mirror. It may be treated as a single surface for which \(n = -1\), and then

\[\begin{align*}
g &= 1, & h &= 0, & k &= -2B, & l &= -1, & \nu &= -2B, \\
\delta g &= 2B^2, & \delta h &= 0, & \delta i &= 0, & \delta j &= 2B, & \delta k &= 0, & \delta l &= 0,
\end{align*}\]

but this leaves the positive axis after reflection opposite to the direction of the ray. It is better to reverse the direction of the axis, and this may best be done by multiplying by the scheme \(\{g, h; k, l\} \equiv \{1, *, *; -1\}\), and gives the following set to represent the mirror:

\[\begin{align*}
g &= 1, & h &= 0, & k &= 2B, & l &= +1, & \nu &= -2B, \\
\delta g &= 2B^2, & \delta h &= 0, & \delta i &= 0, & \delta j &= 2B, & \delta k &= = 0, & \delta l &= 0,
\end{align*}\]

the signs of all terms in \(k, l\) being reversed by this step, while \(g, h, \nu\) remain unchanged. Notice that the convention for the sign of \(B\) has not been altered, so that, \(e.g.,\) for the concave mirror \(B\) is negative, and the new value of \(k = (1 - n) B\) is negative also.

If we write \(\delta k = \delta^2 + k x + E\), we must put \(x = -\frac{1}{2}k^2\).

Besides the simple mirror I shall have also to deal with the system consisting of a meniscus, silvered at the back. Such a system I shall call a Reverser. For neglected thickness the coefficients follow readily from the case above (p. 35), of the juxtaposition of three thin lenses, replacing the middle lens by a mirror, and taking for
the third lens the original lens with the surfaces in reversed order. This reversal of order will replace B, B' respectively by \(-B', -B\). Hence \(k, \psi\) will equal \(k'', \psi''\) respectively, but \(q + q'' = 0\).

Hence in the expressions (9), using ' to denote the mirror surface

\[
\delta g + \delta g'' = -k\psi, \quad \delta g' = 2B^2 = -\frac{1}{2}k'\psi', \\
\delta G = -k\psi - \frac{1}{2}k'\psi' - k(\psi + \psi') - k'\psi = -\frac{1}{2}(2k + k')(2\psi + \psi') = -\frac{1}{2}K\Psi. \quad (11)
\]

The same expression is true of a more complicated reverser of any number of thin lenses with the last surface silvered. Also

\[
\delta K = K^2 + (k\chi + k'\chi') - \frac{1}{2}k'^2 + E \\
- \frac{1}{2}(k + k')\psi (k + q) - 2k(-\frac{1}{2}k'\psi') + \frac{1}{2}(k + k')\psi (k - q) \\
+ k(\psi' + (k + k')k') \\
= K^2 + \frac{2n - n^2}{2(1-n)^2}k^2 + \frac{n^2(1+2n)}{2(1+n)^2}k^2 - \frac{1}{4}k'^2 + E \\
- 2(k + k')\psi q + (k + k')^2\psi. \quad \ldots \ldots . \quad . \quad (12)
\]

To conclude this preliminary discussion of systems of thin lenses in contact I shall introduce a system which consists of two thin lenses in contact, of equal and opposite focal length and of the same glass, and therefore a null system in every respect except for aberrations. The use of such a system will be illustrated hereafter. Its simplicity is such that its aberration-coefficients reduce to very easy forms, and can therefore be handled algebraically in an experimental investigation, in order to discover what system will correct the aberrations of a proposed system; it will supply a useful approximation to a solution when any less idealised system is too complicated to manage.

From the expressions (8) we have for the Thin Corrector

\[
K = k + k' = 0, \quad \Psi = kn + k'n = 0, \\
\delta G = -\frac{1}{2}k^2n(1+q/k) - \frac{1}{2}k'^2n(1+q'/k') - kk'n = -\frac{1}{2}k^2n(q/k + q'/k'), \\
\delta K = (k + k')^2 + E \\
+ k^2\left\{ \frac{2n - n^2}{4(1-n)^2} + \frac{1}{2}n(q/k) + \frac{n^2(1+2n)}{4(1+n)^2}(q/k)^2 \right\} \\
+ k'^2\left\{ \frac{2n - n^2}{4(1-n)^2} + \frac{1}{2}n(q'/k') + \frac{n^2(1+2n)}{4(1+n)^2}(q'/k')^2 \right\} \\
- \frac{1}{2}k'\psi n(1+q/k) + \frac{1}{2}kk'^2n(1+q'/k') + k(k - k')k'n \\
= k^2\left[ n \frac{q/k + q'/k'}{4(1+n)^2} + \frac{n^2(1+2n)}{4(1+n)^2}(q/k)^2 -(q'/k')^2 \right] + E, \quad \ldots \ldots . \quad (13)
\]

and all the rest of the coefficients run in agreement with p. 35, so that
\[ \delta_2 K = \delta_3 L = -\delta_4 G \text{ and the rest are zero. These are the values at the surface of the corrector. We notice that all are zero when } q/k + q'/k' = 0, \text{ that is, when the curvatures of the two surfaces in contact are the same.} \]

In order to illustrate the manner of using these, for example, let it be proposed to find the curvatures of a corrector, which when interposed at a given point of an aberrant beam shall produce assigned changes in it. Let this place be at a distance \( v \) before the beam comes to its focus. After passing through the corrector it will still come to a focus at the same place, so that applying the formulæ of the Memoir, p. 164, (22), we have for the distances from the first conjugate focus to the corrector \( d = v \), which is negative, and from the corrector to the second conjugate focus \( d' = -v \), and transferring from the surface of the corrector to these conjugate foci, we have

\[
\begin{align*}
\delta_i h' & = 2v^3 \delta_i \gamma - v^2 \delta_i \kappa, \\
\delta_j h' & = 3v^2 \delta_j \gamma - v^2 \delta_j \kappa, \\
\delta_i h' & = 4v \delta_i \gamma - v^4 \delta_i \kappa,\
\end{align*}
\]

where \( \delta_i \gamma, \delta_i \kappa \) are written for the values of \( \delta_i G, \delta_i K \) given in (13).

We must now apply the formulæ (4) of p. 32. For the corrector \( g' = 1 \). Let the assigned changes be, say,

\[ \Delta_1 = \delta_2 G - \delta_3 G, \quad \Delta_2 = \delta_3 G - \delta_4 G, \]

so that the equations (4) of p. 32 give

\[
\begin{align*}
k^{-1} \Delta_1 &= - \delta_i h' + k \delta_j h' = (-3 + 4kv) v^2 \delta_i \gamma + (1 - kv) v^2 \delta_i \kappa, \\
k \Delta_2 - l \Delta_1 &= \delta_i h' - k \delta_j h' = (+2 - 3kv) v \delta_i \gamma - (1 - kv) v^2 \delta_j \kappa,\
\end{align*}
\]

therefore

\[
\begin{align*}
v^3 \delta_i \gamma &= - \Delta_2 k^{-1} - \Delta_1 klv/(1 - kv), \\
v^3 \delta_i \kappa &= k^{-1} (2 - 4kvl)/(1 - kv) . \Delta_1 + kv (3 + 4klv)/(1 - kv)^2 . \Delta_2 . 
\end{align*}
\]

From these equations the values of the curvatures of the two lenses may be found with the help of equations (13). An example of their use will be found below, on p. 44.

In connection with the question of assigning a system which will produce definite changes it may be remarked that it is not difficult to solve the equations (17) of p. 160 of the Memoir so as to give explicitly either \( \delta_g, \ldots \) or \( \delta_g', \ldots \) so that we have as may be desired either the antecedent set or the consequent set which combine to produce given aberration coefficients \( \delta_i G, \ldots \). The former are obviously obtained by forming \( \ell^2 \delta G - h^2 \delta K, \ell^2 \delta H - h^2 \delta L, -k' \delta G + g' \delta K, -k' \delta H + g' \delta L \), which give respectively \( n' \delta g, n' \delta h, n' \delta k, n' \delta l \). For the latter coefficients \( \delta_g', \ldots \) we form

\[
\begin{align*}
P_1 \delta_i G - 2k \delta_2 G + k^2 \delta_3 G &= \ldots + n^2 (g \delta_g' + k \delta_h'), \\
P_1 \delta H - 2k \delta_2 H + k^2 \delta_3 G &= \ldots + n^2 (h \delta_g' + l \delta_l').
\end{align*}
\]
which give \( \delta g', \delta h' \); and similarly we have \( \delta k, \delta l' \). Form also

\[
-hl\delta G + (gl + kk) \delta g G - gk\delta h G = ... + n^2 (g\delta g' + k\delta h'),
\]

\[
-hl\delta H + ... = ... + n^2 (h\delta g' + l\delta h'),
\]

and

\[
-h^2\delta G - 2gh\delta G + g^2\delta G = ... + n^2 (g\delta g' + k\delta h'),
\]

\[
h^2\delta H - ... = ... + n^2 (h\delta g' + l\delta h'), \quad \ldots \quad (15)
\]

with similar equations in \( \delta K, \delta L \). These equations, for example, answer the question of what aberrations are shown when a known system is reversed and presented with the opposite face to the beam, the unit-points being simply interchanged so that the normal effect as shown in the position of the focus is the same as before. For if an unaberrated beam originating at \( O \) is brought to a focus at \( O' \) and shows there aberration coefficients \( \delta g, ... \); or, what is the same statement, an aberrant beam with coefficients \( \delta g, ... \) emerging from \( O' \) and passing through the system in the opposite direction is brought to an unaberrated state at \( O \), then if \( \delta g', ... \) are the coefficients introduced by the reversed passage we have the joint effect of \( \delta g', ... \) superposed to \( \delta g, ... \) is null, or \( \delta G, ... \) are all zero. But it must be noted, as was pointed out for the mirror, that as the direction of the axis is reversed the signs of \( \delta k, ... \delta l \) must be reversed before they are brought into the equations with \( \delta g', ... \); further, since \( G = 1, H = 0, K = 0, L = 1 \), we have \( g' = l, h' = -h, k' = -k, l' = g, \) and \( n = 1 \).

The whole question has some general interest, but I shall not pursue it further at present, because it is somewhat beside our mark, and I return to considerations that bear upon the main problem.

Coming now to the immediate object of my paper, which is the Cassegrain telescope, I shall first consider what can be effected with two mirrors simply, which will give opportunities for writing down useful expressions of various forms relating to mirrors.

A mirror with both origins at its surface, and the reversal included, gives the scheme (10) p. 36, or say

\[
g = 1, \quad h = 0, \quad k = k, \quad l = 1, \quad p = -k,
\]

where \( k = 2B \), together with the aberration coefficients

\[
\frac{1}{2}k^2, 0, 0; \quad k, 0, 0; \quad \frac{1}{2}(2 + \epsilon)k^2, \frac{1}{2}k^2, 0; \quad \frac{3}{2}k^2, k, 0. \quad \ldots \quad (16)
\]

With the surface for one origin and the principal focus for the other, these become

\[
g = 0, \quad h = -k^{-1}, \quad k = k, \quad l = 1,
\]

with the coefficients

\[
\frac{1}{2}k^2, -\frac{1}{2}k, 0; \quad -\frac{1}{2}k, -1, 0; \quad \text{ibid.} \quad ; \quad \text{ibid.} \quad \ldots \quad (17)
\]

If by the formulæ of the Memoir, p. 164 (22), we transfer the origins to two
conjugate foci, P, P' respectively, say at distances $PO = u$, $OP' = v$ along the ray from the surface, so that

$$u + v + kvu = 0$$

—where it is to be noted that the positive direction for both $u$ and $v$ is the direction of the ray, which is reversed at the surface, so that if $P, P'$ are found upon the same side of the mirror $u$ and $v$ will have the same sign—we have the scheme

$$g = 1 + kv, \quad h = 0, \quad k = k, \quad l = 1 + ku,$$

with the coefficients

$$\delta g = \frac{1}{2}k^2 [1 + kv + \frac{1}{2}e^2 u^2], \quad \frac{1}{2}k^2 u [1 + \frac{1}{2}e^2 u^2]$$

$$\delta h = k [1 + kv + \frac{1}{2}e^2 u^2], \quad \frac{1}{2}k^2 u [1 + \frac{1}{2}e^2 u^2]$$

$$\delta k = \frac{1}{2}k^2 [1 + \frac{1}{2}e^2], \quad \frac{1}{2}k^2 [1 + \frac{1}{2}e^2 u^2]$$

$$\delta l = \frac{1}{2}k^2 [3 + ku + \frac{1}{2}e^2 u^2], \quad k + 2k^2 d + \frac{1}{2}k^2 d^2 (1 + \frac{1}{2}e), \quad kd [2 + \frac{1}{2}k^2 d + \frac{1}{2}k^2 d^2 (1 + \frac{1}{2}e)].$$

(18)

To obtain the system for a Cassegrain telescope, we must combine two systems, $(gh...)$, $(g'h...)$, as in the Memoir, p. 160 (17), of which the former gives the great mirror at its principal focus, by (17) above, while the latter gives the second mirror between two conjugate foci, by (18). Let $\kappa', e'$ refer to the great mirror, and $\kappa, e$ to the second one. If we confine attention to spherical aberration, coma, curvature, and astigmatism, it will suffice to form $\delta G, \delta H, \delta G$ for the compound system, deriving $\delta H$ with the help of the equation $\delta G - \delta H = H$. The resulting expressions are

$$\delta G = \frac{1}{2}k^2 v [1 + \frac{1}{2}e^2 u^2] - 1 - \frac{1}{2}e' \kappa' u^2,$$

$$\delta G = \frac{1}{2}k^2 u [1 + \frac{1}{2}e^2 u^2], \quad \frac{1}{2}k^2 u [1 + \frac{1}{2}e^2 u^2]$$

$$\delta G = -\kappa' v [1 + \kappa' v], \quad \frac{1}{2}k^2 u [1 + \frac{1}{2}e^2 u^2]$$

$$\delta G = -\kappa' [1 + \kappa' v], \quad \frac{1}{2}k^2 u [1 + \frac{1}{2}e^2 u^2]$$

$$\delta G = -\kappa' [1 + \kappa' v], \quad \frac{1}{2}k^2 u [1 + \frac{1}{2}e^2 u^2]$$

(19)

with

$$\delta G - \delta H = - (\kappa + e') v / ku.$$

The quantities $\epsilon - 1, \epsilon' - 1$ are what SCHWARZSCHILD calls the deformations of the mirrors, from spherical figures; when $\epsilon = 0$, or the deformation $-1$, we have a paraboloid; if we choose them so as to annul coma and spherical aberration we have, from the equations $\delta G = 0, \delta G = 0$ respectively,

$$\frac{1}{2}e' \kappa' u^2 = 2\epsilon' v + 1/(1 - ku),$$

$$\epsilon = -2\epsilon' v^2 (1 - ku);$$

while if we eliminate $\epsilon'$ from $\delta G$, we get

Curvature of field $= -K (\delta G + \delta H),$ $= -\{1 + (1 - ku) \epsilon' (v - u)\}/v,$ $= 1/[u + \kappa' (v - u) + (1 + \kappa (v - u))/v,$

and

$$\delta H = -1/(ku - (1 - ku))/2v.$$
CASSEGRAIN REFLECTOR WITH CORRECTED FIELD.

These expressions are identical, except for notation, with results given by SCHWARZSCHILD; they contain the complete theory of the Cassegrain combination, corrected by figuring for coma and spherical aberration, except as regards distortion, and this could easily be added by calculating $\delta H$.

We read from equations (20) that for a given design of instrument, as specified in the values of $k, k', u$ or $v$, we can adjust the figures of the two mirrors so as to annul spherical aberration and coma at the principal focal plane, and then the curvature of the field and astigmatism amount to determinate quantities. Coma is annulled only for the purpose of getting a larger field for photography, and there is very little use in annulling it if the field possesses pronounced curvature, or in less degree, if the focal circles are not reasonably small. Hence the practical questions are: can the design be made such that curvature is nearly absent and astigmatism small, and can the corresponding values assigned to the deformations be realised in practice? All these questions are treated more or less explicitly by SCHWARZSCHILD, and I shall traverse the ground again only in order to connect the problem with its subsequent development and bring out the points which I require.

Regarding the expression for curvature, $v - u$ is the positive distance from the principal focus of the great mirror to the principal focus of the combination. In the Cassegrain form the latter point is, as a rule, not far beyond the surface of the great mirror, so that $v - u$ is not far from the focal length of the great mirror and $1 + k(-u + v)$ will be a small fraction; also $k' u$ is numerically less than unity. Hence the curvature of the image will differ very little from $1/u$, the reciprocal of the distance from the second mirror to the principal focus of the great mirror, a distance which would seldom be more than one-third or one-fourth of the focal length of the great mirror, or one-tenth to one-twentieth of the focal length of the combination. The common Cassegrain is subject to the same objection. The values of its errors may be read from the equations (19) on p. 40, if we have the means to determine $e, e'$.

As an illustration we may take the great 60-inch reflector of Mount Wilson Observatory, which can be used either as a Newtonian, with a focal length of 25 feet, or in three different forms as a Cassegrain; taking the form designed for direct photography, it has an effective focal length of 100 feet, so that $v/u = -4$. If we take the final focus at the great mirror, which is nearly the case, we have $u = -5, v = +20$, and $k' = +3/20$. Now since the telescope is corrected as a Newtonian, the great mirror is parabolic, or $e = 0$; and therefore taking it as corrected for spherical aberration as a Cassegrain, $\frac{1}{4} + k' uv = 1$, or $e' = -16/9$, which is a hyperboloidal form, the deformation from a sphere being nearly three times that which would produce a paraboloid. Substituting $\frac{1}{4} + k' uv = 1$ in the equation for $\delta G$, we have, after some reductions, $\delta G = \frac{1}{2} k u/v = -\frac{1}{3} K$, or the coma of such an arrangement is the same as for a simple mirror of the same focal length. Also we find $\delta G = -15, \delta G - \delta H = -11$, so that the radius of curvature of the field is one-nineteenth of the focal length or about 5 feet only. As to the astigmatism...

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we have \( \delta s^2 H = -4 \), which may be compared with \( \delta s^2 H = -1 \) for a Newtonian, but since the aperture ratio \( a/f' \) is diminished in the ratio \( 1:4 \) by the increase of effective focal length, the radii of focal circles at all distances from the centre of the field will have the same angular amount that they had in the Newtonian form, neither more nor less. There remains then only the above-found curvature of the field to notice. Taking as a convenient mark a distance \( 34.3' \) from the centre of the field, namely where \( \beta \) in the formulae of p. 30 equals one-hundredth, we should have at this point the field curved back from the plane through the principal focus by more than one inch. In spite of this pronounced curvature, exquisite photographs of the Moon, as well as of small objects like Mars, have been obtained with this telescope in Cassegrain form. The photograph of the Moon (R.A.S. photographs, No. 214) appears to me second only to the Yerkes photographs with the 40-inch refractor and colour screen; but technically it would be more instructive to examine a photograph of a wide field of stars.

It is worth while to demonstrate that curvature of the field cannot be removed by replacing the second mirror by a set of lenses in contact, used as a reverser, as explained on p. 37. By such a replacement we introduce the quantity \( \eta \) which, for a given focal length of the reverser, is adaptable by throwing different proportions of the deviation of the rays upon the lens system and silvered surface respectively.

Then using the formulae (4) of p. 32, in which we may put \( hk = -1 \), \( l = 1 \), \( k \) now referring to the great mirror and \( \kappa \) to the reverser,

\[
\delta s G = g' \delta s g * + k^2 \delta s h',
\delta s G = g' \delta s g * - k^2 \delta s h' + k^2 \delta s h',
\delta s G = g' \delta s g + k^{-1} \delta s h' - 2 \delta s h' + k^3 \delta s h',
\]

where, if \( \delta \gamma, \ldots \pi \) refer to the reverser at its surface,

\[
\delta s h' = \delta s_\gamma + v (\delta s_\lambda + u \delta s_\kappa),
\delta s h' = \delta s_\gamma + \ldots + u (\delta s_\eta + \ldots),
\delta s h' = \delta s_\eta + \ldots + 2u (\delta s_\eta + \ldots) + u^2 (\delta s_\eta + \ldots),
\]

and by (11), p. 37,

\[
\delta s_\eta + \ldots = k^2 v - \pi (1 + \frac{1}{2}xu + \frac{1}{2}xv) + uv \delta s_\kappa,
\delta s_\eta + \ldots = -k u + \frac{1}{2}uv k \pi,
\delta s_\eta + \ldots = uv (k + \pi).
\]

Thus

\[
\delta s G = g' \delta s g + k^2 (\delta s_\eta + \ldots) + k (2ku - 1) (\delta s_\eta + \ldots) + ku (ku - 1) (\delta s_\eta + \ldots),
\delta s G = g' \delta s g + k (\delta s_\eta + \ldots) + 2 (ku - 1) (\delta s_\eta + \ldots) + k^{-1} (ku - 1)^2 (\delta s_\eta + \ldots).
\]

Eliminate \( \delta s_\eta + \ldots \) by forming \( 1 - ku \) \( \delta s G + k^2 u \delta s G \),

\[
(1 - ku) \delta s G + k^2 u \delta s G = g' (1 - ku) \delta s g + g' k^2 u \delta s g + k^2 (\delta s_\eta + \ldots) - k (1 - ku) (\delta s_\eta + \ldots).
\]
Now \( \delta g = -\frac{1}{2} k, \delta s g = 0 \); and if by figuring or otherwise we annul coma, so that \( \delta s G = 0 \), we have

\[
ku^2 \delta G = -\frac{1}{2} g' (1 - ku) - (1 - ku) (-ku + \frac{1}{2} u w + \pi) + ku (k + \pi). 
\]

Also

\[
ku (\delta s G - \delta s H) = H P k u = -(g' / k) (-k + \pi) ku = v (-k + \pi),
\]

so that

\[
ku (\delta s G + \delta s H) = -g' (1 - ku) + kv + u (1 - ku + kv) (2k + \pi),
\]

also \( K = k / g' = -ku / v \); so that the curvature is

\[
-K (\delta s G + \delta s H) = 1/u + (2k + \pi) (1 - ku + kv) u / v. \quad \ldots \ldots \ldots (21)
\]

If we compare this with the expression given in (20) above we see that the sole effect of the change is to replace the reciprocal of the focal length of the second mirror by \((2k + \pi)\) for the reverser, and, since its factor in \( u, v \) is small, this change will not allow any considerable modification of the curvature of the field.

To meet the difficulty of curvature Schwarzschild considers a design of instrument fundamentally altered. Thus in (19) the curvature of the field will vanish if

\[
k' = -v / u^2 \{1 + \pi (-u + v)\}
\]

and this may be secured if \( k' \) is negative as well as \( k \), or if the second mirror is concave; but in order that the curvature of the mirror may not be too great we must then take \( 1 + \pi (-u + v) \) sensibly different from zero, and also \( v / u \) the magnification of the second mirror, not too large. The system to which Schwarzschild is led as generally the best to be found under such conditions has been already described (p. 28). It is so different from anything that has yet been made that it must be regarded merely as an interesting exploration of the possibilities of the theory until an attempt is made to realise it. In particular it is utterly different from the long-focus Cassegrain which I have in mind, and therefore I shall not require to refer to it further.

Returning to the question of the Cassegrain proper we see that if an improvement is to be made it must be by inserting a corrector of some form in the course of the beam. Hence we come to the system which I have indicated on p. 29. To get an approximation to what is required, suppose that the reverser is merely a convex mirror, that the corrector consists of a pair of thin lenses of which the theory is given on pp. 37 and 38, and that all the surfaces are spherical except that of the great mirror which is figured so as to annul spherical aberration. To fix ideas I shall suppose that the unit of length is 100 inches, and that with this unit the aperture of the great mirror is 0'40 and its focal length 2'0000, also that the separation of the two mirrors is 1'3333, that the magnification of the second mirror is 2'4, from which it results that its focal length is \( 1/875 = 1'1429 \), and the principal focus of the combination is thrown beyond the great mirror by '2667, at a distance '6000 from the
second mirror. It will be seen from the expressions (14) that it is desirable that the corrector should be as far as practicable from the principal focus if its aberrations are to be as small as possible, that is to say, if its curves are to be as shallow as possible. It cannot be too far forward or it will cut off some of the rays coming from the great mirror to the reverser. It appears that a convenient distance is 0'9000 from the reverser, or 0'7000 from the principal focus. That is to say, in the formulae (19) of p. 40,
\[ \kappa = -'5000, \quad \kappa' = +'8750, \quad u = -'6667, \quad v = +1'6000, \]
so that, with \( \epsilon' = 1 \), for a spherical reverser,
\[ \delta g = +'3383, \quad \delta g = -3'0301. \]

Now we have to make
\[ \delta g = 0, \quad \delta g + \delta h = 0, \]
and we have
\[ \delta g - \delta h = H \delta = +4'8000 \times -'3750 = -1'8000. \]

Hence the changes \( \Delta_2, \Delta_3 \), which the corrector must introduce, are respectively,
\[ \Delta_2 = -'3383, \quad \Delta_3 = +2'1301. \]

These are the quantities so denoted in (14) p. 38. In the same equation, the values of \( k, l \) to be used come from the scheme resulting from the combination of the two mirrors, viz.,
\[ g = * , \quad h = +4'800, \quad k = -'2083, \quad l = +2'1667, \]
and \( v \) giving the position of the corrector with respect to the principal focus,
\[ v = -'7000. \]

Hence
\[ k^{-1} \Delta_2 = +1'6238, \quad kv \Delta_3 = +'3106, \]
\[ klv = +3160, \quad (1-klv)^{-1} = 1'4620, \quad (-2+4klv)/(1-klv) = -1'0760, \]
\[ (-3+4klv)/(1-klv)^2 = -3'7107, \]
and
\[ v^2 \delta g = -1'6238 - '4541 = -2'0779, \]
\[ v^2 \delta h = -1'7472 - '1525 = -2'8997. \]

Referring now to (13) for the expressions for \( \delta g, \delta h \) for a thin corrector (in which we shall here write \( \kappa, \kappa' \) in place of \( k, k' \)), and remembering that \( E = 0 \) since all the surfaces are taken spherical, we have the equations
\[ (\kappa v)^2 \frac{4}{3} n [q/k + q'/k'] = +2'0779, \]
\[ (\kappa v)^2 n [q/k + q'/k'] \left[ 1 + \frac{n(1+2n)}{4(1+n)^2} (q/k - q'/k') \right] = -2'8997. \]
In order to secure shallow curves the quantities \( q/\kappa, q'/\kappa' \) should be as small as possible. It is therefore evident that \( \kappa \nu \) should be taken negative, that is \( \kappa \) positive. The actual value of \( \kappa \) the reciprocal of the focal length of each member of the corrector has now to be chosen. By increasing \( \kappa, q, q' \) will be made smaller but at the same time the lenses employed will be shortened in focus. As a reasonable trial, take \( \kappa = +1'4286 \), so that \( \kappa \nu = -1 \), and the focus of the combination of the two mirrors is also a focus of either lens of the corrector; then taking, say,

\[
\mu = 1'5200, \quad n = 6579, \quad n(1+2n)/(1+n)^2 = 13857,
\]

we have

\[
\frac{q}{\kappa} + \frac{q'}{\kappa'} = +6'3168,
\]

\[
\frac{q}{\kappa} - \frac{q'}{\kappa'} = -2'1809,
\]

or the equations give

\[
\frac{q}{\kappa} = +2'0680, \quad \frac{q'}{\kappa'} = +4'2488.
\]

The curvatures of the lenses are now found from

\[
\kappa = \left(1 - \frac{1}{n}\right)(B_4 - B'_4) = +1'4286,
\]

\[
q = \left(1 + \frac{1}{n}\right)(B_4 + B'_4) = +2'9543,
\]

or

\[
B_4 = -7'875, \quad B'_4 = +1'9597,
\]

and

\[
\kappa' = \left(1 - \frac{1}{n}\right)(B_8 - B'_8) = -1'4286,
\]

\[
q' = \left(1 + \frac{1}{n}\right)(B_8 + B'_8) = -6'0697,
\]

or

\[
B_8 = +1'693, \quad B'_8 = -2'5779.
\]

These results are a very fair approximation. The final solution, when the thicknesses and consequent separations of all the lenses are allowed for, as well as the introduction of a third weak lens in the reverser to preserve achromatism, with resulting change in the focal length of the second lens of the corrector, is

\[
B_4 = -6'930, \quad B'_4 = +2'0482,
\]

\[
B_8 = -0'242, \quad B'_8 = -2'6120.
\]

The first lens is a double concave, the radii of its two surfaces being 1'270 and 0'510 respectively; the second is double convex, with radii 5'907 and 0'388. The remaining astigmatism is measured by the value of \( \delta_2 H \), which by p. 44 is +0'9000, which is about the same as the residual amount present in the focal plane of a
re refracting doublet. These are all reasonable amounts, so that we are now in possession of a good approximation to a workable solution which corrects coma and curvature of the field, and leaves the figure of the great mirror to correct spherical aberration.

It only remains then to adapt this solution to include consideration of all the secondary factors that have been left on one side.

We must now turn to the question of achromatism in general. A thin corrector, such as is contemplated on p. 37, is, among other properties, achromatic; but when the lenses are made thick and their unit points separated, as must be, to make the system real, this property is lost in greater or less degree. With two lenses only it is not possible to restore it completely. Reserving the quantities \( q, q' \) for adjusting aberrations, we may alter the ratio \( k : k' \) from the value \(-1\), but this gives only one adjustable element, whereas there are two necessary conditions for achromatism for any specified position of the object, namely, identical position for the image and identical magnification. It is true that in the ordinary achromatised refractor, consisting of a doublet, results are obtained with satisfaction of only a single condition, but the achromatism secured is necessarily very imperfect for another reason—the imperfect rationality of the dispersions of the two kinds of glass—and this masks the neglect of the second condition. For the reflector, where we aim at perfect achromatism, we must add a third lens to supply an additional adjustable element. I shall now give the theory of complete achromatism at a chosen point with three lenses of the same glass, separated by given distances. To make all the lenses of the same glass secures achromatism for all colours if it is attained for any two. The lenses are supposed thin, and the results must therefore be considered merely as approximations, since the thickness will alter the positions of their unit points as well as their focal lengths when a ray of different refractive index is considered. But the approximation will be generally close, and an illustration of how to make a complete adjustment will be given later.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Fig. 1.}
\end{figure}

Let the lenses be placed at \( O_2, O_4, O_6 \) and produce images in succession at \( P_3, P_5, P_7 \) of an object at \( P_1 \) as shown in the figure. Then the position and size of the image at \( P_7 \) must be constant.
Write

\[ P_1O_2 = v_1, \quad O_2P_3 = u_2; \quad P_3O_4 = v_2, \quad O_4P_5 = u_5; \quad P_5O_6 = v_5, \quad O_6P_1 = u_7, \]

also

\[ \kappa_2 = \left(1 - \frac{1}{n}\right)(B_2 - B'_2), \quad \kappa_4 = \left(1 - \frac{1}{n}\right)(B_4 - B'_4), \quad \kappa_6 = \left(1 - \frac{1}{n}\right)(B_6 - B'_6); \]

then we have the equations

\[
\begin{align*}
 v_1 + u_3 + \kappa_2 v_2 u_4 &= 0, \\
 v_3 + u_5 + \kappa_4 v_5 u_6 &= 0, \\
 v_5 + u_7 + \kappa_6 v_6 u_7 &= 0;
\end{align*}
\]

and the linear magnification is equal to

\[-\left(\frac{u_4}{v_1}\right) \cdot \left(\frac{u_6}{v_3}\right) \cdot \left(\frac{u_7}{v_5}\right).\]

Varying the system with respect to \(1/n\), the refractive index, and making a condition that \(v_1, u_7\), and the magnification are unchanged, we have

\[
\begin{align*}
 \Delta \kappa_2/\kappa_2 &= \Delta \kappa_4/\kappa_4 = \Delta \kappa_6/\kappa_6, \\
 \Delta u_3/u_3^2 &= \Delta \kappa_2, \quad \Delta u_3 + \Delta v_3 = 0, \\
 \Delta v_3/v_3^2 + \Delta u_5/u_5^2 &= \Delta \kappa_4, \quad \Delta u_5 + \Delta v_5 = 0, \\
 \Delta v_5/v_5^2 &= \Delta \kappa_6,
\end{align*}
\]

and

\[
\Delta u_3/u_3 - \Delta v_3/v_3 + \Delta u_5/u_5 - \Delta v_5/v_5 = 0;
\]

eliminate \(\Delta u_3\), \(\Delta v_3\) and this gives

\[
\Delta v_5 \left(1/u_3 + 1/v_3\right) = \Delta u_5 \left(1/u_5 + 1/v_5\right);
\]

eliminate \(\Delta v_5\), \(\Delta u_5\) and we have the two equations

\[
\Delta \kappa_2 \cdot (u_3 d_3/v_3) = \Delta \kappa_6 \cdot (v_3 d_3/u_3),
\]

\[
\Delta \kappa_2 \cdot (u_5^2/v_5^2) + \Delta \kappa_6 \cdot (v_5^2/u_5^2) = -\Delta \kappa_4;
\]

finally

\[
\frac{\kappa_2}{(v_3 d_3)} = \frac{-\kappa_4}{(u_3 d_3 + v_3 u_3 d_3)} = \frac{\kappa_6}{(u_5 d_5)}.
\]

or

\[
\frac{1/v_1 + 1/u_2}{1/v_5 - 1/d_3} = \frac{1/0_{1} + 1/v_3}{1/u_5 + 1/v_5 - (1/d_3 + 1/d_5)} = \frac{1/u_7 + 1/v_5}{1/v_5 - 1/d_5};
\]

thus, knowing \(d_3, d_5, v_1,\) and choosing, say, \(\kappa_2,\) we determine in succession \(u_3\) the value of the ratio, \(v_3, u_3, \kappa_4, v_5, u_5, \kappa_6.\) But this choice and order is open to modification.

For example, if we take, as on a subsequent page,

\[
d_3 = +'9261, \quad d_5 = +'01694, \quad u_3 = +1'6000, \quad \kappa_4 = +1'4286,
\]
we find

\[ \kappa_2 = -0.01056, \quad \kappa_6 = -1.3704. \]

This is an illustration of the simple corrector (\( \kappa, -\kappa \) in contact) modified by a slight separation of the two lenses and completed by the addition of a weak lens \( \kappa_2 \) at a considerable distance, and adjusted for a point which is nearly at the principal focus of the middle lens. The exact solution on pp. 51–53, gives

\[ \kappa_2 = -0.01152, \quad \kappa_6 = -1.3459; \]

the differences are considerable; this must be expected because the thicknesses of the lenses are of the same order as the separation \( d_0 \) of the unit-points; but in all cases the solution will be close enough to supply a good approximation that will allow the actual case to be adjusted.

The general process, suitable for use when we have obtained an approximation by the method just explained, will be the following. Let the standard scheme and that of the varied refractive indices be

\[ \{G, H; K, L\} \quad \text{and} \quad \{G + \Delta G, H + \Delta H; K + \Delta K, L + \Delta L\} \]

respectively. Then the conditions for complete achromatism at the principal focus are simply

\[ \Delta G = 0, \quad \Delta K = 0, \]

for these imply that the focal length is unchanged and also the distance \(-G/K\) from the origin to the principal focus for either way. Then using the approximation already supposed found, calculate the values of \( \Delta G, \Delta K \) which it shows. Vary the focal length of the first lens and recalculate them. Vary also the third lens and recalculate them. We then have means for interpolating the correct values of the first and third lenses requisite to give an achromatic system in conjunction with the middle lens.

This will be illustrated by the calculation of the actual system which I set out to find and to which I now come. It will be understood that it was obtained by steps of approximation.

It is unnecessary to give details regarding all these steps, which were unnecessarily circuitous, owing to numerical mistakes and ill-judged processes. I shall therefore give the final stage only.

The notation is slightly varied from the standard notation of Seidel, \( O_6 \) is the vertex of the great mirror, \( B_6 \) its curvature, \( O_2 \) the vertex of first surface of the reverser, \( O'_2 \) the vertex of the second or silvered surface, \( O''_2 \), which is the same point as \( O_2 \), is the last surface of the reverser; \( B_2, B'_2, B''_2 = -B_2 \), are the corresponding curvatures; \( O_4, O'_4 \) are the vertices of the first and second surfaces of the first lens of the corrector, with curvatures \( B_4, B'_4 \); \( O_6, O'_6 \) with curvatures \( B_6, B'_6 \) refer to the second lens of the corrector. For the thicknesses of the lenses I employ here
even suffixes, thus \( t_2 = O_2O'_{2} = O'_{2}O''_{2} \), \( t_4 = O_4O'_{4} \), \( t_6 = O_6O'_{6} \); for the separations, \( d_1 = O_0O_{2} \), \( d_3 = O'_{2}O_{4} \), \( d_5 = O'_{4}O_{6} \).

\( H_2, H''_{2} \) are the unit points of the reverser; \( H_4, H'_{4} \) and \( H_6, H'_{6} \), those of the two lenses of the corrector. Similarly \( F_1 \) is the principal focus for the great mirror, \( F_3, F'_{3}, F''_{3} \) for the different surfaces of the reverser, and so on, the final focus of the whole combination being \( F''_{7} \).

Writing, as above,

\[
\kappa = \left(1 - \frac{1}{n}\right)(B - B'), \quad q = \left(1 + \frac{1}{n}\right)(B + B'), \quad q = q/K,
\]

we find by considering the scheme

\[
\{ \begin{array}{c} 1, \\ (n-1)B, \\ n \end{array} \} \{ \begin{array}{c} 1, \\ t \end{array} \} \{ \begin{array}{c} 1, \\ (n^{-1}-1)B', \\ n^{-1} \end{array} \} \equiv \{ \begin{array}{c} G, \\ H \end{array} \}
\]

that for any thick lens

\[
K = \kappa - n(1-n^{-1})^2tBB' = \kappa + \frac{n}{4}ntk^2 - \frac{1}{4}n (1-n)^2(1+n)^2tq^2,
\]

and

\[
OH = (L-1)/K = (n^{-1}-1)tB'/K, \quad O'H' = (1-G)/K = (1-n)tB/K. \quad (24)
\]

For the reverser we have the scheme, including reversal of the ray at the reflection,

\[
\{ \begin{array}{c} 1, \\ (n-1)B_2, \\ n \end{array} \} \{ \begin{array}{c} 1, \\ t_2 \end{array} \} \{ \begin{array}{c} 1, \\ (n^{-1}-1)B'_2, \\ n^{-1} \end{array} \} \equiv \{ \begin{array}{c} G_2, \\ H_2 \end{array} \}
\]

whence

\[
K_2 = 2n^{-1}k_2(1 + t_2k_2) + 2n^{-1}B'_2(1 + t_2k_2)t^2,
\]

\[
O_2H_2 = nt(k + (1 + t_2k_2)) = -O''_2H''_2. \quad \quad \quad \quad \quad \quad (25)
\]

Write \( (K_2) \) for the part of \( K_2 \) which is due to the lens of the reverser, namely,

\[
(K_2) = (1-n^{-1})(B_2 - B'_{2}) - n(1-n^{-1})^2t_2B_2B'_2.
\]

By methods essentially the same as those exposed below I was led to the following approximate values as a system corrected for aberrations:

\[
\begin{align*}
B_0 &= -250000, \quad e_0 = +16502, \quad a_0 = +200000, \\
d_1 &= +1.320133, \\
B_2 &= -B''_{2} = +469009, \quad B'_2 = +450653, \quad t_2 = +020000, \\
d_3 &= +906760, \\
B_4 &= -697845, \quad B'_4 = +2043309, \quad t_4 = +012500, \\
d_5 &= +002500, \\
B_6 &= +003705, \quad B'_6 = -2610677, \quad t_6 = +012500.
\end{align*}
\]
The initial semi-aperture, \( \alpha_0 \), does not enter the calculations, but is carried through at its adopted value, which is recorded here for reference.

It follows that

\[
K_2 = +0.875000, \quad (K_3) = -0.010297, \\
O_2H_2 = -O''_2H''_2 = +0.013200, \\
K_4 = +1.428571, \quad O_4H_4 = +0.006116, \quad O'_4H'_4 = -0.002089, \\
K_6 = -1.359456, \quad O_6H_6 = +0.008211, \quad O'_6H'_6 = -0.000012, \\
\]

and that

\[
O_6H_6 = +1.333333, \quad H''_2F''_3 = +1.600000, \quad \ldots \ldots \ldots (26) \]

and the power of the combination of great mirror and reverser is the same as in the preliminary solution. The achromatism of the system proved also satisfactory, but the numbers had to be recast because of the following defect. As will be seen on p. 63, the semi-aperture of the lenses of the corrector is about \( \alpha = +0.0615 \). Hence the separations of the vertices of the surfaces which are next to one another must be at least \( \frac{1}{2} \alpha^2 (B'_4 - B_4) = +0.00387 \). Hence enough separation has not been allowed, since we have taken \( d_0 = +0.00250 \). I therefore increased \( d_0 \) to the value of \( +0.005000 \).

At the same time I decided to increase the thickness \( t_6 \) also to \( t_6 = +0.015000 \). To change \( d_6, t_6 \) means upsetting the balance of achromatism between the lens of the reverser and the lenses of the corrector. All the quantities then will require adjustment. The first step is to re-establish the achromatism. In doing so I keep the first lens of the corrector unchanged, and two trials at least will be requisite to get material for a proper adjustment of the other two as explained on p. 48. I found by inspection and by previous trials that an alteration of the second lens of the corrector produces its effect almost solely upon the coefficient \( K \) of the final scheme, and hardly at all upon \( G \); hence I first adjust the lens of the reverser so as to make \( \Delta G = 0 \) for variation of refractive index, and then the second lens of the corrector so as to make \( \Delta K = 0 \) also. Since the system \( O''_2 \ldots O_6 \) from the last face of the reverser to the first face of the second lens is unaltered throughout I take it in one piece, taking the lens (4) with the data of p. 49, and

\[
d_3 = +0.906760, \quad d_5 = +0.005000, \\
\]

and taking in succession

\[
\mu = n^{-1} = 1.520000, \quad n = 657.895 \\
\mu + \delta \mu = (n + \delta n)^{-1} = 1.01 \times n^{-1} = 1.535200, \quad n + \delta n = 651.381, \\
\]
we have then for the piece $O''_2\ldots O_6$,

\[
\begin{align*}
\left\{1, +'906760\right\} & \times \left\{1, +'012500\right\} \\
* & 1 \\
* & +'238736, +'657895 \\
* & 1 \\
& +1'062521, +1'520000
\end{align*}
\]

\[
\times \left\{1, +'005000\right\} = \left\{1',010127, +'929210\right\} [n],
\]

and

\[
\begin{align*}
\left\{1, +'906760\right\} & \times \left\{1, +'012500\right\} \\
* & 1 \\
* & +'243282, +'651381 \\
* & 1 \\
& +1'093579, +1'535200
\end{align*}
\]

\[
\times \left\{1, +'005000\right\} = \left\{1',010393, +'929370\right\} [n+\delta n].
\]

Now, taking the reverser first as given on p. 49, we have for the system $O_0\ldots O''_2$,

\[
\begin{align*}
\left\{1, *\right\} & \times \left\{1, +'320133\right\} \\
-500000, 1 & \times * 1 \\
-1'60450, +'657895 & \times * 1 \\
\end{align*}
\]

\[
\times \left\{1, *\right\} = \left\{+330582, +1'361934\right\} [n],
\]

and

\[
\begin{align*}
\left\{1, *\right\} & \times \left\{1, +'320133\right\} \\
-500000, 1 & \times * 1 \\
-1'63505, +'651381 & \times * 1 \\
\end{align*}
\]

\[
\times \left\{1, *\right\} = \left\{+330672, +1'361508\right\} [n+\delta n],
\]

\[n \geq 2\]
also the second lens of the corrector \( O_e \ldots O'_e \) with curvatures, as given on p. 49, but increasing the thickness to \( '015000 \), is

\[
\begin{align*}
\left\{ \begin{array}{c}
1, \\
-0.012067, +657865
\end{array} \right\} \left\{ \begin{array}{c}
1, +'015000 \\
1
\end{array} \right\} = \left\{ \begin{array}{c}
1, \\
-1.357555, +1.520000
\end{array} \right\} \\
\left\{ \begin{array}{c}
+0.999981, +0.009868 \\
-1.359452, +0.986044
\end{array} \right\} [n],
\end{align*}
\]

and

\[
\begin{align*}
\left\{ \begin{array}{c}
1, \\
-0.01292, +651881
\end{array} \right\} \left\{ \begin{array}{c}
1, +'015000 \\
1
\end{array} \right\} = \left\{ \begin{array}{c}
1, \\
-1.397234, +1.535200
\end{array} \right\} \\
\left\{ \begin{array}{c}
+0.999981, +0.009771 \\
-1.399190, +0.986348
\end{array} \right\} [n + \delta n].
\end{align*}
\]

Hence the whole combination gives

\[
\begin{align*}
\left\{ \begin{array}{c}
330582, +1.361934 \\
-208333, +2.166666
\end{array} \right\} \left\{ \begin{array}{c}
+1.010127, +0.929210 \\
+1.428571, +2.304108
\end{array} \right\} = \left\{ \begin{array}{c}
+0.999981, +0.009868 \\
-1.359452, +0.986044
\end{array} \right\} \\
\left\{ \begin{array}{c}
+1.999981, +0.009771 \\
-1.399190, +0.986348
\end{array} \right\} [n]
\end{align*}
\]

and

\[
\begin{align*}
\left\{ \begin{array}{c}
330672, +1.361508 \\
-208522, +2.165574
\end{array} \right\} \left\{ \begin{array}{c}
+1.010393, +0.929370 \\
+1.470392, +2.342197
\end{array} \right\} = \left\{ \begin{array}{c}
+0.999981, +0.009771 \\
-1.399190, +0.986348
\end{array} \right\} \\
\left\{ \begin{array}{c}
+1.999981, +0.009771 \\
-1.399190, +0.986348
\end{array} \right\} [n + \delta n].
\end{align*}
\]

Hence for \( n + \delta n \) there is an excess in the coefficient \( G \) of 26 units; to correct this, guided by previous experiments, I made a trial change in \( (K_2) \), which refers to the lens of the reverser, of \( -1220 \) units, so that

\[
(K_2) = -'010297 -'001220 = -'011517.
\]

This gives, to redetermine the reverser, supposing its power is to remain unchanged,

\[
2n^{-1}k_2(1 + t_2k_2) + 2n^{-1}B'_2(1 + t_2k_2)^2 = +075000,
\]

\[
k_2 = (n-1)B_2, \quad t_2 = +020000,
\]

\[
(1-n^{-1})(B_2 - B'_2) - n(1-n^{-1})^2t_2B_2B'_2 = -'011517,
\]

whence

\[
B_2 = +'472584, \quad B'_2 = +'451898,
\]
and these give for the system $O_0...O''_2$ the schemes, built up just as on the previous page,

$$[n], \begin{cases} +0.330582, & +1.361934 \\ -0.208333, & +2.166667 \end{cases}$$

and

$$[n + \delta n], \begin{cases} +0.330571, & +1.361508 \\ -0.208547, & +2.165477 \end{cases}$$

of which the first is the same as we had before, supplying a verification of the solution of the equations for $B_2, B'_2$.

Substitute these in the schemes $O_0...O'_6$ in place of the values already used; $[n]$ is, of course, unchanged, and we find for

$$[n + \delta n], \begin{cases} +0.140266, & +3.457242 \\ -0.198505, & +2.236634 \end{cases}$$

Hence $G$ has now the same value in both schemes and it is unnecessary to make a further trial or change of the reverter, but there remains an excess in $K$ of $-55$ units; to deal with this, try reducing the curvature of each face of the second lens of the corrector by one-hundredth part. This will give the schemes

$$\begin{cases} 1, & \ast \\ -0.001254, & +0.657895 \end{cases} \begin{cases} 1, & +0.015000 \\ \ast & 1 \end{cases} \begin{cases} 1, & \ast \\ -1.343976, & +1.520000 \end{cases} = \begin{cases} +0.999981, & +0.009868 \\ -1.345856, & +0.986738 \end{cases} [n],$$

and

$$\begin{cases} 1, & \ast \\ -0.001279, & +0.651381 \end{cases} \begin{cases} 1, & +0.015000 \\ \ast & 1 \end{cases} \begin{cases} 1, & \ast \\ -1.383262, & +1.535200 \end{cases} = \begin{cases} +0.999981, & +0.009771 \\ -1.385200, & +0.986484 \end{cases} [n + \delta n].$$

Substituting these in the combination $O_0...O'_6$ we get

$$[n], \begin{cases} +0.140265, & +3.457414 \\ -0.196543, & +2.284718 \end{cases}$$

and

$$[n + \delta n], \begin{cases} +0.140266, & +3.457242 \\ -0.196543, & +2.284996 \end{cases}$$

Hence both $G$ and $K$ are now identical, and, in consequence, both schemes indicate the same principal focus and the same focal length; in other words, complete achromatism at the principal focus.
We now return to the aberrations; we have replaced the numbers of p. 49 by the following:

\[
\begin{align*}
B_0 &= -250000, \\
B_1 &= +1'320133, \\
B_2 &= -B''_2 = +'472584, \quad B'_2 = +'451898, \quad t_2 = +'020000, \\
d_3 &= +'906760, \\
B_4 &= -'697845, \quad B'_4 = +2'043309, \quad t_4 = +'012500, \\
d_5 &= +'005000, \\
B_6 &= +'003667, \quad B'_6 = -2'584570, \quad t_6 = +'015000. 
\end{align*}
\]

(28)

and in these changes the aberrations calculated for the lenses of p. 49 will be changed; we now require to find new values for \( q_6, q_6, \) which will restore the disturbed correction. It may be remarked that the chromatic correction depends very little upon the distribution of the curvatures between the two faces which is indicated in the value of \( q, \) and it might have been reflected that as the surface (6) is nearly plane, and the beam meets it nearly at right angles, while the surface (6') produces almost the whole deviation of the beam for which the second lens is answerable, it would have been better to keep \( B'_6 \) unmodified while the second lens was adjusted for achromatism, but this was not noticed until the solution which follows had been made, and was found to reproduce almost exactly the value of \( B'_6 \) of p. 49.

The aberration coefficients for a thin lens at its surface are given by (6), p. 33. I have not so far succeeded in supplementing these by any algebraic expression containing reference to thicknesses or separations of lenses, which are simple enough to be useful. Hence the procedure for finding \( q_6, q_6, \varepsilon_6 \) must be by approximation, and the following is the method adopted. Calculate at the principal focus of the complete combination given by (28) the numerical values of the aberration coefficients, or at least the essential ones \( \delta G, \delta G, \delta G, \) in three parts, namely, first, the great mirror and reverser together in which \( \varepsilon_6 \) is easily included as an unknown; second, the first lens of the corrector; and third, the second lens of the corrector. The conditions for a corrected system are then

\[
\delta_1 G = 0, \quad \delta_2 G = 0, \quad \delta_3 G = \frac{1}{2}H \Psi;
\]

supposing these are not satisfied we must bring in corrected values of \( q_6, q_6, \varepsilon_6 \) to satisfy them. I assume for the purpose of approximate correction that the quantities \( q, q^2 \) enter the calculated aberrations with the same coefficients as if the lenses were thin; on this supposition I calculate the algebraic values of the aberrations, carrying them from the surfaces of the lenses forwards to \( F'' \) and backwards to \( O \) by a double application of the formulæ (17) of p. 160 of the Memoir. Assuming that these expressions involving the adjustable parameters \( \varepsilon_6, q_6, q_6 \) account for the discrepancies we have equations to determine \( \varepsilon_6, q_6, q_6, \) and in consequence amended values of the
curvatures of (28), that is to say, the material to repeat the approximation, if required, and finally to prove that no further change is necessary.

The numerical calculation of all aberrations follows the model given in the Memoir, pp. 172 et seq., and it will be unnecessary as a rule to give details of the working here, though I may mention that I have found a noteworthy abbreviation of it.

The great mirror and reverser together, the former treated as parabolic, contribute at \( F \),

\[
\delta_1 G = \ldots + 0.057176, \quad \delta_2 G = + 354860, \quad \delta_3 G = -3297523.
\]

We must also introduce the deviation of the mirror from a paraboloid, viz., we have at the surface of the mirror the additional term \( \delta_k = \ldots + 2e_0B_0 = -0.0312500e_0 \), and all the others unaffected. To find the effect of this in the final set \( \delta_i G \), ..., by (17) of the Memoir we must take \( h'\delta_k \) in \( \delta_i G \) merely, where \( h' \) belongs to the scheme \( O_0...F' \) and is simply equal to the final focal length, which comes out +5.087942; hence we must supplement the numerical values above by the unknown term

\[
\delta_1 G = \ldots -158998e_0, \quad \delta_2 G = *, \quad \delta_3 G = *.
\]

Next we find that the other two lenses contribute together at \( F' \),

\[
\delta_1 G = \ldots -0.030167, \quad \delta_2 G = \ldots -342260, \quad \delta_3 G = \ldots +2.494419.
\]

Further, for the three sections

\[
\begin{align*}
\mathfrak{B} &= -417950 + 937763 - 8885449 = -365636, \\
H &= +5.087942, \\
\frac{1}{2}\mathfrak{B}H &= -930167,
\end{align*}
\]

and the three equations to satisfy being

\[
\delta_1 G = 0, \quad \delta_2 G = 0, \quad \delta_3 G = -930167,
\]

we find the actual numbers leave residuals in the left-hand members of the values

\[
+0.027009, \quad +0.012600, \quad +127057. \ldots \ldots \ldots (29)
\]

These are to be brought to zero when supplemented by the proper expressions in \( e_0, q_0, q_0 \), and \( e_0 \) is dealt with above.

Now referring to the expressions for a thin lens and writing \( q = q/k \) so that for the system just computed \( q = +2.3734, q_0 = +4.8325 \), and confining attention to the forms in which \( q \) is introduced at the surfaces of the lenses, these are respectively:

**First lens**—

\[
\begin{align*}
\delta_1 y_4 &= \ldots -671321q_4, \\
\delta_2 k_4 &= \ldots +959031q_4 + 265793q_4^2, \\
\delta_2 k_4 &= \ldots +671321q_4,
\end{align*}
\]

and the rest zero;
Second lens—

\[ \delta_{1\gamma_0} = \ldots -'595833q_{10}, \]

\[ \delta_{1\kappa_0} = \ldots -'801905q_{10}'222246q^2_{10}, \quad \delta_{1\rho_0} = \ldots +'595833q_{10}, \]

\[ \delta_{1\lambda_0} = \ldots +'595833q_{10}, \]

and the rest zero.

For the second lens, the subsequent normal scheme \( O'_p \ldots F'_r \) is

\[ \{g', h'; k' l'\} = \{1, +713667; \quad \ast, 1\} \]

and by (17) of the Memoir, this gives for the second lens from \( O_p \ldots F_r \), the terms in \( q \) :

\begin{align*}
\text{Coefficient, } q_{10} &
\delta_{1\gamma_0} = \ldots -1'168126 \\
\delta_{1\rho_0} = \ldots +'425226 \\
\delta_{2\gamma_0} = \ldots \ast \\
\delta_{1\rho_0} = \ldots \ast \\
\delta_{1\rho_0} = \ldots \ast \\
\delta_{2\rho_0} = \ldots \ast \\
\end{align*}

The preceding normal scheme \( O_o \ldots O_o \) is

\[ \{g, h; k, l\} = \{+1'40346, +3'389017; \quad -'007760, +6'937860\}. \]

We see, by referring to the equations (17) of the Memoir already quoted so frequently, that in order to get \( \delta_1G, \delta_2G, \delta_3G \), we must form \( g\delta_{1\gamma_0} + k\delta_{1\rho_0} (s = 1, 2, 3) \) with these values of \( g, h, k, l \), and multiplying them respectively by

\begin{align*}
g^2 &= +'019697, \\
gh &= +'475635, \\
h^2 &= +11'485436, \\
2gk &= -'002178, \\
gl + hk &= +'947402, \\
2hl &= +47'025051, \\
kl &= -'053838, \\
l^2 &= +48'133901, \\
\end{align*}

take the sums. The values of \( g\delta_{1\gamma_0} + k\delta_{1\rho_0} \) are

\begin{align*}
\text{Coefficient, } q_{10} &
\delta_{1\gamma_0} = \ldots -'167242 \\
\delta_{1\rho_0} = \ldots +'059679 \\
\delta_{2\rho_0} = \ldots \ast \\
\end{align*}

the resulting values are

\begin{align*}
\delta_1G &= \ldots -'003424 \\
\delta_2G &= \ldots -'023006 \\
\delta_3G &= \ldots +'885540 \\
\end{align*}

\[ \quad \text{(31)} \]
In the same way, for the first lens of the corrector, the subsequent normal scheme $O'_1...O_7$ is \{\(g', h'; k', l'\)\} = \{+0.039488, +714268; -1.345856, +980009\}, which gives for the first system, between $O_1...F'_7$:

<table>
<thead>
<tr>
<th>Coefficient, (q_4)</th>
<th>Coefficient, (q_4^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta_1\gamma_4)</td>
<td>+6.58496</td>
</tr>
<tr>
<td>(\delta_2\gamma_4)</td>
<td>+4.79503</td>
</tr>
<tr>
<td>(\delta_3\gamma_4)</td>
<td>*</td>
</tr>
<tr>
<td>(\delta_1\eta_4)</td>
<td>+4.79503</td>
</tr>
<tr>
<td>(\delta_2\eta_4)</td>
<td>*</td>
</tr>
<tr>
<td>(\delta_3\eta_4)</td>
<td>*</td>
</tr>
</tbody>
</table>

The preceding normal scheme $O_0...O_4$ is

\[\{g, h; k, l\} = \{+141675, +3.326584; -208333, +2.166667\},\]

which gives

\[g^2 = +0.020072,\quad 2gk = -0.059031,\quad k^2 = +0.043403,\]
\[gh = +4.71294,\quad gl +hk = -3.86074,\quad kl = -4.51388,\]
\[h^2 = +11.066161,\quad 2hl = +14.415196,\quad l^2 = +4.694446,\]

so that with the values of \(g\delta_1\gamma_4 + k\delta_1\eta_4\), which are

\[s = 1 \quad -0.006604 +0.026897\]
\[2 \quad +0.067934\]
\[3 \quad *\]

we find the contributions of the first lens $O_1...F'_7$

<table>
<thead>
<tr>
<th>Coefficient, (q_4)</th>
<th>Coefficient, (q_4^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta_1G)</td>
<td>-0.004143</td>
</tr>
<tr>
<td>(\delta_2G)</td>
<td>-0.029340</td>
</tr>
<tr>
<td>(\delta_3G)</td>
<td>+0.906201</td>
</tr>
</tbody>
</table>

With the values \(q_4 = +2.3734\), \(q_5 = +4.8325\), the joint contribution of the two lenses in respect to the terms \(q, q^2\) would be, from these expressions,

\[\delta_1G = ... -0.033578,\quad \delta_2G = ... -3.566618,\quad \delta_3G = ... +2.136186.\]

Hence if new values of \(q_6, q_4, q_5\) are to satisfy the conditions exactly, these are determined by the equations

\[\begin{align*}
0 &= -1.58998 -0.004143 +0.005399 -0.003424 -0.000438,5 +0.060587 \\
0 &= * -0.029340 +0.126762 -0.023006 -0.010587,6 +0.369218 \\
0 &= * +0.06201 +0.297642,1 +0.885540 -0.255667,4 -2.009113 . 
\end{align*}\]
the solutions of which are

\[ a_1 = +2.390547, \quad a_8 = +4.936038, \quad e_0 = +164675. \ldots (33A) \]

If with these values of \( a_0, a_5 \) we calculate the curvatures of the two lenses from the formulae (24) of p. 49, we find that the completed approximation directs us to replace the numbers of p. 54 from which we set out by

\[ B_4 = -0.03009, \quad B'_4 = +2.048193, \]
\[ B_6 = -0.24163, \quad B'_6 = -2.612025, \ldots \ldots (34) \]

together with the value of \( e_0 \) just written down.

Turning back to p. 49, where these data from a previous approximation are set down, we see that the chief effect of the step is to restore \( B'_6 \) to the value given on p. 49, throwing the change in focal length which is demanded for achronism, in accordance with p. 53, almost exclusively upon \( B_6 \), which is a surface that contributes very little to these aberrations. The changes are thus in reality smaller than they appear. Following now strictly the plan given on p. 54, the next step is to take the new system as a whole and calculate exactly its numerical aberrations at its principal focus; it is unnecessary to give the details of this step, which contains nothing new; the following numbers show first the normal schemes from the surface \( O_6 \) up to each other point, and then the contribution of each surface to each of the coefficients \( \delta_0 G \ldots \delta_5 H \) at the principal focus \( F'_7 \).

\[ \mu = 1.5200. \]

<table>
<thead>
<tr>
<th>Surface ( O_6 ) to ( O_2 )</th>
<th>( g, h ; k, l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_6 ) to ( O_2 )</td>
<td>([+1'000000, \ldots \ldots 500000, +1'000000])</td>
</tr>
<tr>
<td>( O_6 ) to ( O_2 )</td>
<td>([+339933, +1'320133] \ldots [-300000, +1'000000])</td>
</tr>
<tr>
<td>( O_6 ) to ( O'_2 )</td>
<td>([+339933, +1'320133] \ldots [-383906, +444645])</td>
</tr>
<tr>
<td>( O_6 ) to ( O'_2 )</td>
<td>([+332255, +1'329022] \ldots [-383906, +444645])</td>
</tr>
<tr>
<td>( O_6 ) to ( O'_2 )</td>
<td>([+332255, +1'329022] \ldots [-083615, +1'645630])</td>
</tr>
<tr>
<td>( O_6 ) to ( O'_2 )</td>
<td>([+330583, +1'361935] \ldots [-083615, +1'645630])</td>
</tr>
<tr>
<td>( O_6 ) to ( O'_2 )</td>
<td>([+330583, +1'361935] \ldots [-208333, +2'166667])</td>
</tr>
</tbody>
</table>

\[ \ldots \ldots (35) \]
\( \mu = 1'5200. \)

Aberration Coefficients at \( F' \).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>+'29182</td>
<td>-'31800</td>
<td>*</td>
<td>+'63600</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>2</td>
<td>-'01094</td>
<td>+'06023</td>
<td>-'04251</td>
<td>-'19332</td>
<td>-'16509</td>
<td>+'47099</td>
</tr>
<tr>
<td>2'</td>
<td>-'04216</td>
<td>+'08433</td>
<td>-'16866</td>
<td>-'01984</td>
<td>-'67464</td>
<td>-'50801</td>
</tr>
<tr>
<td>2''</td>
<td>-'01195</td>
<td>-'02233</td>
<td>-'04922</td>
<td>+'19242</td>
<td>-'20279</td>
<td>-'121803</td>
</tr>
<tr>
<td>4</td>
<td>+'00227</td>
<td>+'00722</td>
<td>+'05341</td>
<td>-'04912</td>
<td>+'125414</td>
<td>-'04592</td>
</tr>
<tr>
<td>4'</td>
<td>+'00550</td>
<td>-'00212</td>
<td>+'13140</td>
<td>+'03502</td>
<td>+3'13954</td>
<td>+8'63910</td>
</tr>
<tr>
<td>6</td>
<td>'00000</td>
<td>'00000</td>
<td>'00000</td>
<td>-'00029</td>
<td>+'00006</td>
<td>+'21537</td>
</tr>
<tr>
<td>6'</td>
<td>-'02836</td>
<td>-'01552</td>
<td>-'69898</td>
<td>+'17309</td>
<td>-'1722893</td>
<td>+6'37874</td>
</tr>
<tr>
<td></td>
<td>+'20618</td>
<td>-'20619</td>
<td>-'77456</td>
<td>+'77396</td>
<td>-13'87771</td>
<td>+12'93224</td>
</tr>
</tbody>
</table>

\[ \delta_1 G = -'00001 \quad \delta_2 G = -'00060 \quad \delta_3 G = -'94547 \]

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-'63600</td>
<td>+'127200</td>
<td>*</td>
<td>-'2'54401</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>2</td>
<td>+'05238</td>
<td>-'28821</td>
<td>+'20342</td>
<td>+'92507</td>
<td>+7'9000</td>
<td>-'225380</td>
</tr>
<tr>
<td>2'</td>
<td>+'18850</td>
<td>-'37701</td>
<td>+'75400</td>
<td>+'08871</td>
<td>+3'01601</td>
<td>+6'74169</td>
</tr>
<tr>
<td>2''</td>
<td>+'04992</td>
<td>+'09328</td>
<td>+'20564</td>
<td>-'80386</td>
<td>+8'4719</td>
<td>5'08862</td>
</tr>
<tr>
<td>4</td>
<td>+'00103</td>
<td>+'00327</td>
<td>+'02417</td>
<td>-'02222</td>
<td>+5'6746</td>
<td>-'02078</td>
</tr>
<tr>
<td>4'</td>
<td>+'27142</td>
<td>-'10500</td>
<td>+6'48513</td>
<td>+1'72840</td>
<td>+154'95440</td>
<td>+426'39006</td>
</tr>
<tr>
<td>6</td>
<td>-'00006</td>
<td>-'00021</td>
<td>-'00153</td>
<td>+'17490</td>
<td>-'03694</td>
<td>-132'40118</td>
</tr>
<tr>
<td>6'</td>
<td>-'33991</td>
<td>-'18598</td>
<td>-8'37835</td>
<td>+2'07474</td>
<td>-206'51664</td>
<td>+76'45958</td>
</tr>
<tr>
<td></td>
<td>-'41272</td>
<td>+'41214</td>
<td>-7'0752</td>
<td>+1'62173</td>
<td>-46'37852</td>
<td>+380'00419</td>
</tr>
</tbody>
</table>

\[ \delta_1 H = -'00058 \quad \delta_2 H = +'91421 \quad \delta_3 H = +333'62567 \]

From these results we read the particulars of the field from the data given on p. 30. We have, by pp. 58, 49,

\[ f' = +5'088015, \quad \alpha = +'200000. \]

Hence, from \( \delta_3 G \), we find for the remaining spherical aberration a circle of radius 0''0004 at distance 0'00001 before the axial focus.
For faults that depend on obliquity, I shall take as standard

\[ \beta = 0'1 = \tan^{-1} 34'22''6, \]

but shall also give the results for \( \beta = \tan 30' \), and \( \beta = \tan 60' \). We have then for the radius of the comatic circle

\[
\begin{align*}
\beta &= \tan 30'. & \beta &= \tan 34'.4. & \beta &= \tan 60'. \\
-0''0042 & & -0''0048 & & -0''0083. \\
\end{align*}
\]

For the radius of the focal circle

\[
\begin{align*}
\beta &= \tan 30'. & \beta &= \tan 34'.4. & \beta &= \tan 60'. \\
+0''282 & & +0''370 & & +1''127. \\
\end{align*}
\]

For the radius of the curvature of the field, \(-162'817\); and hence for the displacement of the focal circle from the plane through the axial principal focus

\[
\begin{align*}
\beta &= \tan 30'. & \beta &= \tan 34'.4. & \beta &= \tan 60'. \\
-0'00006 & & -0'00008 & & -0'00024. \\
\end{align*}
\]

Finally, for the distortional displacement

\[
\begin{align*}
\beta &= \tan 30'. & \beta &= \tan 34'.4. & \beta &= \tan 60'. \\
+4''48 & & +6''75 & & +35''89. \\
\end{align*}
\]

It will be recalled that the linear unit is supposed to be 100 inches.\(^*\)

We conclude that spherical aberration, coma, and curvature of the field are now completely insensible, and that stars would be represented by strictly circular images of diameter 0'56 seconds at a distance of 30 minutes from the centre of the field, and 2'25 seconds at 1 degree distance. No images at present obtained with any telescope, at the middle of the field, where all obliquity-faults are absent, are sensibly less than 1 second in diameter. Hence this also is completely satisfactory up to a diameter of field of 1½ degrees, or even more. There remains distortion, which requires examination. This can be calculated precisely and applied as a correction to measures made, along with differential refraction and other unavoidable corrections. Hence, even if its amount is very considerable it can be dealt with in a way that will not vitiate the use of the telescope. It is possible, indeed, that a correction for distortion requires to be applied to other telescopes now in use, especially those in which the lenses of the object-glass are separated. It is instructive to look into the contributions of the different surfaces to the total of \( \delta \)H. The most remarkable is \(-132'4\) units from the surface (6) which is nearly a plane surface. This is an obliquity-constituent, and would be present if the surface were a perfect plane. We see by examining the normal scheme next preceding the surface (6) that the original obliquity, \( \beta \), of the ray is increased nearly

\[^*\text{Note added March 8, 1913.}--\text{It is of interest to add that these conclusions have been checked by trigonometrical calculations also, made by Mr. A. E. Conradi at the instance of one of the Referees.}\]
seven fold before impact upon this surface. It is this that produces the large coefficient.

It might be possible, with these numbers before one, to rearrange the general plan of the surfaces so as to produce a smaller value of $\delta_3 H$, but as explained above, it is not essential to do so in a telescope which is not likely to be used for exact measures over a field of more than 30 minutes radius.

We now return to the question of achromatism. We shall first verify that as far as the normal scheme goes, the achromatism which was secured for the scheme of p. 54, has not been sensibly impaired by the changes since made in the distribution of curvatures between the surfaces. Writing down only the surfaces, we have

$$\mu = 1.5352.$$

Normal Schemes for $n + \delta n$.

| Surface $O_0$ | $+1.000000$ | $-$500000, $+1.000000$, |
| Surfaces $O_0, O_2$ | $+339933$, $+1.320133$ | $-381696$, $+433886$, |
| $O_0, O_2, O'_2$ | $+332299$, $+1.328811$ | $-081365$, $+1.634860$, |
| $O_0...O'_2$ | $+330671$, $+1.361508$ | $-208547$, $+2.165477$, |
| $O_0...O_4$ | $+141569$, $+3325076$ | $-101641$, $+2.213876$, |
| $O_0...O'_4$ | $+140298$, $+3352749$ | $-002245$, $+7.074002$, |
| $O_0...O_6$ | $+140287$, $+3388119$ | $-000281$, $+4.636412$, |
| $O_0...O'_6$ | $+140283$, $+3.457665$ | $-196540$, $+2.284156$, |
| $O_0...F'$ | $+5.088015$ | $-196540$, $+2.284156$. |

By comparing this with the schemes (35), p. 58, it will be seen that the rays of different refractive index separate decidedly in the course of their passage through the instrument before they are brought together at their common principal focus. The final agreement was to be expected as it was within our control, as far as the normal schemes were concerned, but it now remains to be considered whether there is any sensible chromatic difference of aberrations; this is found by recalculating the aberration coefficients with refractive index 1.5352 in place of 1.5200. The results are as follows:

$$n + \delta n.$$

$$\delta G = -0.0018, \quad \delta_2 G = +0.00846, \quad \delta_3 G = -45888.$$

$$\delta H = +0.00844, \quad \delta_2 H = +1.39705, \quad \delta_3 H = +351.826.$$
Interpreting these, as on p. 59, we conclude that for

$$\mu = 1.5352.$$ 

Radius of Least Circle of Aberration, $-0''.007$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Value</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = \tan 30'$</td>
<td>$+0''.0599$</td>
<td>$+0''.0686$</td>
<td>$+0''.1197$</td>
</tr>
<tr>
<td>Comatic radius</td>
<td>$+0''.431$</td>
<td>$+0''.566$</td>
<td>$+1''.724$</td>
</tr>
<tr>
<td>Focal radius</td>
<td>$+4''.74$</td>
<td>$+7''.13$</td>
<td>$+37''.91$</td>
</tr>
<tr>
<td>Distortion</td>
<td>$-000159$</td>
<td>$-000240$</td>
<td>$-000638$</td>
</tr>
<tr>
<td>Displacement of focal circle</td>
<td>$-000159$</td>
<td>$-000240$</td>
<td>$-000638$</td>
</tr>
</tbody>
</table>

Radius of curvature of field, $-5''.423$.

The effect of the distortion at $\beta = '01$ will be to draw out the image into a small spectrum of length $7''.13 - 6''.75 = 0''.38$. The radius of curvature of the field is decidedly changed; but the effect of the change as shown in the corresponding displacement of the image-circle is not considerable.

It will be remarked that all these numbers run in the sense of increasing the aberrations; as there is no minimum property about the original index $1.52$, we conclude that the aberrations for smaller indices would be proportionately diminished, and we see that it would have been better to have secured exact agreement for the larger index in place of the smaller one. In estimating the effect we may, for instance, take the following values, which are the indices for Chance's *hard crown* glass:

<table>
<thead>
<tr>
<th>Ray</th>
<th>C, D, F, G,</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index</td>
<td>1.5150, 1.5175, 1.5235, 1.5284;</td>
</tr>
</tbody>
</table>

that is to say, with such a glass two-thirds of the excesses shown in the table above, over the results of p. 60, would cover all chromatic differences. There appears to be nothing in any of them that calls for a revision of the calculations.

Now let us turn to the question of the actual sizes and places of the mirrors and
lenses in respect to the passage of a ray through the instrument. Calculate from the normal schemes, p. 58, for \( b = +'20, \) and \( \beta = -'01, 0, +'01 \) respectively, the value of \( b' \) at each surface and also at the focal plane \( F' \); this will give the necessary apertures for complete inclusion of all rays from the great mirror, up to these limits of obliquity. We find as follows:

**Value of Semi-aperture.**

<table>
<thead>
<tr>
<th>Surface</th>
<th>( \beta = -'01 )</th>
<th>( \beta = '00 )</th>
<th>( \beta = +'01 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>+'200</td>
<td>+'200</td>
<td>+'200</td>
</tr>
<tr>
<td>2</td>
<td>+'055</td>
<td>+'068</td>
<td>+'081</td>
</tr>
<tr>
<td>2'</td>
<td>+'054</td>
<td>+'067</td>
<td>+'080</td>
</tr>
<tr>
<td>2''</td>
<td>+'053</td>
<td>+'066</td>
<td>+'079</td>
</tr>
<tr>
<td>4</td>
<td>-'005</td>
<td>+'028</td>
<td>+'061</td>
</tr>
<tr>
<td>4'</td>
<td>-'006</td>
<td>+'028</td>
<td>+'062</td>
</tr>
<tr>
<td>6</td>
<td>-'006</td>
<td>+'028</td>
<td>+'062</td>
</tr>
<tr>
<td>6'</td>
<td>-'007</td>
<td>+'028</td>
<td>+'063</td>
</tr>
<tr>
<td>7</td>
<td>-'051</td>
<td>0</td>
<td>+'051</td>
</tr>
</tbody>
</table>

Hence if the great mirror is 40 inches in diameter, the reverser requires to be 16'2 inches, the first face of the corrector 12'2 inches, and the last face 12'6 inches; the diameter of the image at the focal plane would be 10'2 inches.

It is necessary to verify that the corrector does not cut out any rays coming from the great mirror to the reverser. By the data on p. 54, the first face of the corrector is at a distance +'413750 beyond the surface of the great mirror. Calculating the value of \( y' \) along the ray \( y' = \beta'x' + b' \), for this value of \( x' \), where \( b', \beta' \) are taken from the normal scheme for the ray between the surfaces \( O_s \) and \( O_2 \), we have

**Value of \( b \).**

<table>
<thead>
<tr>
<th>( b = -'01 )</th>
<th>( \beta = '00 )</th>
<th>( \beta = +'01 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+'200</td>
<td>+'154</td>
<td>+'159</td>
</tr>
<tr>
<td>+'081</td>
<td>+'061</td>
<td>+'064</td>
</tr>
</tbody>
</table>

Thus the ray which just cleared the reverser on its way to the great mirror would clear the corrector on its return.

Allowing that '085 of the radius of the great mirror is unavailable the effective aperture-ratio is reduced from \( 40/508'8 = 1 : 12'72 \) to \( 36'28/508'8 = 1 : 14'05 \).

The following table shows the inclinations of the ray to the axis of the telescope between the various surfaces:
Inclination of Extreme Ray to Axis.

For \( b = +200 \), \( \beta = -0.01 \), \( \beta = 0.00 \), \( \beta = +0.01 \).

<table>
<thead>
<tr>
<th>Surface</th>
<th>( \beta = -0.01 )</th>
<th>( \beta = 0.00 )</th>
<th>( \beta = +0.01 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before surface ( O_0 )</td>
<td>(-0.6)</td>
<td>0.0</td>
<td>+0.6</td>
</tr>
<tr>
<td>Between ( O_0 ) and ( O_2 )</td>
<td>(-6.3)</td>
<td>(-5.7)</td>
<td>(-5.1)</td>
</tr>
<tr>
<td>( O_2 )</td>
<td>(-4.6)</td>
<td>(-4.4)</td>
<td>(-4.1)</td>
</tr>
<tr>
<td>( O''_2 )</td>
<td>(-1.9)</td>
<td>(-1.0)</td>
<td>0.0</td>
</tr>
<tr>
<td>( O'_4 )</td>
<td>(-3.6)</td>
<td>(-2.4)</td>
<td>(-1.1)</td>
</tr>
<tr>
<td>( O'_4 )</td>
<td>(-2.5)</td>
<td>(-1.2)</td>
<td>(-0.1)</td>
</tr>
<tr>
<td>( O''_6 )</td>
<td>(-4.1)</td>
<td>0.0</td>
<td>+3.9</td>
</tr>
<tr>
<td>( O'_6 )</td>
<td>(-2.7)</td>
<td>0.0</td>
<td>+2.6</td>
</tr>
</tbody>
</table>

The inclinations of the ray to the normals of the surface \( z_2 \) are given by \( \beta_2 + b_2 z_2 \beta' \), which may be calculated at once from the normal schemes; but note that as these include reversals for the case of a mirror we must then take in place of the latter \( \beta_2 - b_2 z_2 \beta' \):

Inclination of Extreme Ray to Normal to Surface.

For \( b = +200 \), \( \beta = -0.01 \), \( \beta = 0.00 \), \( \beta = +0.01 \).

\[
\begin{align*}
O_0 & \{ -3.4, -2.9, -2.3 \} \\
O_2 & \{ -4.8, -3.9, -3.0 \} \\
O'_2 & \{ -3.3, -2.7, -2.1 \} \\
O''_2 & \{ -3.3, -2.8, -2.3 \} \\
O_4 & \{ -3.4, -3.5, -3.6 \} \\
O'_4 & \{ -2.3, -2.3, -2.4 \} \\
O_6 & \{ -3.1, +2.1, +7.3 \} \\
O'_6 & \{ -4.1, -0.1, +3.8 \} \\
O''_6 & \{ -2.7, -0.1, +2.5 \} \\
\end{align*}
\]

Thus the greatest angle of incidence is 11° 0 degrees upon the second surface of the first lens of the corrector. This is much below what is permitted in the construction of the object glass of a refractor; we find, for example, in Steinheil and Voit's 'Handbuch,' with an aperture ratio of 1 : 12, the angle of incidence of extreme rays, originally parallel to the axes, upon the first surface if the flint-lens exceeds 15 degrees.
I would add a few remarks upon the problems presented by the construction of such a telescope, or at any rate, of its optical parts. It requires the production of a great mirror and three lenses which shall be in due relation to one another. None of the sizes or curves go outside what has already been made; and whenever a refractor is made, three of the surfaces must be turned out in agreement with the fourth. Hence there is no new difficulty in making and the problem is essentially a question of testing. The testing must be optical and not mechanical, for the former far outruns the latter in delicacy—it is said ten times. And because there are so many surfaces it would be essential to test them independently of one another. In the lenses, four out of the six surfaces are concave and spherical and can be tested with reflected light. The great mirror is neither a sphere nor a paraboloid, but its radius of curvature for different zones can be laid down, and each zone tested for agreement with this, just as in making a paraboloid. There remain then two convex surfaces, and the question of figuring the lens-surfaces to allow for inequalities of refractive index within the glass. These are matters for the skill of the maker and it would seem a not unreasonably difficult task.

I add a plan of the whole instrument and, upon a larger scale, of the reverser and corrector, and also the final specification, collected from pp. 60, 62, but making the unit 1 inch. For comparison the field of a Newtonian of the same aperture and focal length is added. It may be recalled that the displacement of the centre of the comatic circle is twice the comatic radius. For an uncorrected Cassegrain the field would be very much the same as for a Newtonian of the same aperture but of focal length equal to that of the great mirror, except in respect to curvature and distortion, see p. 41.

I would express my acknowledgments to Mr. R. W. Wrigley who helped me to perform many of the calculations.
Final Scheme.

Great mirror—
Aperture \[ 2a_1 = 40, \]
Radius of curvature \[ R_0 = -400'000, \]
\[ \epsilon_0 = +16468. \]
\[ d_1 = +132'013. \]

Reverser—
Aperture \[ 2a_2 = 16'2, \]
First surface \[ R_2 = +211'603, \]
Silvered surface \[ R'_2 = +221'289, \]
Thickness \[ t_2 = 2'000. \]
\[ d_2 = +90'676. \]

Corrector, 1st lens—
Aperture \[ 2a_4 = 12'2, \]
First surface \[ R_4 = -144'298, \]
Second surface \[ R'_4 = +48'824, \]
Thickness \[ t_4 = 1'250. \]
\[ d_4 = +0'500. \]

Corrector, 2nd lens—
Aperture \[ 2a_6 = 12'6, \]
First surface \[ R_6 = -4138'559, \]
Second surface \[ R'_6 = -38'285, \]
Thickness \[ t_6 = 1'500. \]
\[ d_6 = +71'377. \]

Focal length \[ f' = +508'802. \]
Distance of principal focus beyond surface of great mirror \[ +33'290. \]
Whole length of instrument \[ 167'3. \]

Specification of Field at \( \beta = 0'1 = \tan 34'4. \)

\[ \mu = 1'5200. \quad \mu = 1'5352. \quad [\text{Newtonian.}] \]

\[
\begin{array}{lcl}
\text{Radius of least circle of aberration} & = & 0'000 \quad -0'007 \quad 0'00 \\
\text{Radius of comatic circle} & = & -0'005 \quad +0'069 \quad +0'80 \\
\text{Radius of focal circle} & = & +0'370 \quad +0'566 \quad -0'41 \\
\text{Distortional displacement} & = & +6'75 \quad +7'13 \quad 0'00. \\
\text{Curvature of field} & = & -1/16282 \quad -1/542'3 \quad -1/508'8.
\end{array}
\]
III. The Thermal Properties of Carbonic Acid at Low Temperatures.


Communicated by Sir J. Alfred Ewing, K.C.B., F.R.S.

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PART I.—OBJECT, SCOPE, AND THEORY OF EXPERIMENTS.

The experiments described in the following paper were originally undertaken to determine the Latent Heat of Liquid CO₂ and the Specific Heats of the liquid and of the gas at temperatures below −30° C., which is the lowest temperature for which MOLLIER has calculated them, and also to check MOLLIER'S Entropy-Temperature diagram by direct experiment, as it appeared likely that the calculated results might be appreciably wrong near the limits of their range. The results of the first experiments confirmed this expectation, and it became apparent at the same time that MOLLIER'S θφ diagram could not be modified to agree with the experimental results without some further data. The investigation was therefore extended so as to include the measurement of all the quantities required for the construction de novo of a θφ diagram for saturated gas at low temperatures. Finally, by Sir ALFRED EWING'S suggestion, the range of the experiments was further extended to higher temperatures, to enable the diagram to be constructed nearly to the critical point.

The experiments made to carry out this programme were:

1. The determination of the Pressure-Temperature Curve for Saturated Vapour;
2. Three series of heat measurements, called Series I., II., and III., to determine the Latent Heat L, the Total Heat I of the liquid, and the Specific Heat of the gas;
3. A series of throttling experiments, called Series IV., to determine the Joule-Thomson effect for liquid CO₂;
4. A series of direct volumetric measurements, called Series V., to determine the Dilatation and Elasticity of liquid CO₂.

The experiments also supply data from which may be calculated: Specific Volume of saturated vapour (or its reciprocal, the Density); relative Densities of liquid CO₂ at saturation pressures; Specific Heat of liquid CO₂ either at constant pressure or at saturation pressure.

The pressure-temperature curve has been often observed during the last 50 years; the specific volume of saturated vapour and the specific heat of the liquid at saturation pressure have not been observed before below −25° C.; the latent heat, the specific heat of the gas, and the dilatation and elasticity of the liquid have not been observed before below 0° C. The total heat of the liquid, the specific heat of the liquid at constant pressure, and the Joule-Thomson effect have never been observed before. The latent heat has often been calculated, but the specific volume of the saturated vapour, on which the calculations are based, has not been observed below −25° C., and only once below 0° C., so that all calculations below −25° C. are based on extrapolations.
From the results of these six sets of experiments all the data were calculated for constructing the $\theta_{\phi}$ diagram from $+20^\circ$ C. to $-50^\circ$ C. The diagram is shown in fig. 12, p. 79, and some of the results are given in Table IX.

The experiments were carried out in the Engineering Laboratory at Oxford with a vapour-compression freezing machine, presented to the Laboratory by Brasenose College. In addition to the usual parts the apparatus includes a pair of suspended flasks by which the rate of flow of the CO$_2$ round the circuit can be measured. The following additional apparatus was made for these experiments:—Two electrically heated calorimeters, one of which always replaced the refrigerating tank of the freezing machine; several thermo-junctions for measuring the temperatures of the calorimeters and of the CO$_2$ at various points in the circuit; a graduated glass capillary tube, with regulating valves, for measuring the changes of volume of liquid CO$_2$ under varying pressures and temperatures in Series V. experiments; and a special throttle-valve for Series IV.

The gas used was commercial CO$_2$, supplied by Messrs. Barrett and Elers, Limited, of London, who have kindly informed the authors how it is made. Coke is burnt in a furnace and the products of combustion, after being washed with hot and cold water in scrubbers filled with fragments of limestone to eliminate any SO$_2$, are passed through absorbing towers filled with coke over which a stream of potash lye flows which absorbs the CO$_2$. The enriched lye is then heated in iron boilers and the CO$_2$, driven off by the heat, is compressed in compound pumps into the steel flasks in which it is sold. It is dried by passing over calcium chloride between the first and second stage of compression. The gas made in this way is said by the makers to contain no impurities, except possibly $\frac{1}{2}$ per cent. to $\frac{1}{3}$ per cent. of air and traces of SO$_2$. These traces of SO$_2$ were probably eliminated with the moisture in the special drying appliances used in these experiments, so that the only impurity left was air.

To estimate the amount of air present the gas was analysed in a modified form of HEMPEL’s apparatus specially arranged for this test. About 100 c.c. of gas was measured over mercury in a burette and then passed into the potash absorption bulb. The residue of undissolved gas (air) was then drawn back into a small burette and measured over potash solution. The apparatus was arranged so that the test could be repeated as often as desired while the residue accumulated in the small burette. In this way a sufficient quantity could be analysed to allow of an accurate determination of the small amount of air. The amount of air found was only 0.11 per cent. by volume (0.073 per cent. by weight). ANDREWS,† in his classical experiments, never was able to reduce the air in the gas he used to less than $\frac{1}{20}$ to $\frac{1}{100}$. The

* This is not strictly accurate. In working out Series V. the density of liquid CO$_2$ at one temperature is needed. BEHN’s result has been used. Any possible error in this has no appreciable effect on the result. With this exception every quantity needed has been measured.

† ‘Phil. Trans.,’ 1869, p. 581.
The general arrangement of the apparatus connected for normal working is shown diagrammatically in fig. 1. The gas enters the pump at pressure $p_2$ and is compressed to a higher pressure $p_1$. It is then condensed in the condenser at the corresponding saturation temperature $\theta_1$. From the condenser it flows as liquid through the weighing flasks to the throttle valve on the calorimeter. In passing the throttle valve the pressure falls to $p_2$, some of the liquid evaporates and the temperature falls to $\theta_2$. The mixture of liquid and gas then enters the calorimeter, where the rest of the liquid evaporates, taking up heat at the constant temperature $\theta_2$. From the calorimeter the gas passes back to the pump. In order to make sure that all the liquid has evaporated, the gas is warmed (superheated) a few degrees above $\theta_2$ before it leaves the calorimeter. The approximate $\theta\phi$ diagram* for this cycle is shown in fig. 2.

The line AB represents the expansion of the liquid through the throttle valve from $p_1\theta_1$ to $p_2\theta_2$. AB is a line of constant total heat $I$. The line BE represents the evaporation of the liquid at constant pressure $p_2$ and temperature $\theta_2$. The line EC represents the small amount of superheating from $\theta_2$ to $\theta_3$ at constant pressure $p_2$. The line CD represents the adiabatic compression in the pump from $p_2$ to $p_1$. The

line DFA represents the cooling and condensation in the condenser, both at constant pressure $p_1$.

The pressure-temperature curve was determined with the apparatus working in this way: A pressure gauge was connected to the pipe immediately after the throttle valve and a thermo-junction inserted at the same place, as shown in fig. 1. The gas is always saturated at this point so that the temperature is unaffected by radiation or conduction along the pipes. By varying the adjustment of the throttle valve a series of readings of corresponding pressures and temperatures was obtained; a summary of the observations is given in Table I. The observations were plotted and a smooth curve drawn through them. Figures taken from the smooth curve are given in Table IX. The smooth curve is copied in fig. 18, p. 95, for comparison with previous observations.
Series I. measurements were also made with the apparatus working in the normal way described above, two thermo-junctions and a pressure gauge being connected as shown in fig. 3. In this series the principal quantities measured were the rate of flow of CO\textsubscript{2} and the electrical power supplied to the calorimeter to balance the refrigeration. From these data the refrigeration, \textit{i.e.}, the heat absorbed per lb., represented by the area NBECQ (fig. 2) was calculated. This is the heat required to evaporate the liquid part of the CO\textsubscript{2} and to superheat it all from \(\theta_{2}\) to \(\theta_{3}\), the heat used in superheating being represented by the area PECQ. A series of experiments was made with different values of \(\theta_{2}\). A summary of the observations is given in Table II; the results are also given in column \(a\), Table VI.

Experiments were not made at temperatures above 20° C. owing to the increasing difficulties of manipulation. At the higher temperatures the weighing flask had to be heated to keep the pressure above the evaporation pressure. At the same time its capacity fell off rapidly owing to the great expansion of the liquid. The condenser and pump had to be run at correspondingly higher temperatures and great care exercised lest the condenser and flasks got over-full of the expanded liquid.

![Diagram](image)

**Fig. 4.**

Series II.—For this series the normal arrangement of the apparatus was slightly modified, as shown in fig. 4. The liquid CO\textsubscript{2} before reaching the throttle valve, was led first through the second coil in calorimeter I., so that its temperature was reduced to any required temperature \(\theta_{a}\), and then through calorimeter II., in which it was warmed again at constant pressure to any desired temperature \(\theta_{y}\). The quantities measured were the rate of flow, the rise of temperature of the liquid, \(\theta_{y}-\theta_{a}\), and the electric power supplied to calorimeter II. From these data the change of total heat I of the liquid at constant pressure for the range \(\theta_{2}\) to \(\theta_{y}\) was calculated. Two sets of experiments were made, one at 700 lbs. per sq. in. pressure and one at 900 lbs. per sq. in. pressure. A summary of the observations is given in Table III. The observations were plotted and smooth curves drawn through them. Figures taken from the smooth curves are given in Table VII. This series of experiments does not determine the absolute values of I, but only differences; the zero of the I scale was determined later, see p. 80. The slope of the I curve is the specific heat of the liquid at constant pressure. Values of the specific heat deduced from the slope of the curves are given in Table VII.
Series III.—For this series the normal arrangement of the apparatus was again slightly modified, as is shown in fig. 5, by inserting calorimeter II. between calorimeter I. and the pump. The gas leaving calorimeter I. at a temperature $\theta_3$ was warmed to any desired extent in calorimeter II. The quantities measured in this series were the rate of flow of CO$_2$, the rise of temperature of the gas in calorimeter II., and the electrical power supplied to calorimeter II. From these data the specific heat of the gas at constant pressure was calculated. A series of measurements was made with different values of $\theta_2$. The results are shown in fig. 6, where the specific heat of the gas near the limit curve* is plotted against the pressure.

Combining the results of Series I., II., and III., an approximate value of the latent heat may now be calculated. If we neglect the complications introduced by the changes of volume of liquid CO$_2$ (or what is equivalent, if we assume, as a first approximation, that the limit curve coincides with the constant-pressure curves) then the difference of total heat $I_2 - I_1$ from $\theta_2$ to $\theta_1$ is represented in fig. 2, p. 71, by the area RGAM.

* The limit curve is the boundary of the area on the diagram representing saturated vapour, separating it on the one side from the area representing liquid and on the other from the area representing superheated gas. The two sides of the curve are called the “liquid-limit curve” and the “gas-limit curve”; they meet at the critical point.
Also, by a well-known property of constant-pressure lines,

\[ \text{Area RGBN} = \text{RGAM} = \text{I}_2 - \text{I}_1, \]

which can be read off the I curves.

Taking any experiment of Series I., we have

Area BQ, given by Series I. (column \( \alpha \), Table VI.); 

,, EQ, which may be calculated from the specific heat of the gas obtained by Series III. (column \( \gamma \), Table VI.); 

,, GN = I_2 - I_1, read off the I curves obtained by Series II. (column \( \delta \), Table VI.).

Whence the latent heat, \( L = BQ - EQ + GN \), may be found.

On the same assumptions an approximate \( \theta \phi \) diagram may be constructed. Starting at the zero-point on the \( \theta \phi \) diagram (\( \theta = 273^\circ \) C. abs., \( \phi = 0 \)) plot, step by step, the constant-pressure line corresponding to the curve of I, remembering that on the \( \theta \phi \) diagram the area under each element of the curve is equal to the corresponding difference of I; this is quickly done since the curve is almost straight. Since we are neglecting changes of volume of the liquid, this curve will coincide with the liquid limit curve (as it practically does in the Steam diagram). To obtain the gas-limit curve, mark off the values of \( L/\theta \) for various temperatures, starting at the liquid-limit curve and measuring to the right; joining up the points
so found we have the gas-limit curve. The diagram might be completed by adding the I lines, &c., but it will be convenient to consider first the modifications required to allow for the changes of volume of the liquid CO₂ which have so far been neglected.

Fig. 7 represents the same diagram as fig. 2 with the addition of some lines to show the effects of the dilatation and elasticity of the liquid.

Let AHK be the constant-pressure line through A, in the liquid area,
Let GA represent the limit curve as before,
Let GK be the constant I line through G, meeting AHK in K.

The heating of the liquid CO₂ in Series II. experiments is now represented by the line KA, instead of by GA.

The true value of L is found as follows:

\[ L = GN + BQ - EQ \]

\[ = GN + BQ - EQ \]
as before; also BQ and EQ are given, as before, by Series I. and III. experiments, but GN is no longer \( I_2 - I_4 \). We now have

\[ GN = I_B - I_A, \]
\[ = I_A - I_B, \]
since AB and GK are constant I lines,
\[ = (I_A - I_B) + (I_B - I_K), \]
\[ = (I_A - I_B) + (I_B - I_K), \]
using suffixes to refer to the temperatures \( \theta_1, \theta_2, \) and \( \theta_4. \)

\( I_A - I_B \) is read off the I curve as before, but \( I_3 - I_4 \) can only be read off the curve when \( \theta_4 \) is known. The quantity \( I_3 - I_4 \) may be regarded as a small correction to be applied to the approximate value of L to allow for the elasticity and dilatation of the liquid.

Series IV. experiments were made to determine the difference of temperature \( \theta_3 - \theta_4 \) between H and K, i.e., the Joule-Thomson effect for the pressure drop \( p_1 - p_2 \). It was observed directly by measuring the change of temperature as the liquid passed through a throttle valve. The arrangement of the apparatus is shown in fig. 8.

A summary of the observations is given in Table V. The observations are plotted in fig. 9 and a smooth curve drawn through them. The values of \( I_3 - I_4 \) calculated from
this curve for each of Series I. experiments are given in column 8 of Table VI., and the corrected values of $L$ in the last column. These values of $L$ were plotted and values taken from the smooth curve are given in column 5 of Table IX.

If we proceed now to the construction of the true $\theta \phi$ diagram a further difficulty is met with. If a constant-pressure curve is drawn as before, using the values of $I$ already obtained, the result is the curve $KA$. The difficulty is to draw the true limit curve $GA$, which was previously assumed to coincide with $KA$. The authors have been unable to devise any direct experiment to fix the position of the limit curve relatively to the constant-pressure curve $KA$, and an indirect method had to be used. The fundamental thermodynamic equations give the well-known equation

$$\frac{d\phi}{dp} = -\frac{d\psi}{d\theta}.$$

Let $H$, fig. 7, be the point $p_1, \theta_2$; then

$$HG = \delta \phi = \int_{p_1}^{p_2} \frac{d\psi}{d\theta} dp.$$

Thus the distance $\delta \phi$ between the limit curve and the constant-pressure curve can

be calculated from the temperature coefficient of the liquid. This coefficient* does not appear to have been determined below 0° C., so the experiments called Series V. were undertaken to measure it. **AMAGAT**'s (5) results give a few values above 0° C.

* The specific volume of the liquid at low temperatures for points on the limit curve is known. This enables the rate of change of volume with temperature, along the limit curve, to be calculated, but what is required is \( \frac{dv}{d\theta} \), which is equal to

\[
\frac{dv}{d\theta} + \frac{dv}{dp} \frac{dp}{d\theta},
\]

where \( \frac{dv}{d\theta} \) and \( \frac{dp}{d\theta} \) are taken along the limit curve.
P.V. isothermal curves for liquid CO₂.
(Mean results)

Area scale:
1 sq. cm = 0.00167 th.:u.

Points: Δ and dotted curves show Amagat's observations.

If the area between the $\theta_1$ and $\theta_2$ curves is bounded at the side by the ordinates at pressures $p_1$ and $p_2$, then the area is

$$(\theta_1 - \theta_2) \int_{p_1}^{p_2} \left( \frac{dv}{d\theta} \right)_p dp,$$
and therefore

\[
\frac{\text{area}}{\theta_1 - \theta_2} = \int_p \left( \frac{dv}{d\theta} \right) dp = \delta \phi \text{ between pressure curves } p_1 \text{ and } p_x.
\]

Similarly, if the area is bounded by the limit curve and the ordinate at \( p_1 \), then

\[
\frac{\text{area}}{\theta_1 - \theta_2} = \delta \phi \text{ between limit curve and pressure curve } p_1.
\]

\( \delta \phi \) corresponds in each case with the mean temperature \( \frac{1}{2} (\theta_1 + \theta_2) \).
Values of $\delta \phi$ between the 700-lb. and 900-lb. pressure curves and the limit curve were calculated in this way and are given in Table VIII. From this table we see that the 700-lb. and 900-lb. pressure curves cut the 0° C. temperature line at $\phi = -0.024$ and $\phi = -0.049$ respectively. These two points serve as the starting points for plotting the two constant-pressure curves (700-lb. and 900-lb.) on the $\theta \phi$ diagram.

The corrected $\theta \phi$ diagram may now be constructed. The result is shown in fig. 12. This diagram was drawn as follows:—Starting at the points just found, the 700-lb. and 900-lb. pressure curves were drawn as before, remembering that the area under each curve on the $\theta \phi$ diagram between any two temperatures equals the difference of I for the same temperature range. The liquid-limit curve was then set off on the right of the pressure curves by plotting the small values of $\delta \phi$ given in Table VIII.

The gas-limit curve was then plotted by measuring off the values of $L/\theta$, taken from Table VI., to the right of the liquid-limit curve.

To plot the constant I lines it is first necessary to find a starting point. At the origin ($\theta = 273, \phi = 0$) the value of $I_0$ is given by the equation $I_0 = A \rho v$, where

- $p$ is the saturation pressure at 0° C. = 508 lbs. per sq. in. = 73,200 lbs. per sq. ft.,
- $v$ is the volume of 1 lb. of liquid = ’0173 cub. ft.,
- $A = 17300$; therefore $I_0 = 905$ thermal unit.

The point on the 700-lb. curve having the same I was then calculated as follows:—

The change of temperature at 0° C. is taken from the curve, fig. 9, p. 76, viz., 24° C. per 100 lbs. difference of pressure. The difference of pressure is 700 − 508 = 192. Therefore the temperature of the required point is 24 × 1.92 = 46° C. In other words, I = 905 at a point +46° C. on the 700-lb. pressure curve. Similarly, I = 905 at a point +94° C. on the 900-lb. pressure curve.

Having found in this way the true value of I for one point on each of the pressure lines, we can mark the true zero on the scale of the I curves, so that they shall give absolute values of I instead of only differences (see p. 72). Using the new scales, the points on the 700-lb. curve corresponding to I = 0, −5, −10, −15, −20, and −25 were marked off on the $\theta \phi$ diagram; also the points on the 900-lb. curve corresponding to I = +5, +10, and +15. The corresponding points on the limit curve were then calculated from the difference of pressure multiplied by the rise or fall of temperature, given in fig. 9. Having found these points, the rest of the I curves within the saturated area are easily constructed. Draw a horizontal (constant temperature) line through one of the points, say, where I = 0, on the limit curve. This is at temperature −1° C. = 272° abs. Along this line, starting at the limit curve, mark off a series of points, equally spaced, at distances $\delta \phi = 2\frac{1}{2}$ apart. These will be points on the +5, +10, &c., I lines.

* Bern's (6) value of the density of liquid CO₂ at 0° C. is 0.925.
THERMAL PROPERTIES OF CARBONIC ACID AT LOW TEMPERATURES.

Five horizontal lines were divided in this way and the I lines drawn through them. The values of I for the liquid, given in Table IX., were found by interpolation. The values of I given for the gas were obtained by adding the corresponding values of L to the values of I for the liquid.

Approximate constant-pressure lines in the dry area may be drawn, if the specific heat is assumed to be constant. They are logarithmic curves given by the equation

$$\frac{d\phi}{d\theta} = \frac{\sigma}{\theta},$$

where $\sigma$ is the specific heat of the gas at constant pressure.

These curves are drawn in the diagram as straight lines, since for the short length shown the curvature is imperceptible. The values of $\sigma$ are taken from the curve, VOL. CCXIII.—A.
fig. 6, p. 73. They agree closely with Mollier's curves, as may be seen in fig. 13 (p. 81), where a few of each (at different pressures) are drawn for comparison.

The four dryness curves on the diagram were drawn by dividing the distances between the limit curves into quarters. This completes the construction of the diagram.

**PART II.—DETAILED DESCRIPTION OF THE APPARATUS AND METHOD OF CARRYING OUT THE EXPERIMENTS SERIES I., II., AND III.**

The compressor is a single acting pump made by Messrs. J. and E. Hall, of Dartford. It is driven by a variable speed electric motor. The piston-rod gland is formed of a pair of copper-leathers, between which oil is forced by an auxiliary piston, thus no leakage of CO₂ takes place, but a little oil enters the cylinder, and is pumped over with the CO₂; it is mostly caught in an oil separator, but a trace of oil is carried round the whole circuit with the CO₂. Under ordinary conditions the pump runs cold, but when working under the abnormal conditions of some of the tests it ran hot and gave trouble till a water-jacket was fitted round the cylinder.

The condenser is a coil of pipe in a tank through which cooling water flows.

The drying flask is a steel flask containing a little phosphorus pentoxide. The gas is led in by a pipe leading nearly to the bottom of the flask and leaves by a pipe from the top. A few ounces of P₂O₅ were put in the bottom of the flask and renewed from time to time. When the apparatus was first tried great trouble was experienced with moisture which collected and plugged the throttle valve with ice. The whole apparatus had to be thoroughly dried out and all the gas dried by passing it through calcium chloride in the drying flask before the difficulty was got over. After this all fresh charges of gas were passed through a small flask filled with calcium chloride before they entered the apparatus, and the above described drying flask was kept in circuit to eliminate any traces there might be left. Some of the oil carried round by the CO₂ collected in this flask.

The weighing apparatus (see fig. 1, p. 70) consists of two steel flasks, each capable of holding 40 lbs. of liquid CO₂; both were originally hung on spring balances. Each flask has valves at the top and bottom so that they may be alternately filled and emptied. The connections to the flasks are made of coils of copper pipe, flexible enough to allow of a small vertical motion. The spring balances were calibrated to allow for the stiffness of these coils. This arrangement had certain defects and was subsequently modified. In order to be sure that no CO₂ passed unweighed, it was necessary to stop the supply of CO₂ from one flask before starting it from the other; this inevitably caused a momentary variation in the rate of flow. There was also some doubt as to the effect of the weight of CO₂ in the coils of pipe connected to the flasks, which might be full or empty at the moment of weighing. The spring-balances were divided in pounds, and tenths of a pound could be roughly estimated.
The readings, however, had to be made in haste, and the probable error in the weight of CO₂ might amount to nearly 1 per cent. The balances had to be frequently recalibrated, as the stiffness of the pipes was found to change gradually. Only three of the 17 observations in Table VI., viz., those at -26¹, -14⁹, and -8⁶, were made while working in this way.

To get over these defects the following modification was made:—One flask was hung on a steelyard, the arm of which was allowed only a very minute movement. When the arm fell it made an electric contact and rang a bell. This arrangement was found to be so sensitive that it would turn with '01 lb. When the arm fell it was raised again by hanging a weight (usually 1 lb.) on the flask. When another pound of CO₂ had passed out of the flask the bell rang again and another weight was hung on, and so on. The increased sensitiveness of this arrangement made it possible to record the rate of flow of CO₂ accurately at short intervals and to complete the whole test with one flask full or less of CO₂, for as small a quantity as 10 lbs. could be weighed to ½ per cent. No calibration or allowance for the copper pipes is necessary. A simple dash-pot made of a disc of tin in a vessel of oil got over all difficulties due to vibration without reducing the sensitiveness.

The calorimeters are tin-plate tanks containing coils of copper pipe and electric heaters; the tanks are lagged on the outside to prevent the inflow of heat. The larger calorimeter, fig. 14, contains two copper coils and the smaller one a single coil. The coils can be connected in different ways for the different series of tests. Calcium chloride brine was used to fill the tanks for the first experiments, but was replaced later by methylated spirits, which answered much better. Special care was taken in the design of the calorimeters to provide a definite path for the circulation of the liquid, which was maintained by a screw propeller driven by an electric motor. The large calorimeter was originally lagged with 2 inches of slag wool; this was found to be insufficient and 2 inches of felt were added, covered by varnished calico to keep out the moisture. A wooden top was fitted, covered by felt and calico. The small calorimeter, which was completed after experience had been gained with the larger one, was wrapped in 2½ inches cotton wool surrounded by about 2 inches slag wool, all contained in a wooden box. The cover was formed of 3 inches of wood. Several sorts of heater were tried and failed;
finally coils of No. 16 S.W.G. Eureka wire, insulated with vulcanized indiarubber and wound directly on to the coils of the evaporating pipe, were tried; these answered perfectly. Each coil was about 50 yards long, had a resistance of about 10 ohms, and would absorb 1000 watts, taking 10 amperes from the 100-volt power mains. There are three such coils in the larger and two in the smaller calorimeter.

*Measurement of the Electrical Heat.*—The electrical power entering the calorimeters was calculated from the measured resistances of the heating coils and the observed E.M.F. across their terminals. The E.M.F. was measured by means of a Siemens millivoltmeter with fine pointer and mirror, which was calibrated against a cadmium cell with N.P.L. certificate. The scale is divided in single volts and $\frac{1}{10}$ volts can be accurately estimated. Two of the coils in the large calorimeter were always connected, when in use, on the full supply voltage (100), and the third was used in series with an adjustable resistance. The voltaneter has a two-way switch so that the two E.M.F.'s could be read successively. Readings were taken every minute throughout the tests. At the full voltage of 100 an error of 1 volt means an error of 0.2 per cent. in the power. At the lowest readings, 30 volts, an error of 1 volt means an error of 6 per cent. in the power. The resistance of each heating coil was measured before the tests were begun by a bridge which was checked against a standard ohm with N.P.L. certificate. The resistances were measured again after the tests were completed and had not altered appreciably. No temperature correction was made as the coils were made of Eureka, but corrections were made for the resistance of the leads.

*The temperature measurements* were all made with Eureka-copper thermo-couples. The couples were made of No. 22 gauge double cotton and indiarubber-covered Eureka and copper wire, all cut from the same coils, soldered together at the ends. The couples used in the baths for measuring the temperature of the circulating liquid were put into rubber tubes, the bare ends projecting about 3 inch. The couples used for measuring the temperature of the CO$_2$ were held in the special fittings shown in fig. 15, so that the bare wires projected into the CO$_2$ about 1$\frac{1}{2}$ inches. The wires were carried through the gunmetal plugs in fine rubber tubing (bicycle valve tubing). The holes in the gunmetal were tapered and small brass beads were soldered on the wires, so that when the wires were drawn back the beads jammed in the holes. This simple device made an insulated joint which was gas-tight under the highest pressures used (1100 lbs. per sq. inch).

Preliminary calibrations showed that the relation between the E.M.F. of the
Thermo-junctions and temperature could not be satisfactorily represented by the empirical formula often used, viz., \( \log E = n \log t + m \). A calibration curve was therefore drawn, points on it being obtained as follows. For temperatures between +100° C and 0° C. the junction was compared with a standard mercury thermometer with N.P.L. certificate.

At -20° C. it was compared with a mercury thermometer which had been verified at that temperature by the N.P.L. The melting-point of mercury was then observed. The mercury was purified by dropping it through nitric acid, washing it in water, drying it at 120° C., and finally distilling it in vacuo. About 1 1/2 lbs. of the purified mercury was put in a glass vessel and frozen by packing it with CO₂ snow. The mercury was then gradually melted and the temperature of the melting-point observed. There was no difficulty in keeping the mercury half-melted and half-frozen for any length of time desired. The melting-point was assumed to be -38°/80 C.* A calibration curve was drawn through these points and extrapolated to -50° C. As this extrapolation was open to doubt a further point at -50° C. was afterwards obtained by comparison with a platinum-resistance thermometer which had been carefully calibrated against the standard thermometer. The new point obtained in this way fell within \( \frac{1}{2} \)° C. of the curve, thus confirming it satisfactorily. The curve is believed to be correct to \( \frac{1}{2} \)° C.

To maintain the other junctions of the wires at a steady known temperature they were immersed in a large tin of paraffin oil, well jacketed with slag wool, fitted with a calibrated thermometer and lens so that the temperature of the "cold junction" (in our case usually the warmer of the two) could be read to \( \frac{1}{10} \)° C. The oil was stirred by blowing air into it. An incandescent lamp was placed in the oil, so that the oil could be warmed to approximately atmospheric temperature. The temperature of the oil was read several times during each test and rarely varied more than \( \frac{1}{10} \)° C.

The E.M.F. produced by the thermo-couples was measured against the standard cadmium cell by a potentiometer with twenty 1-ohm coils and a gilt manganin slide-wire, 1 m. long, and of just over 1-ohm resistance. Special precautions were taken to avoid thermo-electric effects. The various thermo-junctions could be switched on in turn to the potentiometer by a two-pole six-way selector switch, designed to avoid thermo-electric effects. The potentiometer was sensitive enough to measure temperature differences of \( \frac{1}{10} \)° C. Such accuracy was of use when measuring the small rise or fall of the temperature of the bath during the run; also in Series IV. experiments, which depend on small differences of temperature, and in measuring the slow temperature rise during radiation tests.

The pressure of the gas was measured by steel tube Bourdon gauges made by Messrs. Schaeffer and Budenberg with specially fine needles and fine scale-divisions. They were calibrated by means of a dead-weight testing machine, in which a dead-

* CHAPPUS, 1900, quoted KAYE and LABY, p. 48. Dr. J. A. HARKER, of the N.P.L., has informed the authors that 38°/86 C. is probably a more accurate figure.
weight rests on a plunger in an oil cylinder. There was some doubt as to the
effective area of the piston, since it was not exactly uniform in diameter, and there
was a small clearance between it and the cylinder. In order to clear up this point
the testing machine was checked against a mercury column about 1 m. high. The
value of the effective area of the plunger was found in this way; it only differed by
\( \pm 0.2 \) from the maximum measured area. The testing machine only gave pressures up
to 400 lbs. per sq. inch. Up to this pressure the gauges showed a practically
constant error, and it was assumed that the error remained the same at the higher
pressures.

The gauge used for the pressure-temperature curve, and for experiments where
accurate high-pressure readings were needed, was subsequently calibrated for its
whole range by the N.P.L.; the results agreed closely with those obtained by the
authors. In a few of the experiments, where accurate low pressures were needed, a
low-pressure gauge was used, which was calibrated over its whole range. The
pressures are believed to be correct to about 1 lb. per sq. in.

Adjustments.—Before beginning any test the apparatus was run for a considerable
time while the conditions were adjusted to what was required; the test was not
begun until a steady régime had been attained and all the parts had reached steady
temperatures; moreover, unless a test was completed without anything more than
trilling changes of any of the conditions, the results were discarded. The conditions
were adjusted by regulating the speed of the pump, opening or closing the throttle
valve, and switching on more or less electrical power to the calorimeters. While the
adjustments were being made the flask B was emptied and the flask A filled, so that
before the actual test began the CO\(_2\) was circulating through A, which was full. The
potentiometer and the temperature of the cold junction were also adjusted.

As soon as everything was ready, the valve on the top of flask A was closed and
that on B opened, and the regular readings of all the instruments was commenced.
In most tests these readings were made every minute. Each time the weighing bell
rang the time was entered to the nearest second—the first ring marking the time of
start of the test, which usually continued till the flask was almost empty.

In Series I. the apparatus was connected as shown in fig. 3, p. 71, the object being
to measure the heat represented by the area NBCQ (fig. 7, p. 74) for a series of tempe-
raturess \( \theta_s \), ranging from the highest to the lowest attainable. To keep the rate of flow
of CO\(_2\) within convenient limits the pump was run as slowly as possible for the higher
values of \( \theta_s \) and as fast as possible for the lower values of \( \theta_s \). The temperature \( \theta_s \)
was not directly measured, but was deduced by the pressure-temperature curve from
the pressure shown by the gauge connected to the pipe leaving the calorimeter; the
throttle valve was adjusted so as to keep this pressure steady at the figure selected
for each test.* The electrical power was adjusted so as to keep the temperature of

* If the percentage of air present was not always approximately constant, this method of estimating
\( \theta_s \) is liable to error.
the bath a few degrees above \( \theta_2 \), so as to make sure that the gas leaving the calorimeter was slightly superheated. This was checked by the direct measurement of the final gas temperature \( \theta_3 \) by a thermo-junction in the pipe.

During the test the following observations were made:

- Times when the weighing bell rang;
- E.M.F. on each heating coil, observed every minute;
- Pressure of gas leaving coil, ""; Temperature \( \theta_1 \) of liquid \( \text{CO}_2 \) before the throttle valve, every three minutes;
- "" of gas \( \theta_3 \) leaving calorimeter, every three minutes;
- "" bath, every three minutes;
- "" cold junction, several times during test;
- "" atmosphere, "".

During the test small adjustments of the throttle valve and of the electric power were made so as to keep the pressure and bath temperature as constant as possible. As an example, the complete records for one experiment are given in Tables XI. and XII. The times when the weighing bell rang were plotted as a check on the uniformity of the rate of flow of \( \text{CO}_2 \) during the experiment. The initial temperatures of the liquid \( \text{CO}_2 \) were plotted to obtain the true mean.

In Series II. the apparatus was connected as shown in fig. 4, p. 72, the object being to measure the total heat \( I \) of the liquid, i.e., the heat represented by the area SKAM, fig. 7, p. 74, for a series of ranges of temperature. The speed of the pump was settled as in Series I. The electrical power entering the large calorimeter was adjusted so as to keep its temperature a few degrees below \( \theta_2 \), so that the liquid \( \text{CO}_2 \) might be cooled to the desired temperature \( \theta_4 \). The electrical power put into the small calorimeter was adjusted so as to keep it at the selected temperature \( \theta_1 \). The throttle valve was adjusted so as to keep the evaporation temperature a little below the temperature of the large calorimeter, but this temperature was of no importance in this series.

During the test the following observations were made:

- Times when the weighing bell rang;
- E.M.F. on each heating coil in calorimeter II., observed every minute;
- Temperature \( \theta_1 \) of liquid \( \text{CO}_2 \) entering calorimeter II., every three minutes;
- \( \theta_1 \) "" leaving ""; "" bath, calorimeter II., observed every three minutes;
- "" cold junction, several times during test;
- "" atmosphere, "".

No observations were entered for the large calorimeter, which was only used as a cooler for the liquid, but the power was adjusted as required to keep its temperature constant.

As an example, the complete records for one experiment are given in Tables XIII. and XIV.
In Series III. the apparatus was connected as shown in fig. 5, p. 73, the object being to measure the specific heat of the gas at various temperatures near the saturation points. The adjustments of the pump, throttle valve, and electrical power for the large calorimeter were made exactly as in Series I. The electrical power put into the small calorimeter was adjusted so as to keep it at a steady temperature, a moderate amount above \( \theta_0 \). It was not possible to start heating the gas exactly at \( \theta_0 \), but \( \theta_0 \) was kept as close to \( \theta_0 \) as possible, so that the range through which the gas was heated began only a few degrees above the saturation temperature. The actual ranges are shown in lines 4 and 5, Table IV. The last line gives the mean specific heat for this range of temperature.

During the test the following observations were made:

Times when weighing bell rang;
E.M.F. on each heating coil in calorimeter II., observed every minute;
Temperature \( \theta_3 \) of gas entering calorimeter II., every three minutes;
" " " leaving " " " " " " " ";
" " " bath, calorimeter II., every three minutes;
" " " cold junction, several times during test;
" " " atmosphere, " " " " " ".

As an example, the complete records for one experiment are given in Tables XV. and XVI. The times when the weighing bell rang were plotted as a check on the uniformity of the rate of flow of \( \text{CO}_2 \) during the experiment.

Corrections.

Before making use of the data obtained in the tests, it is necessary to consider the effects of differences between the actual and theoretical cycles and also the corrections for radiation, conduction, and change of temperature of the calorimeter during the test.

Differences between the Actual and the Ideal Cycle.

(i.) Friction in the evaporation coil produces a small difference of pressure between the two ends. The evaporation, therefore, should not be represented by the constant-pressure line BE (fig. 7), but by a curved line starting a little above B and falling to E. It is easy to show that this has no effect on the heat absorbed, which is always \( I_B - I_A \), and is accurately represented by the rectangle NBEP.

(ii.) The vapour is moving in the pipe with some velocity and consequently possesses kinetic energy. A simple calculation shows that the kinetic energy is always small enough to be neglected.

(iii.) The compression in the pump is not adiabatic and there are other deviations in the condenser, but as this part of the cycle is not included in the measurements they have no effect.
(iv.) Change of Temperature of the Bath during a Test.—The temperatures of the bath at the beginning and end of a test were accurately determined by taking several observations of temperature at short intervals before and after the actual moment of the start or finish and plotting them. The temperature at the actual moment of start or finish was then read from the graph. Any change of temperature of the bath showed that the electric heat supplied had been slightly too much or too little. The balance of heat (excess or shortage) is simply the temperature rise or fall multiplied by the water equivalent of the calorimeter. The water equivalent was determined by a simple heating experiment when the bath was at approximately the atmospheric temperature so that radiation could be neglected. The results obtained in this way are not quite accurate since the heat capacity of the lagging varies with the rate of heating.

Radiation.

A number of experiments were made to determine the rate at which heat entered the calorimeters from the surrounding air before concordant results were obtained. In the end good results were arrived at and the radiation was found to be proportional to the difference of temperature between the calorimeter and the atmosphere. The rate for the large calorimeter was 83° C. per hour with a temperature difference of 40° C. The water equivalent being 97.5 this radiation corresponds to $83 \times 97.5 = 81$ thermal units per hour for 40° C. difference of temperature. The small calorimeter rate was 1°92 C. per hour or $1.92 \times 15 = 28.8$ thermal units per hour for 40° C. difference of temperature. These figures include the mechanical work put in by the stirrer motors, which in the small calorimeter was measured and found to be about 2°2 thermal units per hour.

Conduction along the Pipes.

The calorimeter coils, and most of the connecting pipes, were made of copper, $\frac{3}{4}$ inch external diameter, $\frac{1}{4}$ inch internal diameter. Such a pipe would conduct about 11 thermal units per hour with a temperature gradient of 40° C. per foot. To minimise conduction, the small calorimeter had a piece of thin-walled steel pipe inserted in both the ingoing and outgoing connections which would conduct heat at about one-sixth of the above rate. All the pipes were well lagged. When the apparatus is working it can easily be shown that there can be only a very small temperature gradient beyond the thermo-junctions so that no appreciable conduction can take place. Temperature measurements along the pipes confirmed this. When the vapour is not circulating the conditions are not quite the same and a small amount of conduction takes place. This conduction makes the apparent radiation rather too large, so that the corrections applied for radiation are a little too large. An approximate estimate shows that the error introduced is not greater than 3 thermal units in the value of L at the lowest temperature and that it will not be appreciable above $-30°$ C.
PART III.—DETAILED DESCRIPTION OF THE APPARATUS AND METHOD OF CARRYING OUT THE EXPERIMENTS SERIES IV. AND V.

In Series IV. the apparatus was arranged as shown in fig. 8, p. 75, the object being to measure the Joule-Thomson effect, i.e., the change of temperature $\theta_3 - \theta_4$ (fig. 7), corresponding to a change of pressure from $p_1$ to the pressure on the limit curve $p_x$. In the actual tests the pressure could not be allowed to fall quite down to the limit curve for fear of introducing errors due to the commencement of evaporation. To get the total $\theta_3 - \theta_4$, the observed difference of temperature was increased in the proportion of the observed drop to the total drop of pressure. In fig. 8 the pressure drop is measured by the gauges $a$ and $b$ and the temperature change by the thermo-junctions A and B. The liquid CO$_2$ from the weighing flask first passes through the inner coil of the calorimeter and is there cooled to any desired temperature. It then flows in order past:

- Gauge $a$ and thermo-junction A.
- Throttle valve $V_1$.
- Thermo-junction B and gauge $b$.
- Throttle valve $V_2$.
- Outer coil in the calorimeter and gauge $c$.

The CO$_2$ is liquid up to the second valve, $V_2$.

The pressure of the liquid up to the valve $V_1$ is $p_1$. The valve $V_1$ is adjusted so as to allow it to drop to a pressure $p_x$ a few pounds above $p_2$.

The valve $V_2$ is adjusted so as to allow it to drop from $p_2$ to $p_x$, which is the saturation pressure corresponding to the temperature of the liquid. The difference of temperature $\theta_A - \theta_B$ between A and B is the Joule-Thomson effect corresponding to the drop of pressure $p_1 - p_x$. Therefore

$$\theta_4 - \theta_3 = \frac{p_x - p_2}{p_2 - p_1} (\theta_A - \theta_B).$$

In the first series of experiments the valve $V_1$ was a large bronze hydraulic valve, and the thermo-junctions were inserted in the gunmetal fittings shown in fig. 15, which were connected to the valve by pipes about 6 inches long. The whole apparatus was well lagged, but the amount of heat which leaked in when the temperature was low was sufficient to raise the temperature of the CO$_2$ to an extent which was large compared with the small temperature change which had to be determined; this leakage was allowed for by observing the temperature change due to the leakage only, when the valve $V_1$ was full open, and subtracting this from the temperature change when $V_1$ was throttled, the rate of flow being kept the same in the two experiments. Conduction along the pipe between A and B has no effect. Each observation was repeated a number of times and the means taken. It was not
THERMAL PROPERTIES OF CARBONIC ACID AT LOW TEMPERATURES.

found possible to make observations at temperatures below -30° C., and even at this temperature the results were open to criticism as being small differences between large measurements, see Table V. (first part).

To get reliable results it was, therefore, necessary to design a special apparatus, combining two thermo-junctions and a throttle, which should not allow of appreciable radiation. Experience with the original apparatus showed that the throttle had to be adjustable to enable the required pressure ranges to be obtained. After one or two experiments the apparatus shown in fig. 16 was made and answered perfectly.

The body of the valve is vulcanite, the gland and screw are brass, but the screw is insulated from the passage through the valve by an extension rod of vulcanite. A minute brass valve is inserted under the extension rod, and is the only metal encountered by the CO₂ in passing through the valve. The two thermo-junctions are held in vulcanite plugs inserted in the steel tubes on either side of the valve. They project inside the inner vulcanite tubes, shown in the figure. The outer space forms a jacket of CO₂ at approximately the same temperature as the inner space—which is entirely protected by the vulcanite tubes from any external influence. Vulcanite is one of the best thermal insulators—the specific conductivity is given by Kaye and Laby as '00042. The whole apparatus is held together by an iron yoke, and can be taken to pieces in a moment by slackening one of the set screws at the end. The

Fig. 16.

A. Ebonite valve body.
B. Packing.
C. Brass gland.
D. " valve spindle.
E. " valve.
F. Ebonite distance piece.
G. Brass cap.
H. Steel tube.
K. Ebonite tube.
L. " plug.
M. Copper gauze plug.
N. Inlet.
O. Outlet.
P. Iron yoke.
R. Vulcanite block for control exp. H.
four joints between the steel tubes and the vulcanite pieces were perfectly tight under the maximum pressure used, 900 lbs. per sq. in. The bore of the valve is \( \frac{1}{8} \) inch diameter and the passages are bell-mouthed on both sides. A plug formed of rolled copper gauze serves to dissipate the kinetic energy of the jet issuing from the valve. The time occupied in the passage from one thermo-junction to the other, at the slowest rate of flow, was only a fraction of a second.

The thermo-junctions were connected to the selector switch (p. 85) in such a way that the temperature of either junction, or the difference between the two, could be measured. The latter arrangement was convenient, since it was the difference of temperature which was being investigated.

Before using the new apparatus tests were made to ascertain whether the heat insulation was in fact perfect, and also what difference, if any, there was between the readings of temperature given by the two thermo-junctions. The construction of the apparatus makes these tests very simple.

To avoid the slightest drop of pressure between the thermo-junctions, a simple full-bore block of vulcanite (also shown in fig. 16) was substituted for the adjustable valve, then liquid CO\(_2\) was passed through the apparatus, first with the thermo-junctions in their normal positions, and again when the thermo-junctions had been interchanged, end for end. The apparent difference of temperature between the thermo-junctions in the first case represents the sum, and in the second case, the difference, of the two quantities to be determined, viz., the actual change of temperature of the liquid and the difference of the thermo-junctions.

Experiments were made with liquid CO\(_2\) at temperatures ranging from +20° C. to −50° C. They showed that there was no change in the temperature of the liquid, but that there was a very slight difference between the thermo-junctions. This was allowed for in subsequent measurements.* As a further check, similar tests were made with wet CO\(_2\) vapour instead of liquid CO\(_2\). The difference between the readings of the thermo-junctions was confirmed. There could be no difference of temperature of the vapour in this case, since the pressure was the same at the two junctions.

As the calorimeter could not be cooled much below −40° C., and readings were wanted at −50° C., an “infra-cooler” was inserted in the pipe between the calorimeter and the throttle valve, by which the liquid could be cooled to any extent desired. By this means readings were obtained down to −55° C. The “infra-cooler” consisted of two concentric copper pipes 10 feet long, the inner one \( \frac{1}{4} \) inch external diameter, the outer one \( \frac{3}{8} \) inch internal diameter. The liquid CO\(_2\) passed through the inner tube, while a separate supply of liquid CO\(_2\) was admitted to the outer tube and evaporated at a pressure of about 80 lbs. per sq. in., escaping through a throttle valve into the atmosphere. The concentric pipes were bent into a coil about 9 inches diameter and well lagged with cotton wool. The arrangement worked well, though

* See note on Table V.
it was not possible to keep the temperature of the liquid CO₂ absolutely constant during an observation. Slight variations of temperature account for the irregularities of the observations below -30° C.; above that temperature the "infra-cooler" was not used. The results of the tests with the original and with the new apparatus are summarised in Table V., and both are plotted in fig. 9, p. 76. The two sets of results are in good agreement, considering the smallness of the quantities being measured.

Series V.—The apparatus used for measuring the elasticity and dilatation of liquid CO₂ is shown in figs. 10 and 17:—

α is a capillary glass tube, the lower end of which is closed and the upper end thickened and blown into a thistle funnel. A centimetre scale was etched along its whole length, and it was carefully calibrated by measuring the variation in the length of a thread of mercury in different positions.

β is a gunmetal socket, shown in detail in fig. 17, turned to hold the glass tube, the joint being made by a thin rubber sleeve. The top of the socket is closed by a screw plug.

c is a fine copper pipe, ¼-inch bore, 12 feet long, connecting the socket to the regulating valves, pressure gauge, and, through a drying flask, to the CO₂ flask. This pipe is sufficiently flexible to allow the glass tube to be moved about as required while under pressure.

The glass tube was charged as follows:—The tube was laid in a nearly horizontal position and a small quantity of mercury poured into the thistle funnel, where it lay without obstructing the entrance to the tube. The plug was then inserted in the gunmetal head. The air was removed by means of an air pump, successive charges of CO₂ gas being admitted and exhausted. The lower half of the glass tube was then surrounded with ice, and CO₂ gas was admitted up to a pressure slightly above the saturation pressure at 0° C. The gas then condensed in the glass tube, and the meniscus could be seen travelling up the tube. The rate of condensation could be regulated with ease by modifying the pressure. When sufficient liquid was condensed, the tube was raised into a vertical position so that the mercury in the funnel flowed into it, on to the top of the liquid CO₂. The tube, kept vertical, was then lowered into the calorimeter, the temperature of which had been adjusted to a few degrees below 0° C. The amount of liquid used and the length of the mercury column were chosen so that the whole of the liquid CO₂ was below the level of the bath in the calorimeter, and therefore at a uniform temperature, and the top of the mercury showed above the lid of the calorimeter, so that its position could be observed with a cathetometer.
When making an experiment, the bath was first cooled to the desired temperature. Then readings were taken with the cathetometer as the pressure was varied step by step from the maximum available to a pressure just above the saturation pressure corresponding to the temperature of the bath. These readings give a constant-temperature curve on the p.v. diagram. A series of experiments was made with different temperatures, so that a number of constant-temperature curves were obtained, as shown in fig. 11, p. 78.

The apparatus appeared to work satisfactorily, but the results obtained are not in perfect agreement amongst themselves; the cause of this has not yet been ascertained. The curves shown in fig. 11 have been arbitrarily constructed to represent the mean results of several sets of observations and to agree amongst themselves; they must be taken as only approximate, but they are sufficiently accurate to determine the small correction $\delta$, Table VI., for which they are required. AMAGAT’S (5) curves have been added to fig. 11 for comparison.

By extrapolation the curves may be extended to the left to the saturation pressure, which is only just below the lowest pressure observed. This has been done in fig. 11. By joining up the ends of all the curves, the new curve (named “limit curve” in the figure) is obtained which shows the change of volume along the limit curve. A curve giving BEHN’S (6) observations is also drawn for comparison. The agreement is fairly good.

It is not necessary to describe the lengthy and rather complex calculations required to reduce the data obtained in these experiments. Allowance was made for the changes of volume of the glass and of the mercury indicator at different temperatures, and corrections applied to allow for the variation in the bore of the tube. All these corrections are small compared with the changes of volume of the CO$_2$ due to temperature. Since the actual weight of the column of liquid CO$_2$ was not measured, the density at some one temperature had to be assumed; BEHN’S value, viz., 0’925 at 0$^\circ$ C., was used.

The elasticity and dilatation may be derived from the curves directly. The elasticity $(dv/dp)_t$ is the slope of the constant temperature lines. The dilatation $(dv/dt)_p$ is the distance apart of the constant temperature lines divided by the difference of temperature.

These quantities have not been worked out because of the known inaccuracy of the observations. The authors intend to repeat the experiments and hope to obtain accurate results.

**Part IV.—Discussion of Results.**

*Pressure-Temperature Curve.* (Fig. 18. Tables I. and X.)

The relation between the pressure and temperature of saturated CO$_2$ vapour has been observed by REGNAULT (7), CAILOTET (8), AMAGAT (5), KUENEN and ROBSON (9),
Zeleny and Smith (10), and others. The observations are tabulated in parallel columns in Table X, and all plotted in fig. 18 besides the authors' curve.

The most accurate determinations are probably Kuenen and Robson's below zero and Amagat's above zero. Our curve agrees closely with Kuenen and Robson's,

only differing by 3 lbs. at 510 lbs. pressure. It also agrees fairly closely with Amagat's, differing by 6 lbs. at 830 lbs. Both these differences are almost exactly equal to the calculated effects of the presence of the observed '11 per cent. of air (by volume) under the special conditions of our test.
Zeleny and Smith's points lie close to our curve, but a curve through their points would be definitely flatter than ours. There is a difference of 10 lbs. at their highest pressure, 410 lbs.

Cailletet's figures only go up to \(-34^\circ\) C. They do not agree very closely in position or slope.

Regnault's figures lie well above the others. He states that his experiments did not satisfy him.

Mollier adopted a composite curve, using Amagat's figures above zero and Regnault's below zero, modified so as to make them fit together. His curve is shown dotted in fig. 18.

An accurate determination of the pressure-temperature curve is important because its gradient, \(dp/d\theta\), is one of the factors in Clapeyron's equation, which may be used to calculate the latent heat or the vapour density.

The gradient of Mollier's curve is clearly too small, particularly at low temperatures. It is remarkable how large a difference in the gradient results from a very small divergence between the curves. The values of \(dp/d\theta\), used by Cailletet and Mathias (11), Mollier, and Kuenen and Robson for calculating \(L\), and by the authors who only used it for calculating the specific volume of the gas, are given below:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>lbs./in.(^2) ° C.</td>
<td>lbs./in.(^2) ° C.</td>
<td>lbs./in.(^2) ° C.</td>
<td>lbs./in.(^2) ° C.</td>
</tr>
<tr>
<td>+20</td>
<td>20·4</td>
<td>19·4</td>
<td></td>
<td>19·6</td>
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<tr>
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<td></td>
<td>16·3</td>
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<tr>
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<td>10·69</td>
<td>11·01</td>
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<tr>
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<td>8·73</td>
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<td>6·79</td>
<td>6·87</td>
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<td>-40</td>
<td></td>
<td>-</td>
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<tr>
<td>-50</td>
<td></td>
<td>-</td>
<td>4·06</td>
<td>4·0</td>
</tr>
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</table>

The Total Heat \(I\) and Specific Heat \(C_p\) of the Liquid.

Our observations were plotted and values read from the smooth curve are given in Table VII. No experimental determinations of these quantities appear to have been published. Figures for comparison might be deduced from Mollier’s (3) \(I_\phi\) diagram, but this is beyond the range of this paper.

The probable errors in the values of \(I\) do not exceed about \(\frac{1}{8}\) per cent., from \(+20^\circ\) C. to \(-30^\circ\) C., but rise to 1 per cent. at \(-50^\circ\) C. The presence of 0·073 per cent. of air has no appreciable effect on the results.

Experiments have been made by Dieterici (15) and by Margules (16) on the
differences between the total heats at 0° C. and at a few other temperatures. These quantities refer to heating along the limit curve and not to heating at constant pressure.

Specific Heat of the Gas. (Fig. 6.)

No experimental determinations of this quantity appear to have been made at temperatures below 0° C., though much work has been done at high temperatures.

The specific heat is believed to vary considerably near the limit curve. Our measurements give a mean value for a moderate range of temperature, starting in each case a few degrees above the limit curve; they are probably not correct for higher temperatures than those at which they were measured. They are only used for the small correction given in column \( \gamma \), Table VI, and for plotting the pressure lines in the superheated area.

Latent Heat of Liquid CO\(_2\).

Our observations were plotted and values read off the smooth curve are given in Table VI. The figures are probably correct to about \( \frac{1}{2} \) per cent. from +20° C. to -30° C. and to 1 per cent. at -50° C. One point, at -8°-6 C. lies 1 per cent, off the curve. This was the result of the first experiment made, when the spring balances were still used for weighing the CO\(_2\). The trace of air present only produces a proportionate error in the value of L, i.e., 0.73 per cent., which is negligible.

No experimental determinations of L below zero have been published. Regnault* made a single determination at +17° C.; Chappuis (13) a single determination at 0° C. and Mathias (12) made a series of measurements between +6° C. and +31° C. Cailletet and Mathias (11) calculated L from their own determination of the liquid and vapour densities and Regnault's pressure-temperature curve. Kuenen and Rosson (9) calculated L from Amagat's (5) vapour density (extrapolated) and Behn's (6) liquid densities and their own pressure-temperature curve. Mollier (1) calculated L from Amagat's (5) vapour and liquid densities (extrapolated) and the compound pressure-temperature curve mentioned above, based on Amagat and Regnault's results.

All these results are plotted in fig. 18 beside our curve. If the drawing is examined it will be seen that the mean of all the previous determinations lies above our curve. We have investigated the causes of this divergence in detail as it appeared to cast some doubt on the accuracy of our results. The investigation has shown, first why the previous results tend to agree amongst themselves, secondly why they differ from ours, and, finally, has resulted in an indirect confirmation of our results.

* Recalculated by Mathias (12).
The cause of the rough agreement among the observations is that the same data have been repeatedly used. Thus Cailletet and Mathias’ densities were used by—

- **Mathias Chappuis** \( \text{in working out their observations,} \)
- **Cailletet and Mathias** \( \text{in their calculations.} \)

Amagat’s densities were used by—

- **Kuenen and Robson** \( \text{in their calculations.} \)
- **Mollier** \( \text{in their calculations.} \)

Regnault’s pressure-temperature curve was used by—

- **Cailletet and Mathias** \( \text{in their calculations.} \)
- **Mollier (in part)** \( \text{in their calculations.} \)

The difference between the various results and ours may be traced to the inaccuracy of Regnault’s (and consequently Mollier’s) pressure-temperature curve and to the error in Cailletet and Mathias’ gas densities below \(-20^\circ\) C, and to the uncertainty of the extrapolated values of the densities used by Mollier and Kuenen and Robson.

All the calculated values of \( L \) were obtained by means of Clapeyron’s equation

\[
L = (v - V) \theta \frac{dp}{d\theta},
\]

where

- \( V \) = specific volume of saturated vapour,
- \( v \) = liquid at saturation temperature.

Their accuracy therefore depends on the accuracy of the two factors \( dp/d\theta \) and \( (V - v) \).

The differences in the pressure-temperature curves and their gradients have already been discussed, so that it is only necessary to consider now the other factor \( (V - v) \). The specific volume of the liquid \( v \) is much smaller than of the vapour, and the values obtained by different observers do not differ much; it is, therefore, not necessary to discuss it here. The specific volume of the saturated vapour, \( V \), is much more doubtful. In fig. 19 are plotted—

1. Amagat’s (5) smooth curve from his observations of \( V \).
2. Cailletet and Mathias’ (11) smooth curve from their observations of \( V \).
3. Kuenen and Robson’s (9) extrapolation (by the Law of Corresponding states) of Amagat’s curve.
4. Mollier’s (1) extrapolation of Amagat’s curve.
5. Our values, calculated from our observed values of \( L \) by Clapeyron’s equation, using Behn’s (6) liquid densities.

Values taken from our smooth curve are given in Table IX.
SPECIFIC VOLUME OF SATURATED VAPOUR

\[
\frac{1}{\text{DENSITY}}
\]

Fig. 19.
The observations on which CAILLETET and Mathias' smooth curve is founded lie very erratically and at considerable distances from the curve. The curve is clearly erroneous below $-20^\circ$ C.

It will be noted that our values agree very closely with CAILLETET and Mathias' from $-20^\circ$ C. to $+10^\circ$ C. Our curve also lies almost exactly parallel with Amagat's curve and Kuenen and Robson's theoretical extension of it. The agreement between our curve, the observations, and the theoretical curve is a confirmation of the accuracy of the shape of our L curve.

The inaccuracy of our pressure-temperature curve, due to the presence of 11 per cent. of air, referred to on p. 69, introduces an error into our values of $\frac{d}{d\theta} p/d\theta$ which probably does not exceed 2 per cent. If this were allowed for, it would raise our specific-volume curve by 2 per cent., and bring it closer to Kuenen and Robson's. The value of $V$ does not enter into the construction of our $\theta\phi$ diagram.

**Joule-Thomson Effect.** (Fig. 9.)

No experiments on the Joule-Thomson effect for liquid CO₂ appear to have been published. Figures for comparison might be deduced from Mollier's $I\phi$ diagram (3), but that would be beyond the range of this paper. It is not easy to say what effect the presence of the trace of air may have on these results.

**Dilatation and Elasticity of Liquid CO₂.**

As has been explained, the results of the Series V. experiments were not sufficiently concordant to warrant the publication of values of the dilatation and elasticity derived from them, though they are accurate enough to determine the values of $\frac{d}{d\theta}$ between the constant-pressure curves and the limit curve on the $\theta\phi$ diagram. The confirmation of BEHN's densities, shown in fig. 11, must not have much weight attached to it for the same reason.

The only results previously published are a single curve at 13°·1 C., given by Andrews (14), and three curves given by Amagat (5) at 0° C. +10° C., and +20° C. We have failed to fit Andrews' curve on our figure, but Amagat's curves are shown for comparison in fig. 11. Andrews suggests that the curvature of the lines near the saturation pressure may be due to the presence of air.

**$\theta\phi$ Diagram.** (Fig. 12.)

As stated at the beginning of this paper, the primary object, for which all the quantities already discussed were measured, was the construction of a $\theta\phi$ diagram for comparison with Mollier's.

It may be useful to recapitulate here the steps in the construction of the two diagrams. Fig. 12 was drawn as follows:
The starting points of the 700-lb. and 900-lb. constant-pressure curves (θ = 273° C., φ = −'0024 and −'0049), found on p. 80, were first marked, and the two constant-pressure curves were then drawn in segments of 10 degrees each.

The liquid-limit curve was then set off from these pressure curves at the distances δφ given in Table VIII. The gas-limit curve was then set off from the liquid-limit curve at the distances δφ = L/θ (Table VI).

A constant I line was then drawn through the origin, at a slope determined by the Joule-Thomson effect. Thus the value of I at the origin (calculated on p. 80) was transferred to the two constant-pressure curves. The points I = 0, +5, +10, +15, and −5, −10, −15, −20, −25 were then marked on these curves and transferred back to the limit curve by drawing I lines as before. Thus the starting points of the I lines on the limit curve were determined. Measuring from these, a number of points were then marked off at distances δφ = 5/θ, 10/θ, 15/θ, &c., and I lines drawn through these points. The space between the limit curves was then divided into quarters, thus determining a few dryness lines. The constant-pressure curves in the superheated area were then drawn, starting at the corresponding saturation temperatures at slopes

$$\frac{\delta \phi}{\theta_1 - \theta_2} = \frac{\sigma}{\frac{1}{2} (\theta_1 + \theta_2)}$$

where the values of θ₁ - θ₂ were the actual temperature ranges in the experiments, Series III., and σ was the corresponding specific heat of the gas at constant pressure, given in fig. 6.

The maximum probable error in the liquid limit curve is δφ = '0008 at −30° C. and '0032 at −50° C. At higher temperatures the error is probably not more than '0005.

Molière's φθ diagram was constructed in a very different manner. He assumed that the characteristic equation of the gas might be expressed in Van der Waal's form

$$P = \frac{R \theta}{v - \alpha} - \frac{C \cdot f(\theta)}{(v + \beta)^2}$$

and determined the constants by means of Amagat's observations. He also assumed that $f(\theta)$ was of the form $f(\theta) = e^{(1-\theta/\varphi)}$, and by means of these equations obtained a general expression for the entropy of the gas, which would hold down to the limit curve. With this he plotted the gas limit curve on the φθ diagram.

He then found an empirical mathematical formula for the slope of the pressure-temperature curve, $dP/d\theta$, which, on integration, corresponded well with the pressure-temperature curve constructed from Amagat's and Regnault's observations. With this and Amagat's values for the vapour and liquid densities (extrapolated) he calculated the values of L, and set off the liquid limit curve to the left of the gas limit curve at distances $\delta \phi = L/\theta$. 

THERMAL PROPERTIES OF CARBONIC ACID AT LOW TEMPERATURES. 101
To construct the constant-pressure lines in the superheated area he assumed (2) that the specific heat at constant volume, $C_v$, was constant, which he points out is not quite true. On this assumption the constant-volume lines on the $\theta\phi$ diagram are the same as those for a perfect gas (logarithmic curves). The constant-pressure curves were then determined from these constant-volume curves by the p.v. $\theta$ curves observed by Amagat. Mollier does not draw I lines on his $\theta\phi$ diagram. His values of I are given in his $I\phi$ diagram (3), which is outside the range of the present discussion.

The agreement between the two diagrams, constructed by such widely different methods, is remarkable, the more so because Mollier had no data to go on below 0° C., except Regnault's very imperfect pressure-temperature curve. The following brief comparison between the two methods shows how widely they differ.

Our diagram is based in the most direct manner possible on experiments at the temperatures and pressures represented, whereas Mollier's is based on a mathematical equation obtained from experiments at higher temperatures. Our diagram is constructed from the left, his from the right hand. Our diagram is based on measurements of heat, his on measurements of density and pressure.

The two diagrams are superposed in fig. 13, and values of $\phi$ and L are given in Table XVII for comparison. In Table XVIII our data are arranged for direct comparison with the tables given by Mollier (1) and Ewing (4).

The differences between the diagrams are due to the differences in the various data, which have been already discussed. The authors take this opportunity of expressing their great admiration for the judgment and skill by which Dr. Mollier has selected the most reliable data and devised mathematical methods capable of giving results which direct experiments have confirmed so closely.

**Summary.**

The authors have reconstructed the $\theta\phi$ diagram by a new method and, at the same time, extended it from -30° C. to -50° C. The reconstruction is based on direct heat measurements, and the results are believed to be more accurate than those arrived at indirectly by Mollier.

The observations include the direct measurement of the following quantities:—

- The latent heat;
- The total heat of the liquid;
- The specific heat of the gas;
- The dilatation and elasticity of the liquid;
- The Joule-Thomson effect for the liquid.

From these direct measurements the following quantities have been calculated:—

- The specific volume of the saturated vapour;
- The specific heat of the liquid at constant pressure.
Most of these measurements are new; the details of what has been done before are given on p. 68 and in Part IV., p. 94.

In addition to the ordinary data given in $\theta \phi$ diagrams and tables, the authors have given the values of $I$, so that a complete $I \phi$ diagram might be constructed from the data supplied in the paper. The authors have made this diagram, but before publishing it they intend to make a series of throttling experiments on superheated gas to check the constant-pressure curves, which have so far only been approximately determined.

P.S.—Since this paper was completed the authors have commenced the gas-throttling experiments referred to above. These may be used as an independent check on the accuracy of the $\theta \phi$ diagram. Choosing an experiment as close to the gas-limit curve as possible—to avoid possible errors in the approximate constant-pressure lines—the confirmation obtained is remarkably good.

Starting at the point "V," fig. 12, on the 700-lbs. pressure line, the gas was throttled down to the 150-lbs. pressure line. The point reached is marked "W." Calculated from the $\theta \phi$ diagram, the point W should have fallen exactly on the limit curve, i.e., at "E." If the limit curve be moved .0001 to the left, W and E will coincide. A shift of .0001 corresponds to an error of .3 per cent. in the value of L. This is the accumulated error in the whole set of measurements for the complete cycle KAVWEGK shown in fig. 20.

The authors desire to express their special gratitude to Brasenose College for the gift of the freezing machine with which the experiments were made. They also have to thank Mr. D. H. Nagel, of Trinity College, and Mr. H. B. Hartley, of Balliol College, for advice on chemical questions and for having generously placed the College Libraries at their disposal.
PAPERS, &c., REFERRED TO.

Reference.

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(2) MOLLIER, 'Zeit. für die ges. Kälte-Industrie,' 1896, No. 4, p. 65.
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(4) EWING, 'Mech. Production of Cold,' 1908.
(5) AMAGAT, 'Annales de Chimie et de Physique,' 6th Ser., 1893, vol. 29, p. 68.
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(9) KUENEN and ROBSON, 'Phil. Mag.,' Ser. 6, vol. III., p. 149 (1902) and p. 622.
(12) MATHIAs, 'Thèses à la Faculté des Sciences de Paris,' No. 687 (1890).
(14) ANDREWS, 'Phil. Trans.,' 1869.

Table I.—Pressure-Temperature Observations.

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<th>Temperature, °C.</th>
<th>Pressure, lbs./in.²</th>
<th>Temperature, °C.</th>
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<td>-51·0</td>
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<tr>
<td>128</td>
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<td>387·5</td>
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<td>167</td>
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<td>407</td>
<td>-8·15</td>
</tr>
<tr>
<td>185</td>
<td>-33·3</td>
<td>429·5</td>
<td>-6·2</td>
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<td>206</td>
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<td>-2·6</td>
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Table II.—Series I. Observations. (Refrigeration.)

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<th>-14.9</th>
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<th>-8.0</th>
<th>-1.7</th>
<th>+6.6</th>
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<td></td>
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<tr>
<td>Pressure evaporation, $p_2$</td>
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<td>169.5</td>
<td>205.5</td>
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<td>-10.1</td>
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<td>853</td>
<td>780</td>
</tr>
<tr>
<td>Bath fall</td>
<td></td>
<td>-22.8</td>
<td>-10.4</td>
<td>+0.3</td>
<td>+17.0</td>
<td>-15</td>
<td>+15.4</td>
<td>+13.6</td>
<td>+70.6</td>
<td>+31</td>
<td>-91</td>
<td>-15.0</td>
<td>+3.8</td>
<td>+2.5</td>
<td>+12.5</td>
<td>+1.7</td>
</tr>
<tr>
<td>Radiation</td>
<td></td>
<td>21.7</td>
<td>60.9</td>
<td>31.9</td>
<td>79.8</td>
<td>143</td>
<td>46.2</td>
<td>56</td>
<td>106.3</td>
<td>25.2</td>
<td>69.8</td>
<td>15</td>
<td>32</td>
<td>3.8</td>
<td>2.7</td>
<td>1.6</td>
</tr>
<tr>
<td>Total heat</td>
<td></td>
<td>517.1</td>
<td>477.5</td>
<td>289.2</td>
<td>743.6</td>
<td>1454</td>
<td>901.6</td>
<td>1649</td>
<td>4085</td>
<td>1523</td>
<td>4446</td>
<td>1499</td>
<td>5637</td>
<td>1011.4</td>
<td>859.5</td>
<td>784.1</td>
</tr>
<tr>
<td>Heat per lb</td>
<td></td>
<td>47.0</td>
<td>47.75</td>
<td>48.2</td>
<td>49.5</td>
<td>48.5</td>
<td>50.1</td>
<td>50.0</td>
<td>50.7</td>
<td>50.8</td>
<td>51.3</td>
<td>50.0</td>
<td>53.0</td>
<td>50.37</td>
<td>47.75</td>
<td>46.1</td>
</tr>
</tbody>
</table>
### Table III.—Series II. Observations. (Total Heat of Liquid.)

<table>
<thead>
<tr>
<th>Weight of CO₂</th>
<th>lbs.</th>
<th>8</th>
<th>17</th>
<th>14</th>
<th>20</th>
<th>26</th>
<th>16</th>
<th>15</th>
<th>18</th>
<th>18</th>
<th>19</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration</td>
<td>min.</td>
<td>33.0</td>
<td>32.3</td>
<td>27.0</td>
<td>29.2</td>
<td>24.9</td>
<td>16.4</td>
<td>16.6</td>
<td>12.15</td>
<td>19.05</td>
<td>20.0</td>
<td>17.9</td>
</tr>
<tr>
<td>Initial temperature</td>
<td>° C.</td>
<td>-39.1</td>
<td>-35.2</td>
<td>-30.3</td>
<td>-25.6</td>
<td>-15.3</td>
<td>-11.9</td>
<td>-11.8</td>
<td>-11.3</td>
<td>-11.4</td>
<td>-6.8</td>
<td>-1.4</td>
</tr>
<tr>
<td>Final temperature</td>
<td>° C.</td>
<td>8.3</td>
<td>6.5</td>
<td>7.7</td>
<td>8.3</td>
<td>8.0</td>
<td>19.25</td>
<td>7.1</td>
<td>3.9</td>
<td>7.8</td>
<td>7.7</td>
<td></td>
</tr>
<tr>
<td>Mean bath temperature</td>
<td>° C.</td>
<td>8.5</td>
<td>7.1</td>
<td>7.6</td>
<td>8.2</td>
<td>7.9</td>
<td>19.6</td>
<td>12.2</td>
<td>7.1</td>
<td>3.9</td>
<td>7.7</td>
<td>7.8</td>
</tr>
<tr>
<td>Fall of bath temperature</td>
<td>° C.</td>
<td>13.8</td>
<td>13.1</td>
<td>14.4</td>
<td>16.5</td>
<td>15.4</td>
<td>17.3</td>
<td>16.1</td>
<td>14.6</td>
<td>16.5</td>
<td>13.1</td>
<td>13.5</td>
</tr>
<tr>
<td>Atmospheric temperature</td>
<td>° C.</td>
<td>197.1</td>
<td>379.2</td>
<td>298</td>
<td>377.5</td>
<td>357.9</td>
<td>306.4</td>
<td>212.5</td>
<td>195.3</td>
<td>158.9</td>
<td>164.3</td>
<td>97.1</td>
</tr>
<tr>
<td>Electric heat</td>
<td>Th.U.</td>
<td>12.5</td>
<td>16.6</td>
<td>3.2</td>
<td>2.9</td>
<td>2.8</td>
<td>1.0</td>
<td>1.1</td>
<td>0.1</td>
<td>0.4</td>
<td>2.6</td>
<td>5.7</td>
</tr>
<tr>
<td>Bath fall</td>
<td>° C.</td>
<td>4.3</td>
<td>4.2</td>
<td>3.5</td>
<td>3.4</td>
<td>2.9</td>
<td>0.0</td>
<td>0.7</td>
<td>1.4</td>
<td>2.7</td>
<td>0.5</td>
<td>0.4</td>
</tr>
<tr>
<td>Radiation</td>
<td>° C.</td>
<td>213.9</td>
<td>400.0</td>
<td>298.3</td>
<td>383.8</td>
<td>358.0</td>
<td>305.4</td>
<td>212.1</td>
<td>196.8</td>
<td>162.0</td>
<td>167.4</td>
<td>103.2</td>
</tr>
<tr>
<td>Total heat</td>
<td>° C.</td>
<td>26.75</td>
<td>19.1</td>
<td>19.1</td>
<td>14.15</td>
<td>10.93</td>
<td>9.0</td>
<td>8.81</td>
<td>5.73</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table IV.—Series III. Observations. (Specific Heat of Gas.)

<table>
<thead>
<tr>
<th>Pressure</th>
<th>lbs./sq. in.</th>
<th>138</th>
<th>188</th>
<th>278</th>
<th>408</th>
<th>504</th>
<th>608</th>
<th>733</th>
<th>824</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight of CO₂</td>
<td>lbs.</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>17</td>
<td>19</td>
<td>24</td>
<td>16</td>
<td>18</td>
</tr>
<tr>
<td>Duration</td>
<td>min.</td>
<td>30.5</td>
<td>23.9</td>
<td>18.6</td>
<td>14.6</td>
<td>14.5</td>
<td>18.07</td>
<td>16.45</td>
<td>21.52</td>
</tr>
<tr>
<td>Initial temperature</td>
<td>° C.</td>
<td>-30.4</td>
<td>-22.4</td>
<td>-13.0</td>
<td>-1.9</td>
<td>8.3</td>
<td>14.9</td>
<td>20.4</td>
<td>24.6</td>
</tr>
<tr>
<td>Final temperature</td>
<td>° C.</td>
<td>7.8</td>
<td>8.6</td>
<td>11.2</td>
<td>9.1</td>
<td>19.5</td>
<td>25.6</td>
<td>30.5</td>
<td>35.4</td>
</tr>
<tr>
<td>Mean bath temperature</td>
<td>° C.</td>
<td>7.3</td>
<td>8.3</td>
<td>11.1</td>
<td>9.2</td>
<td>19.6</td>
<td>25.6</td>
<td>30.6</td>
<td>35.6</td>
</tr>
<tr>
<td>Fall of bath</td>
<td>° C.</td>
<td>+0.33</td>
<td>-0.13</td>
<td>0.04</td>
<td>-0.47</td>
<td>-0.33</td>
<td>-0.85</td>
<td>+0.07</td>
<td>+0.04</td>
</tr>
<tr>
<td>Atmospheric temperature</td>
<td>° C.</td>
<td>15.0</td>
<td>15.3</td>
<td>15.3</td>
<td>14.8</td>
<td>14.8</td>
<td>14.8</td>
<td>14.8</td>
<td>14.4</td>
</tr>
<tr>
<td>Electric heat</td>
<td>Th.U.</td>
<td>78.87</td>
<td>112.3</td>
<td>131.9</td>
<td>67.0</td>
<td>93.1</td>
<td>125.2</td>
<td>81.5</td>
<td>111.5</td>
</tr>
<tr>
<td>Bath fall</td>
<td>° C.</td>
<td>+4.98</td>
<td>-2.0</td>
<td>0.6</td>
<td>-7.1</td>
<td>-4.7</td>
<td>12.1</td>
<td>1.0</td>
<td>0.6</td>
</tr>
<tr>
<td>Radiation</td>
<td>° C.</td>
<td>1.12</td>
<td>0.8</td>
<td>0.45</td>
<td>0.36</td>
<td>0.3</td>
<td>1.5</td>
<td>2.4</td>
<td>4.4</td>
</tr>
<tr>
<td>Total heat</td>
<td>° C.</td>
<td>84.97</td>
<td>115.1</td>
<td>132.95</td>
<td>60.36</td>
<td>88.1</td>
<td>111.6</td>
<td>80.1</td>
<td>107.7</td>
</tr>
<tr>
<td>Heat per lb.</td>
<td>Th.U./lb.</td>
<td>8.5</td>
<td>7.68</td>
<td>6.65</td>
<td>3.55</td>
<td>4.63</td>
<td>4.65</td>
<td>5.01</td>
<td>5.99</td>
</tr>
<tr>
<td>Specific heat.</td>
<td>—</td>
<td>0.22</td>
<td>0.25</td>
<td>0.275</td>
<td>0.32</td>
<td>0.41</td>
<td>0.435</td>
<td>0.495</td>
<td>0.55</td>
</tr>
</tbody>
</table>
Table V.—Series IV. Observations. (Joule-Thomson Effect.)

With Original Apparatus.

<table>
<thead>
<tr>
<th></th>
<th>°C.</th>
<th>10</th>
<th>5</th>
<th>0.5</th>
<th>11</th>
<th>22</th>
<th>30</th>
<th>30.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial temperature</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pressure before throttle valve</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pressure after throttle valve</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Drop of pressure at valve</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Change of temperature, throttling</td>
<td>°C.</td>
<td>-0.66</td>
<td>-0.575</td>
<td>-0.49</td>
<td>-0.10</td>
<td>0.62</td>
<td>1.47</td>
<td>1.38</td>
</tr>
<tr>
<td>Change of temperature, not throttling</td>
<td>°C.</td>
<td>-0.00</td>
<td>-0.055</td>
<td>0.11</td>
<td>0.35</td>
<td>0.73</td>
<td>1.36</td>
<td>1.30</td>
</tr>
<tr>
<td>Change of temperature, due to throttling</td>
<td></td>
<td>-0.66</td>
<td>-0.63</td>
<td>-0.60</td>
<td>-0.45</td>
<td>-0.11</td>
<td>0.11</td>
<td>0.08</td>
</tr>
<tr>
<td>Change of temperature per 100 lbs. drop of pressure</td>
<td>°C.</td>
<td>-0.45</td>
<td>-0.33</td>
<td>-0.22</td>
<td>-0.12</td>
<td>-0.024</td>
<td>+0.02</td>
<td>+0.03</td>
</tr>
</tbody>
</table>

With Special Throttle Valve (fig. 16, p. 91).

<table>
<thead>
<tr>
<th></th>
<th>°C.</th>
<th>+15.1</th>
<th>+10.6</th>
<th>+4.9</th>
<th>-2.1</th>
<th>-11.3</th>
<th>-20.7</th>
<th>-31</th>
<th>-35</th>
<th>-37.9</th>
<th>-41.7</th>
<th>-43.2</th>
<th>-53.6</th>
<th>-55</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial temperature, °C.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial pressure, $p_1$</td>
<td></td>
<td>857</td>
<td>799</td>
<td>802</td>
<td>788</td>
<td>668</td>
<td>664</td>
<td>651</td>
<td>651</td>
<td>648</td>
<td>649</td>
<td>650</td>
<td>646</td>
<td>647</td>
</tr>
<tr>
<td>Final pressure, $p_2$</td>
<td></td>
<td>745</td>
<td>688</td>
<td>630</td>
<td>562</td>
<td>433</td>
<td>360</td>
<td>312</td>
<td>295</td>
<td>268</td>
<td>265</td>
<td>273</td>
<td>245</td>
<td>243</td>
</tr>
<tr>
<td>Fall of pressure</td>
<td></td>
<td>112</td>
<td>111</td>
<td>172</td>
<td>226</td>
<td>235</td>
<td>304</td>
<td>339</td>
<td>356</td>
<td>380</td>
<td>384</td>
<td>377</td>
<td>401</td>
<td>404</td>
</tr>
<tr>
<td>Change of temperature*</td>
<td></td>
<td>-0.72</td>
<td>-0.53</td>
<td>-0.56</td>
<td>-0.44</td>
<td>-0.26</td>
<td>-0.135</td>
<td>+0.04</td>
<td>+0.18</td>
<td>+0.345</td>
<td>+0.595</td>
<td>+0.38</td>
<td>+0.81</td>
<td>+0.73</td>
</tr>
<tr>
<td>Change of temperature per 100 lbs.</td>
<td>°C.</td>
<td>-0.64</td>
<td>-0.48</td>
<td>-0.325</td>
<td>-0.195</td>
<td>-0.11</td>
<td>-0.04</td>
<td>+0.01</td>
<td>+0.05</td>
<td>+0.09</td>
<td>+0.15</td>
<td>+0.10</td>
<td>+0.20</td>
<td>+0.18</td>
</tr>
</tbody>
</table>

* These figures include the correction for the difference between the thermo-junctions. This correction varies from zero at +15° C. to 25 per cent. of the reading at -55° C.
TABLE VI.—Calculation of Latent Heat.

<table>
<thead>
<tr>
<th>Evaporation temperature, $\theta_p$</th>
<th>$\alpha$. (Refrigeration.)</th>
<th>$\beta$. (Series I., NBCQ.* )</th>
<th>$\gamma$. (Series II., THAM.* )</th>
<th>$\delta$. (Series III., PECQ.* )</th>
<th>$\theta_2 - \theta_1$. (Series IV., SKHT.* )</th>
<th>Latent heat. $L = \alpha + \beta - \gamma + \delta$, RGEF.*</th>
</tr>
</thead>
<tbody>
<tr>
<td>-53.4</td>
<td>47.0</td>
<td>37.9</td>
<td>4.4</td>
<td>+0.69</td>
<td>81.2</td>
<td></td>
</tr>
<tr>
<td>-51.6</td>
<td>47.75</td>
<td>35.7</td>
<td>3.5</td>
<td>+0.51</td>
<td>80.45</td>
<td></td>
</tr>
<tr>
<td>-48.3</td>
<td>48.2</td>
<td>34.8</td>
<td>3.72</td>
<td>+0.45</td>
<td>79.75</td>
<td></td>
</tr>
<tr>
<td>-43.2</td>
<td>49.5</td>
<td>31.2</td>
<td>3.7</td>
<td>+0.35</td>
<td>77.35</td>
<td></td>
</tr>
<tr>
<td>-40.9</td>
<td>48.5</td>
<td>28.6</td>
<td>1.5</td>
<td>+0.3</td>
<td>75.9</td>
<td></td>
</tr>
<tr>
<td>-35.6</td>
<td>50.1</td>
<td>24.7</td>
<td>1.49</td>
<td>+0.18</td>
<td>73.5</td>
<td></td>
</tr>
<tr>
<td>-30.0</td>
<td>50.0</td>
<td>22.8</td>
<td>1.3</td>
<td>+0.08</td>
<td>71.6</td>
<td></td>
</tr>
<tr>
<td>-26.1</td>
<td>50.7</td>
<td>20.6</td>
<td>1.87</td>
<td>0</td>
<td>69.4</td>
<td></td>
</tr>
<tr>
<td>-21.4</td>
<td>50.8</td>
<td>17.9</td>
<td>1.7</td>
<td>-0.07</td>
<td>66.9</td>
<td></td>
</tr>
<tr>
<td>-14.9</td>
<td>51.3</td>
<td>14.0</td>
<td>1.7</td>
<td>-0.1</td>
<td>63.5</td>
<td></td>
</tr>
<tr>
<td>-11.6</td>
<td>50.0</td>
<td>14.4</td>
<td>2.0</td>
<td>-0.21</td>
<td>62.2</td>
<td></td>
</tr>
<tr>
<td>-8.6</td>
<td>53.0</td>
<td>11.0</td>
<td>3.1</td>
<td>-0.2</td>
<td>60.7</td>
<td></td>
</tr>
<tr>
<td>-8.0</td>
<td>50.57</td>
<td>14.0</td>
<td>4.22</td>
<td>-0.25</td>
<td>60.1</td>
<td></td>
</tr>
<tr>
<td>+6.6</td>
<td>47.75</td>
<td>10.0</td>
<td>2.27</td>
<td>-0.32</td>
<td>55.2</td>
<td></td>
</tr>
<tr>
<td>+13.4</td>
<td>44.1</td>
<td>6.7</td>
<td>3.61</td>
<td>-0.37</td>
<td>48.8</td>
<td></td>
</tr>
<tr>
<td>+20.05</td>
<td>39.35</td>
<td>1.7</td>
<td>4.2</td>
<td>-0.35</td>
<td>36.50</td>
<td></td>
</tr>
</tbody>
</table>

* These letters denote the areas shown in fig. 7, p. 74.

TABLE VII.—Total Heat $I$ and Specific Heat $C_p$ of Liquid CO$_2$ at Constant Pressure. Taken from Curves.

<table>
<thead>
<tr>
<th>Temperature.</th>
<th>$I$, 700 lbs.</th>
<th>$I$, 900 lbs.</th>
<th>$C_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>°C</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-50</td>
<td>-25.9</td>
<td>—</td>
<td>0.47</td>
</tr>
<tr>
<td>-40</td>
<td>-21.0</td>
<td>—</td>
<td>0.49</td>
</tr>
<tr>
<td>-30</td>
<td>-16.0</td>
<td>—</td>
<td>0.515</td>
</tr>
<tr>
<td>-20</td>
<td>-10.7</td>
<td>—</td>
<td>0.54</td>
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TABLE VIII.—δφ between Pressure Curves and Limit Curve.

<table>
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<tr>
<th>Mean temperature, °C.</th>
<th>δφ between limit curve and 700 lbs./sq. inch line.</th>
<th>δφ between 700 and 900 lbs./sq. inch lines.</th>
<th>δφ between limit curve and 900 lbs./sq. inch line.</th>
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<tr>
<td>-31.5</td>
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Note.—700 lbs. curve meets limit curve at +12°.5 C. 900 lbs. " " " +23°.2 C.

TABLE IX.—Collected Results.

<table>
<thead>
<tr>
<th>Temperature, °C.</th>
<th>Pressure, lbs./in.² Authors</th>
<th>Liquid.</th>
<th>Latent heat, I.</th>
<th>Vapour.</th>
<th>Vapour density, gr./cm.²</th>
<th>Pressure, K. and R. and Amagat, atmospheres</th>
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* The pressures given in the last column, taken from Kuenen and Robson's figures from -50° C. to 0° C. and from Amagat's from 0° C. to +20° C., are probably more accurate than the authors', for reasons explained on p. 95.
TABLE X.—Comparison of Pressure-Temperature Observations for Saturated CO₂ Vapour.

Pressure in Atmospheres (760 mm. Hg).

<table>
<thead>
<tr>
<th>Temperature</th>
<th>Authors.</th>
<th>REGNAULT.</th>
<th>CAILLETET.</th>
<th>AMAGAT.</th>
<th>K. and R.</th>
<th>Z. and S.</th>
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<td>°C.</td>
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THERMAL PROPERTIES OF CARBONIC ACID AT LOW TEMPERATURES.

TABLE XI.—Series I. Test I. on March 25, 1912.

C.F.J. Record.

<table>
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<tr>
<th>Time.</th>
<th>Gauge pressure.</th>
<th>Voltmeter.</th>
<th>Time of</th>
<th>Weight on</th>
<th>Cold junction</th>
<th>Atmospheric</th>
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<td></td>
<td>h. m.</td>
<td></td>
<td>Coil A</td>
<td>Coil C.</td>
<td>ring.</td>
<td>temperature.</td>
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<tr>
<td>12 18</td>
<td>266</td>
<td>98.2</td>
<td>99.3</td>
<td>18 21</td>
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<td>9.4</td>
<td>—</td>
<td>7</td>
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<td>3</td>
<td>99.4</td>
<td>100.3</td>
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<td>11</td>
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<td>21</td>
<td>4</td>
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<td>—</td>
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<td>—</td>
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<td>9.2</td>
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<td>—</td>
<td>—</td>
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<td>—</td>
<td>—</td>
<td>—</td>
</tr>
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<td>5</td>
<td>6.1</td>
<td>9.8</td>
<td>26 34</td>
<td>19</td>
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<td>6.1</td>
<td>9.8</td>
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<td>27</td>
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<td>9.3</td>
<td>36 45</td>
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<td>9.9</td>
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<td>31</td>
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<td>9.5</td>
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<td>35</td>
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<td>3</td>
<td>5.8</td>
<td>0.2</td>
<td>43 34</td>
<td>12.51</td>
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</tbody>
</table>

Mean pressure } 263.3  Duration, 25.217 mins.

Mean readings, Coil A  . \{ \begin{align*} 97.5 & \text{ for } 9.65 \text{ mins.} \\ 89.48 & \text{ " } 10.0 \text{ " } \\ 95.28 & \text{ " } 5.57 \text{ " } \\ 97.14 & \text{ for } 9.65 \text{ mins.} \\ 89.06 & \text{ " } 10.0 \text{ " } \\ 94.80 & \text{ " } 5.57 \text{ " } \end{align*}  \}

Mean volts, Coil A  . \{ \begin{align*} 99.646 & \text{ for } 25.22 \text{ mins.}, \text{ Coil C.} \\ 95.28 & \text{ " } 5.57 \text{ " } \end{align*}  \}

Mean absolute pressure, 272.

Mean $\theta_b$ = 21.4.

Heat = 1484 Th.U.
TABLE XII.—Series I. Test I. on March 25, 1912.

D.R.P. Record.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Bath temperature.</td>
<td>Exit gas temperature.</td>
<td>Liquid temperature.</td>
</tr>
<tr>
<td>Time</td>
<td>Potentiometer</td>
<td>Time</td>
</tr>
<tr>
<td>h. m. s.</td>
<td>ohms cm.</td>
<td>h. m. s.</td>
</tr>
<tr>
<td>12 10 0</td>
<td>-5 74½</td>
<td>12 12 0</td>
</tr>
<tr>
<td>15</td>
<td>71½</td>
<td></td>
</tr>
<tr>
<td>18 30</td>
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</tr>
<tr>
<td>22</td>
<td>71½</td>
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<td>41</td>
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<td>75½</td>
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<tr>
<td>46 30</td>
<td>76</td>
<td>43</td>
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<tr>
<td>48</td>
<td>75½</td>
<td></td>
</tr>
</tbody>
</table>

Means. . . . -5 73½ -5 53 ———— Plotted.

Temperature from calibration curve

\[
\text{Bath temperature at start} \ldots \ldots \ldots -5 \quad 72 \cdot 4
\]

\[
\text{" " finish} \ldots \ldots \ldots -5 \quad 75 \cdot 75
\]

\[
\text{" " fall} \ldots \ldots \ldots \ldots \ldots 3 \cdot 35 \text{ cm.} = 0 \cdot 19 \text{ C.}
\]

\[
\text{Correction for change of cold junction temperature} \ldots \ldots = -0 \cdot 05 "
\]

Therefore nett bath fall \ldots \ldots \ldots 0 \cdot 14 "

Note.—D.R.P. watch was 26s. fast on C.F.J. watch.
Table XIII.—Series II. Test III. on March 26, 1912.

C.F.J. Record.

<table>
<thead>
<tr>
<th>Time</th>
<th>Voltmeter, coil D.</th>
<th>Time of ring</th>
<th>Weight on flask</th>
<th>Cold junction temperature</th>
<th>Atmospheric temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>h. m.</td>
<td>volts.</td>
<td>m. s.</td>
<td>lbs.</td>
<td>° C.</td>
<td>° C.</td>
</tr>
<tr>
<td>4:46</td>
<td>63:6</td>
<td>—</td>
<td>14</td>
<td>14:51*</td>
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<tr>
<td>5:09</td>
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<td>—</td>
<td>14:51</td>
<td>16:7</td>
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</table>

Mean ... 63:96

Corrected volts } 63:70

Duration, 29:18 mins.

Heat = \frac{(63:7)^2 \times 29:18 \times 3:189}{10^5} = 377:5 \text{ Th.U.}
### Table XIV.—Series II. Test III on March 26, 1912.

**D.R.P. Record.**

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<th>T.J. No. 4</th>
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<td><strong>Final temperature.</strong></td>
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<td>ohm cm.</td>
<td>h. m. s.</td>
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<td>4 43 0</td>
</tr>
<tr>
<td>45</td>
<td>26\textsuperscript{\textfrac{3}{4}}</td>
<td>46</td>
</tr>
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<td>26\textsuperscript{\textfrac{3}{4}}</td>
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<td>52</td>
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<tr>
<td>54</td>
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<td>27\textsuperscript{\textfrac{3}{4}}</td>
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<td>9</td>
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<td>18</td>
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<tr>
<td><strong>Means.</strong></td>
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<td>-8 6.6</td>
</tr>
<tr>
<td><strong>Temperature</strong></td>
<td>+ 8\textsuperscript{\textfrac{1}{2}}C.</td>
<td>-25\textsuperscript{\textfrac{3}{4}}C.</td>
</tr>
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</table>

**Ohm cm.**

- Bath temperature at start: 26·3
- Bath temperature at finish: 29·85
- Fall: 3·55 cm. = 0·195 C.
- Correction for cold junction: 0·0

**Nett fall:** 0·195
TABLE XV.—Series III. Test IV. on March 28, 1912.

C.F.J. Record.

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<td>volts.</td>
<td>m. s.</td>
<td>lbs.</td>
<td>° C.</td>
<td>° C.</td>
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<td>—</td>
<td>—</td>
<td>12</td>
<td>14·32*</td>
<td>15·0†</td>
</tr>
<tr>
<td>20</td>
<td>46·9</td>
<td>20 32</td>
<td>—</td>
<td>At 5h. 9m.</td>
<td>Start.</td>
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<td>21</td>
<td>.9</td>
<td>—</td>
<td>14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>.9</td>
<td>22 23</td>
<td>18</td>
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<td>23</td>
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<tr>
<td>33</td>
<td>46·9</td>
<td>33 31</td>
<td>30</td>
<td>14·36</td>
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<td>.9</td>
<td>—</td>
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<td>35</td>
<td>47·0</td>
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<td>—</td>
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<td>37</td>
<td>.0</td>
<td>37 18</td>
<td>32</td>
<td>14·365</td>
<td>15·2 Stop.</td>
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<td>.2</td>
<td>39 12</td>
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</table>

Mean . . 47·15
Volts . . 46·96

Duration, 18·67 mins.

Heat = \(\frac{(46·96)^2 \times 18·67\text{m.} \times 3·189}{10^5}\) = 131·2 Th.U.
TABLE XVI.—Series III. Test IV. on March 28, 1912.

D.R.P. Record.

<table>
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<td><strong>Bath temperature.</strong></td>
<td><strong>Initial temperature.</strong></td>
<td><strong>Final temperature.</strong></td>
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<tr>
<td>h. m. s</td>
<td>ohm cm.</td>
<td>h. m. s</td>
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<td>05 17 0</td>
<td>-0 57</td>
<td>05 17 30</td>
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<td>19</td>
<td>57 1/4</td>
<td>20</td>
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<td>58</td>
<td>23</td>
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<tr>
<td>25</td>
<td>58 3/4</td>
<td>26</td>
</tr>
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<td>28</td>
<td>58 3/4</td>
<td>29</td>
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<tr>
<td>34</td>
<td>59 1/4</td>
<td>35</td>
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<td>37</td>
<td>59</td>
<td>38</td>
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<tr>
<td><strong>Means</strong></td>
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<tr>
<td><strong>+ 11°1 C.</strong></td>
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<td>-13°0 C.</td>
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Bath temperature at start \[\text{ohm cm.} = -0 57.75\]
Bath temperature at finish \[\text{ohm cm.} = -0 59.0\]

Bath temperature fall \[1.25 \text{ cm.} = 0.07 \text{ C.}\]
Correction for change of cold junction temperature \[\text{ohm cm.} = -0.03\]

Nett fall \[0.04\]
TABLE XVII.—Comparison of $\phi$ Diagrams.

<table>
<thead>
<tr>
<th>Temperature, °C</th>
<th>$\phi$ liquid, Authors'</th>
<th>$\phi$ liquid, MOLLIER'S</th>
<th>Latent heat, Authors'</th>
<th>Latent heat, MOLLIER'S</th>
<th>$\phi$ vapour, Authors'</th>
<th>$\phi$ vapour, MOLLIER'S</th>
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<td>+20</td>
<td>+0.0435</td>
<td>+0.045</td>
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<td>+0.021</td>
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<td>55.45</td>
<td>0.1981</td>
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<td>79.9</td>
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TABLE XVIII.—New Data arranged as in MOLLIER's paper (1) and EWING's 'Mechanical Production of Cold' (4).

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<tr>
<th>Temperature, °C</th>
<th>Pressure, kg./cm.²</th>
<th>Behn's volume liquid, c.c./gr.</th>
<th>Volume of vapour, c.c./gr.</th>
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<th>Latent heat, r.</th>
<th>$\phi$.</th>
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Liquid. | Vapour. |
<table>
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<th>Porosity (%)</th>
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</tr>
<tr>
<td>Metal C</td>
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<td>8.80</td>
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</tr>
<tr>
<td>Metal D</td>
<td>0.10</td>
<td>9.20</td>
<td>2.5</td>
</tr>
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</table>

Table XIX: Properties of the Material
IV. The Capacity for Heat of Metals at Different Temperatures, being an Account of Experiments performed in the Research Laboratory of the University College of South Wales and Monmouthshire.

By E. H. Griffiths, Sc.D., F.R.S., and Ezer Griffiths, B.Sc., Fellow of the University of Wales.

Received April 1,—Read May 1, 1913.

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<tr>
<td>II. Outline of apparatus</td>
<td>122</td>
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<tr>
<td>III. Measurement of temperature</td>
<td>126</td>
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<tr>
<td>IV. Measurement of the resistance of the heating coil</td>
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<tr>
<td>V. Measurement of potential difference</td>
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<td>VI. The thermal capacity of accessory substances</td>
<td>138</td>
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<td>VII. Measurement of mass</td>
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<td>VIII. Measurement of time</td>
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<td>(2) Intersection</td>
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<td>XI. Experimental results—</td>
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<td>(1) Cu, (2) Al, (3) Fe, (4) Zn, (5) Ag, (6) Cd, (7) Sn, (8) Pb</td>
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<td>XII. Summary of results</td>
<td>170</td>
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<td>XIII. Nernst's observations at low temperatures</td>
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</tr>
<tr>
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<tr>
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</tbody>
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SECTION I.

Introductory.

A study of the published determinations of the capacity for heat of the elements leads to the conclusion that further information of an accurate nature is desirable. It will be found that, in most cases, the values are deduced by observations of the heat absorbed or given out when the changes of temperature are large, and the conclusions
derived therefrom are based on the assumption that the relation between the specific heat and the temperature is of a linear order.

Again, some, in fact a large majority, are comparative determinations and dependent on the capacity for heat of other bodies, as, for example, those which assume Regnault’s values for the capacity for heat of water at ordinary temperatures — values which we now know to be inaccurate.

The experimental difficulties connected with the method of mixtures are considerable and that method has probably been pushed to its extreme limits of accuracy. The agreement between the results obtained by different observers, and also between those resulting from repetition by the same observer, is rarely satisfactory. The temperature changes have, as a rule, been measured by means of mercury thermometers, without a proper appreciation of the difficulties attendant upon the use of those instruments for accurate work.

A further possible source of uncertainty is the effect of the sudden chilling of a metal when rapidly cooled from a high temperature.

The experimental conditions have not been varied sufficiently to demonstrate the absence of unsuspected causes of error and, according to the chemists, sufficient care has not been devoted to the detection and elimination of the impurities present in the samples used.

It is true that there are determinations of a high order of accuracy which may not justly be subject to this criticism, but such examples are few and it is difficult, when comparing the evidence, to assign to each determination its due weight.

As an illustration of the divergences which exist, we append the values given by leading authorities in the case of copper, a metal which does not appear to present any peculiar difficulties and one in which the values obtained by different observers are, on the whole, in better agreement than is the case with other metals.

All the observers agree with the conclusion that the capacity for heat of copper is a function of the temperature, but they differ markedly as to the value of the function. For example Tomlinson (‘Roy. Soc. Proc.,’ 1885) gives

\[ S_t = 0.09008 + 0.0000648t. \]

Lorenz gives values at 0° C., 50° C. and 75° C., from which the following expression is obtained:

\[ S_t = 0.0898 + 0.000204t + 0.000000317t^2, \]

an expression which denotes that the specific heat increases more rapidly than the temperature.

Gäede (‘Phys. Zeitschr.,’ 4, 1902) gives values at various temperatures between 17° C. and 92° C., from which the following expression can be deduced:

\[ S_t = 0.090135 + 0.000058t - 0.00000017t^2. \]

This would give a maximum capacity for heat at a temperature of 341° C.
If we consider the endeavours to ascertain the mean capacity over the range 0° C. to 100° C., the same lack of agreement is evident. For copper we have—

<table>
<thead>
<tr>
<th>Temperature range</th>
<th>Specific heat</th>
<th>Observer</th>
</tr>
</thead>
<tbody>
<tr>
<td>15–100</td>
<td>0.09331</td>
<td>BEEDE.</td>
</tr>
<tr>
<td>0–100</td>
<td>0.09332</td>
<td>TOMLINSON.</td>
</tr>
<tr>
<td>At 50</td>
<td>0.09169</td>
<td>LORENZ.</td>
</tr>
<tr>
<td>17–100</td>
<td>0.09333</td>
<td>NACCARI.</td>
</tr>
<tr>
<td>23–100</td>
<td>0.0940</td>
<td>TROWBRIDGE.</td>
</tr>
<tr>
<td>15 100</td>
<td>0.09232</td>
<td>TILDEN.</td>
</tr>
<tr>
<td>At 50</td>
<td>0.09261</td>
<td>GAED.</td>
</tr>
</tbody>
</table>

The methods described in this paper, although they doubtless present their own peculiar difficulties, are, we believe, free from many, if not all, of the sources of error above referred to. The method is briefly indicated in the following numbered paragraphs:

1. The energy was supplied electrically and the conclusions are not dependent upon any assumption concerning the capacity for heat of other bodies than those under consideration.

2. The substances were raised across a given temperature through very small ranges of temperature (extreme limit of range, about 1.4° C.).

3. These temperature changes were measured by means of differential platinum thermometers, for which purpose these instruments are admirably adapted.

4. Large masses of the substances were used, ranging from 1 to 4 kg.

5. The apparatus was constructed with all its parts duplicated. The metals examined were suspended by quartz tubes in similar air-tight brass cases which were placed side by side in a large tank containing rapidly stirred water or oil. This tank was electrically controlled with great constancy at any given temperature, \( \theta_e \).

One of the metal blocks remained at the tank temperature throughout an experiment while the other, having been previously cooled below \( \theta_e \), was raised to a somewhat similar temperature above it by a supply of heat electrically developed in the centre of the block, the difference in temperature between the two blocks being determined at regular intervals by means of the differential platinum thermometers.

All changes in the surrounding conditions would therefore affect both blocks equally; hence, by measuring the difference of temperature only, many possible causes of error were eliminated.

6. The equation connecting the various quantities is

\[
M \cdot S \cdot (\theta_2 - \theta_1) = \frac{E^2 \cdot t}{J \cdot R} \pm Q,
\]

where \( M \) = total mass, \( S \) its specific heat; \( \theta_1 \) the initial temperature, and \( \theta_2 \) the final temperature; \( E \), the potential difference at the extremities of the resistance.
coil $R$; $J = 4.184 \times 10^7$; and $Q$, the number of thermal units lost or gained during time $t$ from sources other than the electrical supply.

In these experiments the values of $\theta_1$ and $\theta_2$ were so arranged that $Q$ was in every case small or negligible, and, if necessary, could be estimated with sufficient accuracy.

7. With two exceptions, the samples of metals used were supplied by Messrs. Johnson and Matthey, to whom we wish to express our sincere thanks for the trouble they have taken in the matter. Their certificate concerning the degree of purity is in each case appended. Information regarding the remaining metals (Cu and Fe) will be found in the sections dealing with those two elements.

8. Experiments on identical samples at the same temperature were repeated under very varied conditions, in order to enable us to detect unsuspected sources of experimental error. Two separate methods of experiment, involving different data and methods of reduction, were employed. Three different sets of differential platinum thermometers were used. The rate of heat supply was varied in the ratio of 9:1. The determination of $S$ at a given temperature with a particular sample was in several cases repeated after the lapse of some months; the quartz tubes and cover were replaced by others of different masses &c. We were thus enabled to ascertain causes of error which would otherwise have remained undetected (see p. 139).

9. The results of our observations have been deduced from the actual experimental numbers and in no case from “smoothed curves.”

The most serious difficulty presented by this method of experiment is that of determining the mean temperature of the block of metal when its temperature is altering. Temperature gradients must necessarily exist, since equalisation of temperature by stirring is an impossibility. The manner in which this difficulty was surmounted is described in later sections.

When embarking on this investigation we proposed to extend our range of temperature to the lowest point obtainable by means of liquid air, limiting the inquiry to the study of two or three metals only. Owing, however, to delay by the contractors in the delivery of the liquid-air plant, we were compelled to postpone that portion of our investigation dealing with temperatures below 0° C. to a later date, and therefore enlarged the scope of our inquiry so as to include the following metals, namely, Aluminium, Iron, Copper, Zinc, Silver, Cadmium, Tin and Lead.

As the data already accumulated concerning the capacity for heat of these metals over the range 0° C. to 100° C. may be useful to other observers, we see no reason for delaying the publication of the work already completed.

**Section II.**

*Outline of Apparatus and of the Method of Experiment.*

A diagrammatic sketch of the apparatus within the tank is indicated in fig. 1.

CAPACITY FOR HEAT OF METALS AT DIFFERENT TEMPERATURES.

Fig. 1.

Dry Air (while cooling tube in position)
As the left-hand portion is a replica of the right, it will suffice to describe the latter only.

The metal (A), whose capacity for heat was to be determined, was cast and then "turned" into the form of a cylinder 15·2 em. long and 5·7 cm. in diameter.

This cylinder fitted accurately into a thin copper case (C) of mass, apart from the lid, of 149 grms. Thus the actual radiating surface surrounding the metal blocks was similar throughout all experiments.

Two small copper pins attached A in its proper position to the copper lid to which the case was fastened by a copper ring bearing a screw thread.

Three quartz tubes passed through the brass pipes fixed in the lid of the external case, and supported the copper case and block within the outer brass cylinder.

These quartz tubes having been previously platinized were soldered at their lower ends into short copper ferrules which formed parts of the copper lid and at their upper extremities to the top of the brass tubes which, for 7 cm. of their length, were washed by the tank liquid.

Between the case-lid and the outer brass lid a mica disc of nearly the diameter of the brass cylinder was placed and through it passed the three quartz tubes.

After the metal block had been fixed in its case and pins and ring firmly screwed home, the case and contents were lowered into the outer brass vessel, the edges of the mica disc (which were slightly padded with cotton wool) resting on a projecting circular ring (H) about 3·5 cm. above the top of the copper lid. Thus the effect of the flow of convection currents from inner to outer case, or vice-versa, was diminished. The brass lid was firmly screwed down over a lead ring by eight bolts.

After the parts were assembled, the air-tightness of the apparatus was ascertained both by pressure and exhaust tests. The lateral clearance between the inner and outer cases was 2 cm.; the vertical, between the lids, 6·5 cm.; and between the bases, 6·5 cm.

The volume of air contained in the brass case after insertion of the metal block was about 1500 c.c. A pressure gauge containing a light oil was connected by means of a 3-way tap with one of the tubes leading from this case to the exterior of the tank. Observations of the air pressure within the case were taken immediately before and after an experiment. The air being slightly warmed, the pressure rose during that interval, and thus the presence of any leakage could be detected.

The annular air space could be regarded as the bulb of a constant volume air thermometer, and from the change in pressure during an experiment, the change in the average temperature of the enclosed air could be deduced.

It was found that this change was about one-sixth of that of the contained block.

The approximate magnitude of any correction rendered necessary by the capacity for heat of the contained air could thus be ascertained (see Section VI.).

In our preliminary experiments the copper case was placed within specially constructed "vacuum vessels," the exterior walls of which fitted closely into the
surrounding brass vessels. The reasons for discarding their use are given on p. 159.

A cylindrical hole (K), 9 mm. in diameter and 14 cm. in length, was bored down the centre of each metal block, co-axial with the central quartz tube.

Into this was fitted the "heating coil," the wire of which was wound on a light mica frame of the X section used for platinum thermometers. The edges of the frame were deeply serrated to prevent any possible contact between the wire and the surrounding metal walls. This hole was filled, at low temperatures, with liquid paraffin (previously boiled and placed in vacuo to drive off volatile constituents), and at the higher temperatures with a heavy hydro-carbon oil.

Small mica "baffle plates" were inserted at intervals into the triangular sections of the mica rack, in such a manner as to deflect the convection currents outwards.

The hole O (depth 10.3 cm., diameter 11 cm.), co-axial with the left-hand quartz tube, contained one of the differential thermometers, the other being inserted into the corresponding hole in the left-hand block.

The position of the hole O was such that about half the total mass of metal was contained in the annular ring whose outer surface was in contact with the copper case, and whose inner passed through the centre of the hole. The various precautions taken to secure accuracy in the use of these differential thermometers will be described in Section III.

The third hole (G) was used for the purposes of cooling the block below the surrounding temperature, by the insertion of a thin-walled glass tube containing ether and connected with a water pump.

When the bath temperature was high the cooling process was a rapid one, but somewhat tedious at lower temperatures.

To prevent the entrance of laboratory air within the brass cases during cooling—which, by the deposition of moisture, might have had a serious effect, especially when the tank temperature was 0°C.—a current of well-dried air was passed by a branch tube into a larger one (F) which formed a continuation of the quartz tube leading to the cooling hole. This rapid up-flow of dry air was continued until the cooling tube had been withdrawn and replaced by a glass stopper, the lower end of which reached within 3 cm. of the inner copper lid, and thus prevented convection currents.

Our methods of experiment involved measurements of the following quantities:

(1) Temperature;
(2) Resistance of heating coil;
(3) Potential difference at ends of heating coil;
(4) Mass;
(5) Time;
(6) Thermal capacities of such bodies as oil, quartz, &c., whose temperature changed with that of the metal blocks.
The validity of our final conclusions is dependent upon the accuracy with which these quantities were determined, and in the following sections will be found a description of the methods adopted for their measurement. An error of 1 in 1000 in Nos. 1, 2, 4 and 5 supra, and an error of 1 in 2000 in No. 3 would affect our final results by 0.1 per cent. The thermal capacities of the bodies mentioned in (6), however, were so small, as compared with the capacities of the blocks, that the effect of an error of 1 to 5 per cent. in their valuation would fall below the 0.1 per cent. referred to.

We have, however, no reason to suspect that errors approaching such limits exist in any of the measurements above enumerated.

Section III.

Measurement of Temperature.

The platinum thermometers were of the standard form, thick platinum leads and compensators connecting the coil with the heads. All connections, both to the thermometers and the bridge, were made by means of small cups hard soldered to the ends of the leads and containing a fusible metal which expanded on solidification. The electrical connection thus formed was a perfect one and easily disconnected and re-made.

In our earlier experiments two thermometers, labelled AB and CD, were used. The constants of these thermometers have been previously published, and as far as we can detect, show no signs of change over a lapse of 15 years.

Their resistance was ascertained by means of a Callendar and Griffiths "self-testing" bridge, containing bare Pt-Ag coils immersed in rapidly stirred oil.

Thus, the temperature of the coils could be ascertained with great accuracy.

The bridge was carefully calibrated at the beginning of this work and all its coils and bridge wire divisions expressed in terms of the mean box unit; the absolute value of which, for the purposes of temperature measurements, was of no consequence.

The slight inequality (but 27 parts in 1,000,000) of the "equal arms" \( s_1 \) and \( s_2 \) was ascertained in the usual manner by observations of the apparent alteration in the resistance of a platinum thermometer immersed in ice, caused by exchanging the positions of \( s_1 \) and \( s_2 \).

The resulting correction has been applied to all our measurements of resistance taken with this box.

All the precautions previously published by one of us were observed, and we do not

* Hereafter referred to as "Box A." This bridge was last used in 1900, and it was then observed that one of its larger coils was showing signs of change. That coil was replaced by another one, and hence a recalibration of the whole bridge was necessary.
think it necessary to encumber the paper with a full table of the results. By the introduction of the new coil, the mean bridge unit (approximately \(1/T_0\) of an ohm) suffered alteration. Hence the values of \(R_i\) and \(R_o\) for AB and CD differ somewhat from those previously published. The alteration, however, is in the unit employed, rather than in the thermometers themselves, and the value of \(R_i/R_o\) and \(\delta\) may be regarded as unchanged.

One addition to the bridge, however, is worthy of mention, as it may be found useful in other cases. To obtain good contact by means of plugs, considerable pressure has to be exerted. As the insulating surface holding the brasses (in this case marble) is always somewhat yielding, the tendency of the brasses is to gape and thus contact is only made round a small horizontal section of the plug.

We therefore affixed to each pair of brasses an additional contact maker of the kind shown in fig. 2. The spring of the strip AB caused the wedge fastened on its lower surface to spring clear of its twin wedge on the block C when the screw-head E was sufficiently raised. The upright holding the screw passed through a slot in the strip without making contact therewith. The plug being firmly inserted in its hole, the head E was tightly screwed down; thus the wedge surfaces were firmly pressed together, and that with a sliding movement. The strain on the insulating surface was thus relieved, excellent contact was made by the wedge surfaces, also by the screw E connecting the strip with C and also by the plug itself. By this arrangement consistent results were obtained and one of the troubles of exact resistance measurements eliminated.

The constants of AB and CD were found to be as follows:

<table>
<thead>
<tr>
<th>Thermometer</th>
<th>(R_i)*</th>
<th>(R_o)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>2449·201</td>
<td>1763·279</td>
</tr>
<tr>
<td>CD</td>
<td>2449·044</td>
<td>1763·308</td>
</tr>
</tbody>
</table>

The difference at 100° C. resulting from the separate standardisations is therefore \(AB - CD = +0\cdot157\).

The thermometers were then coupled up differentially and together immersed in the hypsometer. The differential reading was then found to be \(+0\cdot158\).

* The barometer used when observing \(R_i\) was one whose scale coefficients, &c., had been determined by comparison with a French Standard of the Bureau International in 1896. Owing to an accident we had to refill it prior to this work; this was done with all the usual precautions.
In the same manner when in ice:

\[
\text{Difference when determined separately} = +0.028. \\
\text{Difference when determined differentially} = +0.027.
\]

In the course of such a comparison, eight connections had to be undone, re-made separately in sets of four, and then replaced in the first position, i.e., 12 removals and replacements. In the one case, a large number of the box coils were in use; in the other, the bridge wire only. The identity of the results is sufficient proof of the accuracy of the methods employed. These thermometer coils were surrounded by very thin walled tubes of Jena glass which fitted closely into the holes in their respective blocks and thermal connection between these tubes and the surrounding walls was assisted by a thin film of oil of known weight.

Although the observations with AB and CD appeared satisfactory, it was decided, after a considerable number of determinations of specific heats had been made by their means, to alter the conditions and replace them by two other thermometers labelled AA and BB. In these, the platinum leads were fused through glass heads, while the protecting tube was cut off just above the top of the coils, leaving about \(1\frac{1}{2}\) cm. to 2 cm. of glass projecting into the hole in the metal block.\

These thermometers were standardised by temporarily surrounding them with thin tubes containing sufficient oil to completely cover their coils, as we proposed to immerse them similarly when in the blocks. We afterwards found, however, that the effect of the oil was to increase, rather than decrease, the temperature "lag." Their coils were therefore freed from any traces of oil by washing with ether.

Their temperature then very rapidly responded to changes of temperature in the walls of their cavities, their heat capacity being very small as compared with their areas. The constants of AA and BB were as follows:

<table>
<thead>
<tr>
<th>Thermometer</th>
<th>(R_1)</th>
<th>(R_0)</th>
<th>(R_1 - R_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA</td>
<td>2582.983</td>
<td>1863.367</td>
<td>719.616</td>
</tr>
<tr>
<td>BB</td>
<td>2582.341</td>
<td>1863.307</td>
<td>719.034</td>
</tr>
</tbody>
</table>

After a considerable number of experiments had been performed, the glass head of AA was fractured. It was replaced by another thermometer of the same type which had been in our possession for the last 13 years.

Its resistance, however, was slightly less than that of BB. It was necessary, therefore, to reduce the latter until the two became approximately equal. These thermometers were labelled AA' and BB'. Their constants were:

* The tubes being wrapped round with threads of asbestos to prevent the passage of convection currents from the cavity.
CAPACITY FOR HEAT OF METALS AT DIFFERENT TEMPERATURES.

<table>
<thead>
<tr>
<th>Thermometer.</th>
<th>R₁</th>
<th>R₀</th>
<th>R₁ - R₀</th>
<th>δ*</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA'</td>
<td>2576.422</td>
<td>1859.579</td>
<td>716.843</td>
<td>1.50</td>
</tr>
<tr>
<td>BB'</td>
<td>2576.984</td>
<td>1860.052</td>
<td>716.932</td>
<td></td>
</tr>
</tbody>
</table>

It may here be stated that the results deduced from experiments performed with different pairs of thermometers were in excellent agreement, and afforded strong evidence of the accuracy of the temperature measurements.

The twin thermometers were connected in the usual manner, i.e., the coil of the first in series with the compensator of the second on one side of the bridge, the compensator of the first and the coil of the second being placed on the opposite side.

For the remaining sides of the bridge, several forms of approximately equal arms were used. All our later experiments were performed with two Pt-Ag coils, wound together on a mica rack and placed in a brass tube containing oil, the tube itself being immersed in a constant temperature tank. Their resistances at 0°C were

\[ S_1 = 1533.618, \quad S_2 = 1533.685. \]

Their resistances could be taken separately, and were determined in ice and also differentially, both at 0°C and at higher temperatures. Their continued equality with change of temperature was remarkable.

The galvanometer contact with the junction of \( S_1 \) and \( S_2 \) was made on a Pt-Ag wire connecting their ends and situated near the bridge wire. As both ends of the galvanometer circuit were connected with similarly situated Pt-Ag wires, the magnitude of the thermo-electric effect having its origin in these contacts was diminished. The Pt-Ag wires were further shielded by the massive casting of brass which carried the contact maker. During all standardisations and experiments the current through the bridge battery circuit was maintained at 0.013 ampere.

Fig. 3 shows the general arrangement and the approximate resistance of the various arms is indicated when the thermometers are at 0°C, the resistances being so arranged as to give nearly the maximum sensitiveness for a given current.

The current through the thermometer coils was less than 0.006 ampere; its heating effect was so small that it could be disregarded, both thermometers and, therefore, both blocks of metal being equally affected.

The battery key was of the type described in 'Phil. Trans.,' vol. 184, p. 398, and re-established the galvanometer circuit after the battery one was broken. Hence, the

* The value of \( \delta \) was determined in the vapour of boiling aniline (184·13 C.), as the resistance of thermometer AA' in sulphur vapour would have exceeded the total range of our standard marble top box A. The value of \( \delta \) was of secondary importance, as we were concerned only with the value of \( d\theta/d\theta t \) at various tank temperatures.
position of the galvanometer spot when the battery was disconnected was that due to any thermo-electric currents existing in the bridge and its connections, and thus any movement visible on establishing the battery circuit was attributable to that circuit only. The key, however, presented some novel features. Brass segments were fixed on a vertical spindle in such a manner that when the pointer was at 0, the galvanometer circuit alone was complete, when rotated through 120° both galvanometer and battery circuit were established, and on a further rotation through 120° the battery was reversed. The segments were so devised that induced currents during the "makes" and "breaks" would not affect the galvanometer. The whole series of operations could thus be performed very rapidly by one turn of the spindle.

The galvanometer was one of the original Paschen type. Its four coils were wound with wire whose diameters increased with their distance from the centre.

Four coils, each of about 5 ohms and separately adjustable, were used. It was desirable for the present work to obtain a system whose period of oscillation would be small, which would rapidly settle to its final position and yet have great sensitiveness.

For such a purpose it is advisable to use a system whose moment of inertia is reduced as far as possible. The type, constructed by ourselves, consisted of two groups, each containing 18 magnets astatically arranged. The extreme length of the longer magnets was about \(1\frac{1}{2}\) mm. The whole system, together with the mirror and the connecting glass fibre, weighed less than 11 mgr. It was suspended by a quartz thread about 17 cm. long and between 3 and 4 \(\mu\) diameter. The clearance allowed by the ovals in the coils was but a fraction of a millimetre and the faces of the coils were almost in contact, these faces being coated with tinfoil, to promote the damping of the oscillations by electro-magnetic induction. Reckoned on the usual scale, the sensitiveness of this galvanometer could have been easily raised beyond \(10^{-10}\), but by exterior magnetic control we reduced it until by one reversal of battery a deflection of 1 mm. indicated about \(\frac{1}{15,000}\) Pt as we found that, owing to the wandering of the zero point, a higher degree of sensitiveness detracted from, rather than increased, the accuracy of our observations.

The galvanometer had to be placed at a considerable distance (about 13 m.)
from the tank, as, if nearer, it responded to the flashing on and off of the heating lamps, the changes in the magnetic system of the chronograph, &c.  It stood on the top of a massive pillar of masonry which passed through the laboratory floor without contact, and whose foundations were embedded deeply below the base of an underground chamber.  The traffic in Cardiff is heavy, but by taking special precautions, the galvanometer in these circumstances was but little affected.  We found it necessary, however, carefully to guard the system against convection currents.  Every small opening near the suspension was blocked with slips of mica, and the whole galvanometer was enclosed within two separate chambers.  Many of the oscillations usually attributed to earth vibrations are, we believe, due to insufficient attention to the effect of convection currents.

As all our temperature measurements were observed in terms of lengths of the bridge wire, it is evident that the accuracy obtainable was dependent upon the accuracy of the calibration of that wire.  We have notes of a calibration made some 14 years ago.  Before these observations were begun a careful re-calibration was made in terms of the "mean unit" of the marble-top box (A) previously referred to.  The d'Arsonal galvanometer used on that occasion, however, was not sufficiently sensitive to enable the determination of the smaller inequalities.

The calibration was made in terms of 3 coils in Box A, of the approximate value of 1, 5, and 10 hundredths of an ohm.  Near the conclusion of our present work, a very careful re-calibration was conducted, with the object of ascertaining the accuracy of the earlier one and also of ascertaining if the bridge wire had suffered any alteration through use.  The Paschen galvanometer was employed; two separate and independent calibrations were conducted by the two observers and the results were in remarkable agreement.

It appeared that the calibration over the longer intervals on the former occasion was correct, thus showing that the wire had not suffered in the interval.

Each unit of the wire was then expressed in terms of a "mean Box A unit" (the same unit as that used in the standardisation of the thermometers), and a table was formed showing the value of a bridge wire unit at regular intervals, in terms of one Pt degree of each pair of thermometers.

It should here be stated that until the final steps in the reduction of our results, all temperatures are expressed in the platinum scale.

Section IV.

Resistance of Heating Coil.

Our methods of reduction demanded a knowledge of $R$ under the actual conditions prevailing during an experiment.  As it was impracticable to stir the oil in which the coil was immersed, a wire of small temperature coefficient was chosen to reduce to its smallest limits the correction for the heating effect of the current on the wire.

s 2
For preliminary experiments a 10-ohm coil of constantan wire was used, but was replaced in the final form of apparatus by a 20-ohm coil of bare manganin wire, as it was essential to eliminate, as far as possible, sources of thermo-electric forces in the potential circuit.

The diameter of the circle in which the wire was formed was approximately 7 mm.; the number of turns being 59. The upper end of the wire coil was situated about 20 mm. below the surface of the block; two straight manganin leads (1 mm. diameter) projecting from the coil terminated at their upper extremities at the junctions with the current and potential leads.

Both potential and current leads were of manganin, the latter being 1 mm. diameter, and to further diminish the heating effect of the current, two leads were connected in parallel. Thus six leads extended up the central quartz tube to a distance of 30 cm. These leads were insulated by perforated mica discs. A solid wad of such discs was fixed between the top of the rack and the junctions to the current and potential leads, in order to diminish the passage for convection currents.

As the resistance of the coil had to be observed in situ at each temperature and at frequent intervals, four brass cups, amalgamated inside and containing mercury, were soldered on the current and potential leads outside the apparatus; plugs enabled us to isolate these circuits, when a resistance had to be taken, from the various connections to battery, &c. Heavy leads from a dial resistance box terminated in a pair of brass cups alongside those above referred to.

If

\[ R \] be resistance of coil,

\[ r_1 \text{ and } r_3 \] ,, ,, current leads,

\[ r_2 \text{ and } r_4 \] ,, ,, potential leads from cups to coil,

then, if

\[ N_1 = R + r_1 + r_3, \]

\[ N_2 = R + r_2 + r_4, \]

\[ N_3 = r_1 + r_2, \]

\[ N_4 = r_3 + r_4, \]

we have

\[ R = \frac{1}{2} \{(N_1 + N_2) - (N_3 + N_4)\}. \]

As the absolute value of \( R \) was required, the resistance of a reference heating coil (of the same construction and about the same value as the one used) was determined in ice by means of the dial box, and then forwarded to the National Physical Laboratory, where its value was determined in international ohms; on its return we checked our previous determination by the dial box.

This enabled us to reduce our determinations of the resistances of the heating coils used in the work to international ohms.
CAPACITY FOR HEAT OF METALS AT DIFFERENT TEMPERATURES.

Resistance of Reference Coil.

National Physical Laboratory Report.

"Coil Immersed in Unstirred Paraffin Oil cooled to 0°C.

<table>
<thead>
<tr>
<th>Resistance in international ohms.*</th>
<th>Testing current through resistance.</th>
</tr>
</thead>
<tbody>
<tr>
<td>20·1360</td>
<td>0·025</td>
</tr>
<tr>
<td>20·1362</td>
<td>0·050</td>
</tr>
<tr>
<td>20·1370</td>
<td>0·100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Resistance in legal ohms in terms of our dial box.</th>
<th>Testing current through resistance.</th>
</tr>
</thead>
<tbody>
<tr>
<td>20·2010</td>
<td>0·001</td>
</tr>
</tbody>
</table>

Hence, the factor to convert our box readings to international ohms = 0·999678.

As the same ratio arms and plugs in the thousand and hundred dials were used in the determination of both coils, any change in the relative values of the dial box since calibration ('Phil. Trans.,' vol. 184, p. 409) would not appreciably affect our results. Fig. 4 shows the variation with temperature of the heating coil and the permanent change by use and exposure to temperatures of about 100°C.

* "Probable error of resistance values is not greater than 2 parts in 100,000."
The value of \( R \) so determined requires a small correction, as it includes the entire resistance of the two straight leads, previously mentioned, connecting the upper ends of the coil with the potential junctions.

Of the heat generated in these leads, a portion is lost by radiation, &c., owing to their projecting 24 mm. above the surface of the oil.

The resistance of the leads could not be diminished beyond a certain limit, on account of thermal conduction along them of the heat from the hot oil. A diameter of 1 mm. was decided upon, as the thermal conductivity of a manganin wire of this size would be negligibly small.

A certain amount of heat was developed in these short leads by the current. That generated in the 20 mm. below the surface of the oil would undoubtedly be absorbed by the oil. Of the heat generated in the 24 mm. above the oil-surface, it is probable that about one-half would pass into the block, &c., by conduction and by radiation to the ferrules and quartz tubes.

Taking the actual figures:—Resistance per millimetre of the wire = 0'000642. Hence, resistance of portion above the oil surface = 0'0154 ohm.

On the above assumption only the heat generated in half of this was, in the case of either lead, effective in heating the block. We confess that this is merely an assumption, but, with our knowledge of the actual conditions, it appears to be a reasonable one; moreover an error of 10 per cent. therein would only affect the absolute value of our results by less than 1 part in 10,000.

**Change of Resistance due to Change in the Current.**

If \( R' \) is the resistance of the heating coil when a certain current (defined later) is passing through it and \( R \) is the value determined in the usual manner by the dial resistance box, then we define \( \Delta R \) by the relation

\[
\frac{R'}{R} = 1 + \Delta R.
\]

The effect of the temperature rise (produced by the heating current) on the resistance of the wire was of course very small in the case of an alloy like manganin; the resulting correction, however, could be determined with considerable accuracy in the following manner (see fig. 5).

A series of observations was made in which the current was measured by the ordinary potentiometer method. Included in the circuit was a 3-ohm coil (W) of bare manganin wire immersed in stirred paraffin oil. It consisted of 4 strands of 0'4 mm. diameter, in parallel, wound on eight projecting mica plates fixed longitudinally on a wooden drum. The passage of the maximum current (0'45 ampere) for intervals of several minutes did not produce any appreciable change in the temperature of the oil.

One observer adjusted the current in the circuit until the potential difference at
the ends of the heating coil R' was balanced against that of a number of cadmium cells, as in the ordinary method of experiment.

The second observer measured the potential difference at the ends of the 3-ohm coil, by means of a Thomson-Varley potentiometer (P), the readings being taken to about 1 part in 50,000, by interpolation by galvanometer swings.

Fig. 5.

R'. Heating coil in metal block.
W. Manganin coil (about 3 ohm) immersed in oil.
P. Thomson-Varley potentiometer.
S and S'. Rheostats to adjust current.
B. Main storage battery.
Cd. Standard cells balanced at ends of R'.
G₁. High-resistance galvanometer.
G₂. Paschen galvanometer.

Observations were taken when the potential difference at the ends of the heating coil was varied in steps from that of three to eight standard cadmium cells. The temperature of the block was maintained approximately constant by cooling with the ether tube.

Calculation of \( \delta R \).

If \( \omega \) is the resistance of the 3-ohm coil, W, and the current in the circuit is caused by a potential difference of \( nE \) at the ends of the heating coil, and R' is the resistance
of the heating coil for that value of the current, then

\[ C = \frac{nE}{R'} \]

Potential difference at the ends of 3-ohm coil is

\[ \frac{nE}{R'} \times \omega \]

If \( s_1, s_2, \ldots, s_n \) be the potentiometer readings corresponding to 1, 2, 3, 4, \ldots, \( n \) number of Cd cells balanced at the ends of the heating coil, then

\[ s_n = K \cdot \frac{nE}{R'} \times \omega \; \text{; where } K \text{ is a constant} \]

Hence,

\[ R' \propto \frac{n}{s_n} \]

By plotting \( n^2 \) horizontally—since the heating effect is proportional to the square of the electromotive force—and the quantity \( n/s_n \) vertically, but reduced in such proportion that for \( n = 0 \) it is unity, we get the relation between \( \delta R \), the increment of resistance, and \( n^2 \). The resulting points fall (within the limits of experimental error) on the straight line, \( \delta R = kn^2 \). These observations were repeated when the tank temperatures were 0° C. and 97° C., and for both the copper and iron blocks.

In the locality of 0° C. the temperature coefficient of manganin is positive, as shown by the relation

\[ \text{At 0° C., } \delta R = 0'0.52n^2. \]

About 100° C. manganin has a negative coefficient and it was found on reducing the results that \( \delta R \) was negative.

\[ \text{At 97° C., } \delta R = -0'0.86n^2. \]

These equations represent the extremes of \( \delta R \) in our range, for at intermediate temperatures, owing to the locus of R being concave downwards, the factor \( k \) was smaller and vanished altogether between 50° C. and 60° C.

It may be pointed out that for the highest rate usually employed, viz., that due to 8 cells, the correction on account of \( \delta R \) amounted to only 3 parts in 10,000, corresponding to a temperature change in the wire of 10° C. As the values of \( \delta R \) at both 0° C. and 97° C. indicated that the rise in temperature of the wire depended on \( n^2 \) only and was independent of all other conditions, we could, from the curve giving the relation between temperature and resistance, calculate the relation between \( \delta R \) and \( n^2 \) for any temperature within our range.

The value of \( R \) was the one directly determined by the dial box when the heating effect was insignificant, the current through the coil being in that case 0'0015 ampere.
The potential difference at the ends of the heating coil was always balanced against an integral number of standard Weston cells in series. A batch of 25 cells was constructed for use in this investigation, according to the method described by G. A. Hulet ("The Construction of Standard Cells, &c.," 'Physical Review,' Vol. XXXII., 1911).

The glass work was of the usual H form, the platinum leads not being sealed through the glass. The two limbs extended about 15 cm. above the cross tube, and were closed by corks; through these corks passed the electrodes sheathed by capillary tubing, the fine platinum wires projecting 5 mm. beyond the sealed ends.

This mode of construction admits of the cells being directly immersed in water, with the limbs projecting about 7 cm. above the surface.

The water tank containing them was of considerable capacity and well lagged, the temperature rarely varying by one-fifth of a degree Centigrade per day.

The leads from the cells passed to a switchboard across well insulated supports.

The cells were frequently compared by means of a Thomson-Varley potentiometer with two standard Weston cells constructed by Mr. F. E. Smith of the National Physical Laboratory. (We take this opportunity of thanking Mr. F. E. Smith for presenting us with these cells.) Table I gives the values of the cells in terms of the N.P.L. standard. All our results are expressed in terms of these standards.

From an examination of the comparisons at various times during the course of fifteen months, we can find no change greater than that which might be attributed to the experimental errors.

**Table I.**

Temperature 17° C.

The National Physical Laboratory Standards are denoted by symbols BC–1, BC–2.

<table>
<thead>
<tr>
<th>No. of cell.</th>
<th>E.M.F. in international volts.*</th>
<th>No. of cell.</th>
<th>E.M.F. in international volts.</th>
</tr>
</thead>
<tbody>
<tr>
<td>BC – 1</td>
<td>1.0184</td>
<td>10</td>
<td>1.0184</td>
</tr>
<tr>
<td>BC – 2</td>
<td>1.0184</td>
<td>11</td>
<td>1.0185</td>
</tr>
<tr>
<td>1</td>
<td>1.0183</td>
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<td>1.0183</td>
<td>17</td>
<td>1.0184</td>
</tr>
<tr>
<td>9</td>
<td>1.0184</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* The tables used in the reduction of the observations express the E.M.F. of our cells in terms of the standards correct to 1 part in 20,000.
Statement received with Weston normal cells BC—1, BC—2:

"E.M.F. = 1'0183, international volts at 20° C.

Temperature coefficient:—

\[ E_t = E_{20} - 0'0000406(t-20) - 0'000000095(t-20)^2 + 0'000000001(t-20)^3 \, \]

While an experiment was in progress, the current in the heating coil was continually adjusted to keep the balance exact. This was effected by two rheostats in parallel; the shunt being of fairly high resistance. After the preliminary adjustment, the potential balance could be maintained by use of the shunt alone.

The sensibility of the high resistance Thomson galvanometer (7000 ohms) in the standard cell circuit was such that a deflection of 1 mm. on the galvanometer scale corresponded to a change in the potential difference of 1 part in 20,000.

During the course of an experiment, the potential balance could be maintained with great steadiness, the slight oscillations rarely amounting to more than 1 part in 10,000.

**Section VI.**

**Minor Corrections for the Thermal Capacity of Accessory Substances.**

In order to facilitate transmission of heat from the heating coil to the metal block, the central hole was, as previously stated, filled at low temperatures with boiled paraffin oil, and at higher temperatures with a heavy hydrocarbon.

The quantity of liquid thus inserted varied slightly with different blocks, the average volume being about 7 c.c. It was therefore necessary to ascertain the thermal capacity of the oils and their approximate variation with temperature.

In the construction of the apparatus 15'06 gr. of alloy were used in fixing the quartz tubes to the lid of the copper case first employed, and 8'25 gr. of solder in the second and somewhat heavier lid.

Again, the thermal capacity of the portions of the glass sheaths of the thermometers which entered the block had to be allowed for; the mass thus inserted amounted, when thermometer CD was in use, to 3'1 gr., and, in the case of AA and AA', to 1'67 gr.

Lastly, allowance had to be made for the heat absorbed by the lower ends of the quartz tubes which supported the apparatus. This was a most difficult correction to determine, as it was not possible, à priori, to specify what mass of the quartz could be regarded as raised through the same temperature as the copper lid.

1. **Specific Heat of Glass and Oil.**

The mean specific heat of the paraffin oil between 0° C. and 100° C. was determined with a Bunsen's calorimeter, by the introduction of about 2'8 c.c. of oil sealed in a thin glass bulb.
The mercury drawn into the Bunsen was directly determined from the loss in weight of a small capsule; in the first experiment, 2'7347 gr. of mercury; in the second experiment, 2'7335 gr. of mercury.

The constant assumed was 15'44 mgr. mercury per calorie.

The resulting value of the mean specific heat for this paraffin was 0'491, its density being 0'813.

The mean specific heat of glass was determined in a similar manner and was found to be 0'194.

The specific heat and temperature variation of the hydrocarbon oil had already been determined ('Phil. Trans.,' vol. 186, p. 338), viz., \( S_t = 0'466 + 0'0009t \), its density being 0'865. As both oils were paraffins, the temperature coefficients were assumed to be the same, to a first approximation.

2. Alloy and Solder.

The alloy-fixed lid was used only in our earlier experiments and when determining the specific heats of certain metals at 0° C.

The mean specific heat of this alloy over the range 0° C. to 47°'6 C. (its melting point being 97° C.) was also ascertained by the Bunsen calorimeter and was found to be 0'0348.

The soldered lid was used over the range 0° C. to 125° C., hence, both its specific heat and its temperature variations were required.

A block of the sample of solder (3 kgr.) was cast and machined to the same size as the other metal blocks. Its specific heat at 0° C. and 97° C. was determined in our apparatus in the same manner as copper, &c.,

\[ S_t = 0'0422 + 0'000038t. \]

On analysis, the composition of the block was found to be 53'7 per cent. tin and 46'0 per cent. lead, with bismuth and antimony as impurities. The density of the solder block was 8'77. The platinum deposit on the ends of the quartz tubes was negligible, the weight of three coats being only 0'0105 gr.

3. Heat Absorbed by the Supporting Quartz Tubes.

As previously indicated, two different copper lids were used. The copper alone in the first weighed 51'6 gr. and thin quartz tubes were fixed into its ferrules (which were 15 mm. in depth) by fusible alloy. The copper alone in the second lid weighed 68'5 gr.; the tubes were of much heavier make and fixed with solder. The masses of the quartz tubes per unit length in the first lid were only three-fifths of those used in the second; advantage was taken of this difference to determine the effective capacity for heat of that portion of the quartz which might be regarded as rising through the same range as the block of metal.
Two series of experiments with copper at 0° C. were performed under precisely the same conditions, except that the lid with the lighter tubes was employed in the first, and that with the heavier, in the second, series.

From the differences in the capacity for heat resulting from these two series, the difference in the amount which had passed into the quartz tubes could be determined, the capacity of the alloy, solder and copper, being known. Let \( m_1s_2 \) and \( m_2s_2 \) be the respective quartz capacities. Then, from the experimental results we found

\[
m_1s_1 = 1.37; \quad m_2s_2 = 2.28.
\]

No doubt, a temperature gradient existed along these tubes, but the value of \( sm \) thus obtained gave the "effective" capacity, i.e., the number of calories which flowed into the quartz tubes as the temperature of the block was raised by 1° C.

A comparison of other experiments at 0° C., where the conditions were similarly altered, indicated that the accuracy of this "quartz correction" was sufficient.

4. Heat Absorbed by the Air Within the Brass Vessel.

As stated in Section II, we were able to measure the average increase in pressure, and therefore in temperature, of this air during an experiment. The volume being approximately 1500 c.c., the average rise of pressure was 0.7 mm. Hg, indicating an increase of temperature of 0.25 C. The number of calories thus expended equals 0.08.

As the average heat supply during an experiment was about 400 calories, it is evident that this correction would not amount to more than 1 part in 5000; we did not consider, therefore, that the accuracy of our experiments necessitated the inclusion of this correction, especially as it could be only roughly determined.

Section VII.

Mass and Density of the Metals.

The masses of the blocks varied from 1 to 4 kgr. The balance used for the larger masses was capable of weighing to 0.01 gr.; masses under 100 gr. were determined by a Verbeek short-beam balance. The method of double weighing was used and a correction was applied for the displaced air.

A calibration of the box weights proved that, relatively, they were correct to a high degree of accuracy; as, however, the absolute values were required, the 2 kilos. (from another box), and the 1 kilo. and 100 gr. from this box were forwarded to the National Physical Laboratory for standardisation. The kilogram and the 100 gr. weights of our box were found to be exact and the correction on the 2 kilos was given as +0.29 gr.
CAPHAT FOR HEAT OF METALS AT DIFFERENT TEMPERATURES.

Densities.

The densities of the metals were found by weighing in air and water, the usual precautions being observed. The values were checked by calculation from the dimensions and weight.

SECTION VIII.

Measurement of Time.

The only records made during the course of an experiment were the times of transit of the temperature of the thermometer in the block past bridge-wire divisions, and as these were effected mechanically by the depression of a key, the observer's attention could be concentrated on the transits.

The time-recording arrangements may be briefly described as follows:—

An electrically driven seconds pendulum bob, suspended by an Invar rod, at each swing tilted over an exhausted tube about 2 inches long by $\frac{1}{2}$-inch diameter, fixed in a frame capable of oscillating about an axis perpendicular to the length of the tube.

As the carriage bearing the tube was unstable about this axis, a slight impulse sent it over from one stop to the other, causing a small mercury pellet to run down the tube and make momentary connection between two platinum wires fused midway into it, thus completing the electro-magnet circuit in the chronograph.

A series of equally spaced dots about 2 cm. apart on the tape indicated seconds, while the marks of the respective observer's keys were recorded on opposite sides of the tape. By counting and measuring the fraction, the times of transit could be obtained to $\frac{1}{10}$ second.

Although the seconds pendulum kept a fairly constant rate—being fitted with a cut-out device to keep its amplitude to a definite limit—the absolute rate was determined for each experiment by comparison with a rated chronometer.*

This comparison of the total time also afforded a check on the accuracy of the reading of the tape, always a somewhat laborious process, the lengths of tape used varying from 20 to 60 metres.

SECTION IX.

Temperature Control of the Baths.

The absolute steadiness of the bath temperature was of prime importance, as our conclusions were based on the assumption that the temperature of the walls surrounding the blocks remained constant throughout an experiment.

* We are indebted to Mr. T. J. Williams, 63, Bute Street, Docks, Cardiff, for the loan of this chronometer, and for kindly checking its rate from time to time.
When the values of the specific heat at 0° C. were required, a special tank of 15 gallons capacity and lagged with asphalt was used. Two screws—protected by strong metal cages—caused a rapid circulation of water through the powdered ice. For work at higher temperatures this tank was replaced by an oil or water bath, of capacity about 20 gallons, the heat being supplied by immersed electric radiator lamps.

Some difficulty was encountered in insulating the 200-volt leads of these lamps when the tank-temperature was above 50° C., owing to the softening of the stretched rubber tubing by continued exposure to hot water. The cement fastening the caps of the lamps frequently broke down and entailed the loss of several experiments.

The most satisfactory method of insulation hitherto tried was by clipping a discarded motor tyre tube over the end of the "radiator" lamp, the leads being separated within the tube by lengths of glass tubing.

The "lux" lamps used for the purposes of fine adjustment, were insulated by fixing glass tubing of slightly larger diameter over their ends, the joint being closed by a short length of rubber tubing well covered with adhesive tape.

**Thermostats.**

We tried a considerable variety of thermostats which proved defective from one cause or another. The two most satisfactory ones may be briefly described. The first was composed of thin solid-drawn copper piping \( \frac{3}{4} \) inch internal diameter and 16 feet long.

This tube was wound into an oval spiral, so as to surround the two brass cases in the tank (see fig. 1). Two glass tubes were soldered into reduction pieces at each end, one terminating in a tap, the other in a U-tube containing mercury. As the method of constructing these soldered joints is both simple and effective, we have given in Appendix III. a brief description of the process.

Another form of thermostat used in a considerable number of determinations consisted of a large branchwork of glass tubes fused together and so distributed as to take the mean temperature of the tank. The thermostat was filled with toluol, which however proved unsatisfactory at high temperatures, and was replaced by commercial aniline, which in every way seems to be a suitable liquid for thermostats. It has a high coefficient of expansion, low viscosity and a high boiling point (184° C.).

We found it necessary to keep the tap closing the thermostat well greased, otherwise slow leakage and consequent drift of temperature took place.*

The motion of the mercury in the U-tube operated a relay, which in turn switched on and off the lamps in the tank. Both the make and the break in the main circuit

* We have some reasons for suspecting that, owing to neglect of this precaution, the temperature of the bath was not maintained with its accustomed steadiness during some of the group of observations about 67° C.
and the relay circuit had a pair of aluminium plates in water as shunts, to diminish sparking.

A considerable fraction of the heat necessary to maintain the tank at the required temperature was given by a constant supply, while the relay operated the fine adjustment.

The intermittent lamps were placed close to the stirrer, and thus the whole arrangement tended to keep the oscillations of the temperature within narrow limits. At some temperatures we had thermometers by which we could detect changes of \( \pm 0.01 \)° C., but when the apparatus was working satisfactorily we at no time observed oscillations of this magnitude.

Another circumstance which perhaps assisted in diminishing the oscillations was the fact that the stirring was sufficiently vigorous to cause a continual vibration of the U-tube of the thermostat and so prevent any adhesion of the mercury to the platinum point which established connection with the relay.

**SECTION X.**

(1) *The Total Heat Method.*

The metal under examination was cooled to a temperature lower than that of the tank \( (\theta_0) \) and the fall being observed by means of bridge-wire observations, it was stopped when it had passed below the range of the bridge.

The contact-maker was then set at a certain reading, which, for clearness, we will specify as \(-9\).* Meanwhile, the "heating" current was adjusted on an auxiliary coil enclosed in a tube containing oil. This auxiliary coil was a duplicate of the coil in the metal block and the change-over from the one to the other could be effected by the depression of a recording key. Before the transference of the current, the temperature of the block rose very slowly by radiation, &c., and could be followed by the gradual approach of the galvanometer spot to its zero mark.

The rate of rise was of the order of 0°-0.36 Pt per second, consequently the temperature throughout the block was practically uniform.

The moment the temperature had reached the bridge reading \(-9\), as indicated by the transit of the spot across its zero mark, the heating current was switched over, the key† at the same time recording the time on the chronograph tape. A slight readjustment of the rheostat was usually required to maintain exact potential balance when the change-over was effected. The contact-maker of the bridge was then set at the next integer, \(-8\) (the temperature interval from \(-9\) to \(-8\) being roughly

* This was the customary starting point.

† The key was so constructed that any time lag between the marking of the tape and the actual switch on was compensated for during the operation of switching off.
and after slight adjustment of the galvanometer spot to its zero mark, the
galvanometer key was turned so as to re-establish the bridge current (0.013 ampere).

The rise of temperature, as indicated by the movement of the spot, was uniform,
and its transit was recorded by a tap on the chronograph key. The cycle of
operations was repeated; the transits of the temperature across each bridge-wire
reading being recorded in succession until the temperature had risen to +8 bridge-
wire reading, when the current was switched over to the auxiliary coil.

After the current had ceased to impart any heat to the metal, the observed
temperature continued to rise, on account of excess of heat in the oil, the gradient
from the interior to the surface of the metal and the temperature lag of the thermo-
meter.

The metal would, however, after its temperature had risen to a maximum, part
with its heat by radiation, &c., only, the resulting fall being slow and regular. This
"rise above," as we termed it, could be accurately determined by the following
procedure:—

The bridge contact-maker was set above the switching-off point by an amount such
that the galvanometer spot moved to near the centre of the scale before the regular
cooling began. The galvanometer deflections on reversal of the bridge current were
noted, and also (on the chronograph tape) the times of the observations, until the
deflections had increased beyond the range of the scale. The value of 1 mm. scale
deflection in terms of a bridge-wire unit being known,† the rate of fall in temperature
could be determined, and the temperature time-curve P...ABCD could be constructed.

One of the resulting diagrams for the "rise above" is shown in fig. 6.

If P is the point at which the current was switched off, the slope of the line CD, i.e.,
the rate of uniform cooling, gives the data required for the determination of the
horizontal line EG, and thus the temperature which the metal would have attained, in
the absence of radiation, &c., can be ascertained.

If GE be produced backwards to meet the temperature ordinate at F, then it will
be evident that F falls on DC produced.

Thus PF, the rise due to the residual heat in the block, could be determined at the
close of each experiment with considerable accuracy. The value rarely exceeded
0.1 Pt and could be measured to 1 part in 1000, that is, about 1 in 15,000 of the
whole range.

It may be mentioned that the "rises above" for a series of experiments with the
same metal under the same conditions were proportional to $n^2$, i.e., to the rate of supply.

* The galvanometer system generally required this slight readjustment between each observation of
transit in order to maintain the spot on the scale zero when the bridge current was broken. This was
affected by the movement of a small subsidiary control magnet on the table by the observer, and about
1/4 m. distant from the galvanometer. The changes of zero were chiefly those due to variations in the
thermoelectric effects in the circuit, and with considerable attention to shielding the various junctions we
succeeded in diminishing such changes to small dimensions, but could not altogether eliminate them.

† Ascertained for each set of experiments.
We may here point out that the two fundamental observations which determined the temperature range were taken when the temperature of the metal was steady and practically uniform, the only change taking place being that due to the very small rate of rise or fall consequent on radiation, &c.

Fig. 6. "Rise above" for Experiment IV., June 2, 1912 (7 standard cells).

We have next to consider any other necessary correction for the effect of radiation during an experiment. When the current was established at −9 bridge-wire reading, the oil had first to rise in temperature, then a gradient established from centre to surface of block and, when the temperature began to rise, the thermometer would undoubtedly lag behind the temperature of the surrounding walls.

For these reasons, the time of rising through the first bridge-wire division would considerably exceed the times of passing over succeeding equal divisions.

It was found that when the temperature had reached the end of the first bridge-wire division, the conditions had become practically steady, as shown by the fact that in subsidiary experiments in which the current was switched off at the end of the first bridge-wire division, the "rise above" was found to be very nearly the same as when the experiments were completed in the customary manner.

We also investigated, with the smaller currents, the curve showing the rate of rise of the thermometer throughout this first interval and it appeared that, during the first half of the time of passing through the interval, the thermometer only rose from −9 to −8.7 bridge-wire. Consequently, this reading −8.7 bridge-wire may be regarded as approximately the mean temperature throughout the time of the first interval.
The temperature ranges above and below $\theta_0$ were so selected that, excluding the first interval, the two ranges were equal and as they were small (about 0°6 C.), the times over these ranges were so near equality that the losses and gains due to radiation might be neglected.* Hence, the only radiation correction required was that which expressed the heat thus received as the metal rose through the first bridge-wire interval. The true time, however, over that interval was less than the time recorded between the switching on the current and the first transit, owing to the causes of lag above referred to. As this lag was known in terms of temperature, by the "rise above," it was possible to obtain from it an expression involving time.

If

$$\theta_t \text{is the "rise above" in degrees Pt,}$$

$$t_1 = \text{the average time of rising through 1° Pt when the temperature of the block is rising steadily on account of the heating current,}$$

then $\theta_t \times t_1$ would be the approximate time, at any part of the range, of moving through the "rise above"; this we term the "time lag" $= \tau$.

$\tau$ was found to be practically the same for all rates of heating for the same metal. For example, in the case of copper, 36 seconds; of silver, 40 seconds.

Hence, the actual time over the first interval was equal to the observed time diminished by $\tau$.

The rate of rise per second due to radiation alone was obtained by two distinct methods, namely:--

I. By subsidiary experiments in which rate of rise due to radiation alone over the range $-9$ to $-8$ was observed;

II. From the observations of the transits taken during the actual experiment when the conditions were settled.

For, if

$$M = \text{mass of the substance,}$$
$$S = \text{specific heat at } \theta_0,$$
$$ms = \text{thermal capacity of oil, copper case, \\&c.,}$$
$$\sigma = \text{rate of rise due to radiation alone for a difference of 1° Pt between the block and the surroundings,}$$
$$t = \text{time in seconds,}$$
$$E = \text{E.M.F. of a standard Weston cell,}$$
$$n = \text{number of cells balanced,}$$
$$R = \text{resistance at this temperature corrected for heating effect of the current,}$$

* In a previous communication ('Phil. Trans.,' vol. 184, p. 500) it was shown that if $t_1$ is the time of rising to $\theta_0$ from any temperature below that of the tank, and $t_2$ is the time from the lower temperature to an equal range above the tank, then the sum of the losses and gains due to radiation, \\&c., is zero at a time $= 2t_1 + \frac{1}{2}(t_2 - 2t_1)$.

In the conditions above indicated, the error due to the assumption that the radiation, \\&c., was zero at time $t_2$ was found to be negligible.
then
\[
\left( \frac{d\theta}{dt} \right)_{at \theta} - \sigma \theta' = \frac{(n E)^2}{JR(MS + ms)}
\]
and
\[
\left( \frac{d\theta}{dt} \right)_{at \theta} - \sigma \theta'' = \frac{(n E)^2}{JR(MS + ms)}
\]

Hence
\[
\sigma (\theta'' - \theta') = \left( \frac{d\theta}{dt} \right)_{at \theta} - \left( \frac{d\theta}{dt} \right)_{at \theta}.
\]

The table below shows the values of \(\sigma\) deduced from the two methods, and their agreement affords strong evidence of the accuracy of the resulting correction.

<table>
<thead>
<tr>
<th>Metal</th>
<th>Method I</th>
<th>Method II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silver</td>
<td>0.000078</td>
<td>0.000079</td>
</tr>
<tr>
<td>Copper</td>
<td>0.000057</td>
<td>0.000053</td>
</tr>
<tr>
<td>Cadmium</td>
<td>0.000083</td>
<td>0.000096</td>
</tr>
</tbody>
</table>

Hence, if \(\theta_1\) is the range below \(\theta_0\), corrected for radiation, and \(\theta_2\) the range above \(\theta_0\), and \(\theta\), the "rise above" after correction for radiation, then
\[
(M \cdot S + ms)(\theta + \theta + \theta) = \frac{n^2 E^2 t}{JR}.
\]

Table II. (p. 148) represents a typical series of experiments by the "total heat" method, the metal being copper at 0° C., and the thermometers AB and CD.

(2) The Intersection Method.

The metal having been cooled a considerable distance beyond the limits of the bridge, the current and potential balance were established from five to ten minutes before the temperature came within the bridge range. This preliminary heating up under the normal conditions of the experiment was essential, as it ensured a steady state of gradient, lag, &c., being established before the commencement of the observations. The time of transit of the temperature across each bridge wire division was recorded on the chronograph tape, as in the "total heat" experiments.

The current was switched off and the "rise above" taken in the usual way. Similar experiments over the same range were performed with various values of \(n\) (the number of standard cells balanced at the end of the heating coil).

From these observations the value of \(\partial\theta/\partial t\) at the centre of each scale unit of the
TABLE II.—"Total Heat" Method.

Copper at 0° C. Thermometers AB, CD.

<table>
<thead>
<tr>
<th>I.</th>
<th>II.</th>
<th>III.</th>
<th>IV.</th>
<th>V.</th>
<th>VI.</th>
<th>VII.</th>
<th>VIII.</th>
<th>IX.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Feb</td>
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<td></td>
<td></td>
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<td></td>
<td></td>
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<td></td>
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<td>1374-3</td>
<td>109-5</td>
<td>0-0027</td>
<td>0-0355</td>
<td>1-3356</td>
<td>0-09070</td>
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<tr>
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<td>6</td>
<td>965-5</td>
<td>85-5</td>
<td>0-0018</td>
<td>0-0510</td>
<td>1-3519</td>
<td>0-09062</td>
</tr>
<tr>
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<td>3</td>
<td>3743-5</td>
<td>231-1</td>
<td>0-0071</td>
<td>0-0141</td>
<td>1-3095</td>
<td>0-09068</td>
</tr>
<tr>
<td>&quot;</td>
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<td>8</td>
<td>557-4</td>
<td>60-1</td>
<td>0-0009</td>
<td>0-0843</td>
<td>1-3863</td>
<td>0-09070</td>
</tr>
<tr>
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<td>9</td>
<td>446-8</td>
<td>52-3</td>
<td>0-0006</td>
<td>0-1036</td>
<td>1-4059</td>
<td>0-09073</td>
</tr>
<tr>
<td>&quot;</td>
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<td>4</td>
<td>2125-9</td>
<td>150-5</td>
<td>0-0042</td>
<td>0-0239</td>
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<td>717-8</td>
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<td>0-0012</td>
<td>0-0662</td>
<td>1-3678</td>
<td>0-09073</td>
</tr>
<tr>
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<td>557-6</td>
<td>59-9</td>
<td>0-0009</td>
<td>0-0841</td>
<td>1-3861</td>
<td>0-09073</td>
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<td>0-0006</td>
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<td>0-09072</td>
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<td>7</td>
<td>718-3</td>
<td>70-3</td>
<td>0-0012</td>
<td>0-0658</td>
<td>1-3674</td>
<td>0-09070</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>4</td>
<td>2126-9</td>
<td>150-9</td>
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<td>0-0241</td>
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<tr>
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<td>150-2</td>
<td>0-0042</td>
<td>0-0234</td>
<td>1-3219</td>
<td></td>
</tr>
</tbody>
</table>

Mean ... 0-09068 + 0-00016*

M = 3395-80, \[ R = 20-599, \]
\[ ms = 6-489, \]
\[ E = 1-01843 \] (17° C.),

where

Column    | I. = date of experiment,
          | II. = n (number of standard cells),
          | III. = t (seconds),
          | IV. = time over first interval,
          | V. = radiation correction on range,
          | VI. = "rise above" in degrees Pt,
          | VII. = true range,
          | VIII. = specific heat,
          | IX. = percentage difference from mean.

* The value 0-09068 for S is obtained on the assumption that \( \theta_0 \) coincides with 0 bridge-wire reading; if, however, the balancing point was at +0-1 bridge-wire reading, and the range from -9 to +8 bridge wire, then, on account of radiation gain, the above value of S requires a correction of +0-00005.

At the time these experiments were made we did not realise the importance of this correction and, consequently, did not determine the balancing point on the bridge with sufficient care (see p. 151). In our rough notes made at the time we have values ranging from +0-1 to +0-4 bridge wire.

Applying the "intersection method" (see Section X. (2)) to the above experiments, omitting the first two or three transits in each case, we find from the calculated value of \( \theta_s \) that the balancing point should be at +0-33 bridge wire. If we assume this value, the correction on S is +0-00016.

Hence \[ S = 0-09064. \]
bridge-wire division could be calculated, and also \( \tan \phi \), the slope of the resulting straight line obtained by plotting \( \frac{d\theta}{dt} \) against \( \theta \).

If there were no losses or gains by radiation, the resulting lines would be horizontal.

As the rate of rise due to radiation depends solely on the difference of temperature between the metal and the surroundings, the lines representing the observed values of \( \frac{d\theta}{dt} \) for the various rates of electrical supply have all the same inclination to the horizontal, within the limits of experimental error.

The equation of the line representing an experiment, where \( n \) standard cells are balanced at the ends of the heating coil, is seen to be

\[
\frac{d\theta}{dt} + \sigma (\theta + \theta_i - \theta_s) = \frac{n^2 E^2}{JR(MS + ms)},
\]

where

- \( \frac{d\theta}{dt} \) is the observed rate of rise,
- \( \theta \) is the temperature indicated by the thermometer,
- \( \theta_i \) is the temperature of surrounding envelope,
- \( \theta_s \) is the lag of the observed temperature for the particular rate behind the temperature of the "radiating" surface.

(The determination of this lag is discussed below.)

Hence, by dividing throughout by \( n^2 \), we have

\[
\frac{1}{n^2} \frac{d\theta}{dt} + \sigma \left( \frac{\theta + \theta_i - \theta_s}{n^2} \right) = \frac{E^2}{JR(MS + ms)},
\]

The right-hand side would represent the rate of rise due to the electrical supply with a potential difference of one standard cell.

Hence, if we can determine the particular value of \( \frac{1}{n^2} \frac{d\theta}{dt} \) at the temperature which we denote by \( \theta_N \), when the second term of the equation vanishes, we have the rate of rise due to the electrical supply only.

Plotting \( \frac{1}{n^2} \frac{d\theta}{dt} \) against the observed temperature due to the various values of \( n \), we obtain a series of straight lines whose tangents vary inversely as \( n^2 \).

Now, for each experiment thus plotted, there is a certain point on the line where \( \frac{1}{n^2} \frac{d\theta}{dt} \) represents \( \frac{E^2}{JR(MS + ms)} \) alone, and this would correspond to the temperature \( \theta_N \) at which there are no losses or gains by radiation, i.e., when the mean temperature of the surface subject to radiation is coincident with the temperature of the surroundings. As the co-ordinates of this point are the same for all rates, the lines would intersect at one point if either the observed \( \theta \) was the actual temperature of the "radiating" surface, or the lag was constant for all.
The dotted lines in fig. 7 represent a typical case—that of copper at 0° C. with thermometer AA.

It will be noticed that the lines representing the higher rates of supply are markedly to the left of those obtained from the lesser values of \( n \), indicating that the "lag" increases with the rate of supply, as might be expected.

A study of the "total heat" experiments led us to the conclusion that the "rise above" was intimately connected with this "lag." Although the entire "rise above" on switching off could not be solely due to thermometer lag, yet, as a first approximation, it represents the superior limit.

Hence, by shifting each line parallel to itself to the right by the value of \( \theta \), determined at the close of the experiment, we obtained the figure shown in full lines, the result, of course, being the same as if \( \frac{1}{n^2} \frac{\partial \theta}{\partial t} \) had been plotted against \( \theta + \theta \).

Owing to observational errors, the lines do not intersect in a single point, but enclose a small area. In cases where a really satisfactory series of observations has been obtained, however, the area of the triangle (when three experiments are considered) is vanishingly small even when the results are plotted on such a scale.
that 1 cm. vertically represents a change of 1 in 2000 in \( \frac{1}{n^2} \frac{\partial \theta}{\partial t} \) and there are many instances where the ordinates of the vertices do not differ from the mean by more than 1 part in 5000.

This, in our opinion, is the strongest evidence in support of the assumption that the "rise above" is practically equal to the "lag" to the degree of accuracy to which the horizontal scale is required. We may state here that when plotting the results, we used a scale such that 5 cm. abscissae represented 0.1° Pt, the vertical scale, of course, being considerably greater, enabling the fifth figure in the value of \( \frac{1}{n^2} \frac{\partial \theta}{\partial t} \) to be determined. In our earlier reductions, we ascertained the mean ordinate by reading the ordinates of all the points of intersection; for example, for 4 values of \( n \), we obtained 6 intersections. In cases, however, where the angle \( \phi \) resulting from two experiments differed but by a small amount, as in the case of \( n = 7 \) and \( n = 8 \), a slight error in the inclination of either line might cause a large displacement in the point of intersection. We therefore adopted a method of reduction* which enabled us to calculate the co-ordinates of the point such that, measured along the ordinate passing through this point, the sum of the moments of inertia of the points of intersection of the several lines with this ordinate is a minimum about this point. Or, stated otherwise:—The point so calculated gives, by the method of least squares, the most probable value of the ordinate of the point of intersection of all the lines (for a typical example see p. 157).

A large number of determinations of the specific heats of Cu at 0° C. were made by both the "total heat" and the "intersection" methods (see Section XI.).

The correspondence between the final results obtained was remarkably close (the differences in no case exceeding 1 in 1000), and indicated the validity of both methods. Having satisfied ourselves on this point, we adopted the latter method for all our remaining experiments, as it avoided the following cause of difficulty and delay which was unavoidable in the former.

The removal of our metal block and its replacement by another was a lengthy business, requiring considerable care, as all the soldered joints in the various electrical circuits had to be separated and remade, the brass case removed and opened, &c.

It was not possible to complete the operation in less than several hours, and the temperature of the tank necessarily suffered some alteration in the process. On re-establishing the system, small consequential changes in the balancing point on the bridge might have occurred, or, at all events, the absence of any such changes had to be ascertained. Thus, it was necessary to allow time for the newly inserted block to settle to the tank temperature, and, as its approach to that temperature was slow and asymptotic, at least a day or two had to elapse before the "zero" point could be ascertained with certainty. The importance of this matter is indicated by the fact

* For this suggestion we are indebted to Mr. G. M. Clarke, M.A.
that an error of 0'1 bridge-wire division (= 0'007 Pt) in the estimation of the zero point would affect conclusions derived from an experiment of average length by the total heat method by (in the case of Cu, for example) 5 parts in 9000. In our earlier "total heat" experiments we had not realized the importance of this zero reading, and this no doubt is the cause of certain discrepancies.

The position of the zero point was, however, of little importance when the intersection method was adopted, for so long as the temperature of the reference block remained unchanged, the effect of any alteration in the zero point was self-eliminated.

The method of reduction is shown by one example, namely, that of copper at 0° C., with thermometers AA, BB.

The only reason which has guided us in the selection of this out of the 48 similar groups, is that it happens to be first of the groups given in Table XI. The large amount of arithmetic involved in the reduction of our observations is well illustrated by this example.

Explanation of Tables.

n = number of standard cells balanced on heating coil.

Column I., bridge readings.—The successive points on bridge wire across which transits were taken.

Column II. (t).—Times of transit from chronograph tape.

Column III. (dt).—Interval between successive transits. (If transits observed every 1½ bridge wire, as in Experiment IV., then dt for 1 bridge wire calculated.)

Column IV. (δθ).—Value in Pt degree of bridge-wire division corresponding to δt.

Column V. \( \left( \frac{δθ}{δt} \times 10^7 \right) \).

Column VI. (θ).—Temperature at mid-point of δθ, measured from centre of bridge wire.

Column VII.—The letters denote the values of δθ/δt taken in pairs, for the purpose of obtaining the slope of the line.

Column VIII.—Change in δθ/δt for equal intervals of temperature.
### Table III.—Experiment I, June 3, 1912. Number Standard Cells, 5.

<table>
<thead>
<tr>
<th>Bridge readings</th>
<th>t</th>
<th>$\delta t$</th>
<th>$\delta \theta \times 10^3$</th>
<th>$\frac{\delta \theta}{\delta t} \times 10^2$</th>
<th>$\theta$</th>
<th>$\frac{\delta \theta_a}{\delta t_a}$</th>
<th>$\frac{\delta \theta_a}{\delta t_a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-9.5</td>
<td>215.25</td>
<td>71.65</td>
<td>7.2820</td>
<td>10163</td>
<td>-0.657</td>
<td>A</td>
<td>A - A'</td>
</tr>
<tr>
<td>8.5</td>
<td>286.90</td>
<td>72.55</td>
<td>7.2845</td>
<td>10041</td>
<td>-0.584</td>
<td>B</td>
<td>A - A'</td>
</tr>
<tr>
<td>6.5</td>
<td>359.45</td>
<td>72.75</td>
<td>7.2882</td>
<td>10018</td>
<td>-0.511</td>
<td>C</td>
<td>A - A'</td>
</tr>
<tr>
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<td>7.2901</td>
<td>10021</td>
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<td>D</td>
<td>A - A'</td>
</tr>
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<td>4.5</td>
<td>504.95</td>
<td>73.80</td>
<td>7.2874</td>
<td>9875</td>
<td>-0.365</td>
<td>E</td>
<td>A - A'</td>
</tr>
<tr>
<td>3.5</td>
<td>578.75</td>
<td>74.00</td>
<td>7.2742</td>
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<td>F</td>
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<tr>
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<td>73.80</td>
<td>7.2736</td>
<td>9994</td>
<td>-0.219</td>
<td>G</td>
<td>A - A'</td>
</tr>
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<td>74.00</td>
<td>7.2736</td>
<td>9766</td>
<td>-0.146</td>
<td>H</td>
<td>A - A'</td>
</tr>
<tr>
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<td>9806</td>
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<td>A'</td>
<td>A - A'</td>
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<td>74.40</td>
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<td>75.55</td>
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<td>-0.073</td>
<td>C'</td>
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<td>A - A'</td>
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<td>-0.073</td>
<td>E'</td>
<td>A - A'</td>
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<td>9581</td>
<td>-0.073</td>
<td>H'</td>
<td>A - A'</td>
</tr>
</tbody>
</table>

Mean . . . 9789 at -0°C. 110 C. Mean . . . {\{ 350 for difference of 0°C. 584 C.}

### Table IV.—Experiment II, June 1, 1912. Number Standard Cells, 4.

<table>
<thead>
<tr>
<th>Bridge readings</th>
<th>t</th>
<th>$\delta t$</th>
<th>$\delta \theta \times 10^3$</th>
<th>$\frac{\delta \theta}{\delta t} \times 10^2$</th>
<th>$\theta$</th>
<th>$\frac{\delta \theta_a}{\delta t_a}$</th>
<th>$\frac{\delta \theta_a}{\delta t_a}$</th>
</tr>
</thead>
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<tr>
<td>-9.5</td>
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</tr>
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<td>110.55</td>
<td>7.2845</td>
<td>6580</td>
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<td>A - A'</td>
</tr>
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<td>222.00</td>
<td>110.70</td>
<td>7.2845</td>
<td>6507</td>
<td>-0.511</td>
<td>C</td>
<td>A - A'</td>
</tr>
<tr>
<td>-6.5</td>
<td>334.00</td>
<td>112.00</td>
<td>7.2901</td>
<td>6446</td>
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<td>A - A'</td>
</tr>
<tr>
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<td>447.10</td>
<td>113.10</td>
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<td>-0.365</td>
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<td>A - A'</td>
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<td>7.2902</td>
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Mean . . . 6290 at -0°C. 110 C. Mean . . . {\{ 328 for difference of 0°C. 584 C.}
TABLE V.—Experiment III., June 2, 1912. Number Standard Cells, 6.

<table>
<thead>
<tr>
<th>I.</th>
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<th>V.</th>
<th>VI.</th>
<th>VII.</th>
<th>VIII.</th>
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<td>t</td>
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Mean . . . 14062 at \(-0.110\) \(\text{C}\). Mean . . \{348 for difference of \(0.584\) \(\text{C}\).}

TABLE VI.—Experiment IV., June 2, 1912. Number Standard Cells, 7.

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<td>(\frac{\delta \theta}{\delta t} \times 10^7)</td>
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Mean . . . 19113 at \(-0.110\) \(\text{C}\). Mean . . \{347 for difference of \(0.584\) \(\text{C}\).}
CAPACITY FOR HEAT OF METALS AT DIFFERENT TEMPERATURES.

Table VII.—Experiment V., June 2, 1912. Number Standard Cells, 8.

<table>
<thead>
<tr>
<th>I.</th>
<th>II.</th>
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<th>IV.</th>
<th>V.</th>
<th>VI.</th>
<th>VII.</th>
<th>VIII.</th>
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<td>$\frac{d\theta}{dt} \times 10^2$</td>
<td>$\frac{d\theta}{dt} \times 10^3$</td>
<td>$\frac{d\theta}{dt} \times 10^2$</td>
<td>$\frac{d\theta}{dt} \times 10^3$</td>
<td>$\frac{d\theta}{dt} \times 10^2$</td>
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<td>7.3320</td>
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<td>+0.383</td>
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</table>

Mean . . . 24947 at +0°-110 C. Mean . . . 306 for difference of 0°-547 C.

In Experiment I, 5 cells, $\frac{d\theta_A}{dt} - \frac{d\theta_A'}{dt} / \theta_A - \theta_A' = 350/0.584 = 59.9 \times 10^{-7}$ for 0°-1 C.

   II., 4 " " " " " = 382/0.584 = 56.2 " " " " "
   III., 6 " " " " " = 348/0.584 = 59.6 " " " " "
   IV., 7 " " " " " = 347/0.584 = 59.4 " " " " "
   V., 8 " " " " " = 306/0.547 = 55.9 " " " " "

Hence mean difference for change of 0°-1 Pt = 58 \times 10^{-7}.

Reducing the mean $d\theta/dt$ in each experiment from -0°-110 C. to 0° C. by this mean tangent and then calculating the values at -0°-1 C. and +0°-1 C., we obtain the following results:

<table>
<thead>
<tr>
<th>Experiment No.</th>
<th>n.</th>
<th>$\frac{d\theta}{dt} \times 10^9$ at 0° C.</th>
<th>$\frac{d\theta}{dt} \times 10^9$ at -0°1 C.</th>
<th>$\frac{d\theta}{dt} \times 10^9$ at +0°1 C.</th>
<th>$\frac{1}{n^2} \frac{d\theta}{dt} \times 10^9$ at -0°1 C.</th>
<th>$\frac{1}{n^2} \frac{d\theta}{dt} \times 10^9$ at +0°1 C.</th>
</tr>
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<tbody>
<tr>
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<td>9667</td>
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<td>6168</td>
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<td>13940</td>
<td>39044</td>
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<td>19107</td>
<td>18991</td>
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<td>24941</td>
<td>24825</td>
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<td>38789</td>
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</table>

x 2
Two corrections are necessary to make the values of \( \frac{1}{n^2} \frac{\partial \theta}{\partial t} \) comparable—

(1) The change in resistance by change of current;
(2) The departure of the mean E.M.F. of the group of standard cells used from the standards.

The correction to \( \frac{1}{n^2} \frac{\partial \theta}{\partial t} \) for these is designated by \( \text{Cd cell} \) and \( \delta R \), the experimental results are now arranged in order of \( n \).

<table>
<thead>
<tr>
<th>Experiment No.</th>
<th>( n )</th>
<th>( \text{Cd cell} )</th>
<th>( \delta R )</th>
<th>( \frac{1}{n^2} \left( \frac{\partial \theta}{\partial t} \right) \text{at } -0^\circ \text{C.} \times 10^9 )</th>
<th>( \frac{1}{n^2} \left( \frac{\partial \theta}{\partial t} \right) \text{at } +0^\circ \text{C.} \times 10^9 )</th>
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<td>38563</td>
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<td>+5</td>
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<tr>
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<td>+7</td>
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<td>+10</td>
<td>39010</td>
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</table>

Before proceeding with the next step—the determination of the mean point of intersection either graphically or by calculation—we require the value of the "rise above" in each experiment.

As an example of the method, we quote that of Experiment IV., June 2, 1912.

Number standard cells, 7.

Contact maker of bridge set up 0°.088 Pt beyond switching-off point.

**Table VIII.**

<table>
<thead>
<tr>
<th>Time (from instant of switching off).</th>
<th>Millimetre scale deflection.</th>
<th>( \theta \times 10^8 )</th>
<th>( \theta ) after radiation correction ( \times 10^8 )</th>
<th>Time (from instant of switching off).</th>
<th>Millimetre scale deflection.</th>
<th>( \theta \times 10^8 )</th>
<th>( \theta ) after radiation correction ( \times 10^8 )</th>
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<td>9</td>
<td>0.6</td>
<td>9.9</td>
<td>453</td>
<td>91</td>
<td>-6.6</td>
<td>9.7</td>
</tr>
<tr>
<td>267</td>
<td>4</td>
<td>0.3</td>
<td>9.9</td>
<td>467</td>
<td>97</td>
<td>-7.1</td>
<td>9.7</td>
</tr>
<tr>
<td>275</td>
<td>1</td>
<td>0.1</td>
<td>10.0</td>
<td>483</td>
<td>104</td>
<td>-7.6</td>
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</tr>
<tr>
<td>283</td>
<td>-5</td>
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<td>9.8</td>
<td>498</td>
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<td>-8.1</td>
<td>9.8</td>
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<td>291</td>
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<td>-0.6</td>
<td>9.9</td>
<td>512</td>
<td>-118</td>
<td>-8.6</td>
<td>9.8</td>
</tr>
<tr>
<td>302</td>
<td>-14</td>
<td>-1.0</td>
<td>9.9</td>
<td>528</td>
<td>-128</td>
<td>-9.3</td>
<td>9.7</td>
</tr>
<tr>
<td>313</td>
<td>-19</td>
<td>-1.4</td>
<td>9.9</td>
<td>546</td>
<td>-135</td>
<td>-9.8</td>
<td>9.9</td>
</tr>
</tbody>
</table>
Hence

"Total rise above" = 0'088 + 0'0099 = 0'0979 C.

The figure on p. 145 represents the above data.*

**Table IX.**—Calculation of the Co-ordinates of the "Most Probable Common Point of Intersection."

<table>
<thead>
<tr>
<th>Experiment No.</th>
<th>n</th>
<th>( \frac{1}{n^2} \frac{d\theta}{dt} ) × 10⁸</th>
<th>( \frac{1}{n^2} \frac{d\theta}{dt} ) + 1C. × 10⁸</th>
<th>Shift ( \times 10 )</th>
<th>Equation of line after shift applied.</th>
</tr>
</thead>
<tbody>
<tr>
<td>II.</td>
<td>4</td>
<td>39288</td>
<td>38563</td>
<td>0'334</td>
<td>( y = 39046 - 362x )</td>
</tr>
<tr>
<td>I.</td>
<td>5</td>
<td>39147</td>
<td>38683</td>
<td>0'512</td>
<td>( y = 39034 - 232x )</td>
</tr>
<tr>
<td>III.</td>
<td>6</td>
<td>39060</td>
<td>38773</td>
<td>0'729</td>
<td>( y = 39021 - 144x )</td>
</tr>
<tr>
<td>IV.</td>
<td>7</td>
<td>39010</td>
<td>38807</td>
<td>0'979</td>
<td>( y = 39008 - 116x )</td>
</tr>
<tr>
<td>V.</td>
<td>8</td>
<td>38988</td>
<td></td>
<td>1'254</td>
<td>( y = 39011 - 91x )</td>
</tr>
</tbody>
</table>

Equation of mean line \( y = 39024 - 189x \). . . . . . . . (1)

Multiplying each term by the coefficient of \( x \) in the same equation

\[
362y + 131044x = 141346, \\
232y + 53824x = 90559, \\
144y + 20736x = 56190, \\
118y + 13924x = 46029, \\
91y + 8281x = 35500,
\]

Mean

\[
y + 240x = 39031. . . . . . . . (2)
\]

Solving equations (1) and (2) for \( x \) and \( y \) we have the co-ordinates of the required point

\[
x = +0'137, \\
y = 38998.
\]

Expressing \( x \) and \( y \) in absolute measure we have

\[
\begin{align*}
x \times 0'1 & = \text{degrees Pt}, \\
y \times 10^{-9} & = \frac{1}{n^2} \frac{\partial \theta}{\partial t}.
\end{align*}
\]

Hence

\[
\theta_n = +0'0137 \text{ Pt}, \\
\frac{1}{n^2} \frac{\partial \theta_n}{\partial t} = 38998 \times 10^{-9}.
\]

* This figure indicates how the "rise above corrected" can be obtained more simply by the prolongation of a straight line.
A small uncertainty in the value of $\theta_s$ has but little effect on $\frac{1}{n^2} \frac{\partial \theta_n}{\partial t}$; for example, an error of 0.1 in $x$ would only produce an error of 1 in 2000 in the above value of $y$. A correction of $-3$ has to be applied to $y$ for the clock rate, which was a losing one of 0.05 sec. per 1000.

The distribution of the results of the individual experiments about the "most probable point of intersection" may be determined by solving the equation of each line for its intersection point with the ordinate through $x = +0.137$.

**Table X**

<table>
<thead>
<tr>
<th>Experiment No.</th>
<th>II.</th>
<th>III. ($y$ (mean).)</th>
<th>IV. ($y$ (calculated).)</th>
<th>V. Difference</th>
<th>VI. (Difference)$^2$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>II.</td>
<td>4</td>
<td>38995</td>
<td>38993</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>I.</td>
<td>5</td>
<td></td>
<td>38999</td>
<td>+4</td>
<td>16</td>
</tr>
<tr>
<td>III.</td>
<td>6</td>
<td></td>
<td>38998</td>
<td>+3</td>
<td>9</td>
</tr>
<tr>
<td>IV.</td>
<td>7</td>
<td>38997</td>
<td>38989</td>
<td>-6</td>
<td>36</td>
</tr>
<tr>
<td>V.</td>
<td>8</td>
<td></td>
<td>38996</td>
<td>+1</td>
<td>1</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td><strong>38995</strong></td>
<td></td>
<td>66</td>
<td></td>
</tr>
</tbody>
</table>

Probable observational error $= \pm \frac{3}{\sqrt{\frac{\Sigma (\text{difference})^2}{\text{No. experiments}}}} = \pm 3$.

Coefficient of variation per cent. $= \pm 0.01$.

In Tables XI to XVIII, pp. 161 to 169, we give the value of $y$ (calculated) for each group in Column VII.; the error per cent. (Column VIII.) being the coefficient of variation obtained as above.

Fig. 7, p. 150, represents the above group of experiments; the most probable point of intersection obtained by calculation is shown by a large cross.

Attention may be drawn to the fact that no "smoothed curves" have been used in the above reduction.
Reduction to Specific Heat.

\[ \frac{1}{m_s + MS} \frac{d\theta}{dt} = \frac{E^2}{JRf}. \]

M = mass copper block + case = 3392'37 grms.,
ms = thermal capacity of oil, quartz, glass and solder = 4'898,
R = resistance of coil = 20'599,
E = E.M.F. standard cell at 15° C. = 1'01848,
f = d(\theta)/d(Pt) at 0° C. = 0'98480.

Hence

\[ S = 0'09094. \]

Note. \( f = \frac{\partial \theta}{\partial Pt} \). The values of \( \frac{\partial \theta}{\partial Pt} \) at temperature \( \theta \) are obtained from Chappuis and Harker's tables, 'Phil. Trans.' vol. 194, p. 114. Assuming \( \delta = 1'54. \)

Section XI.

Experimental Results. Preliminary Experiments.

A considerable number of preliminary experiments were performed with a view of testing the apparatus employed and deciding on the most suitable conditions. Some were carried out with a constantan heating coil of 10 ohms resistance, which was replaced by a manganin coil of wider section and greater resistance.

A large number of experiments were performed with silvered vacuum vessels interposed between the metal blocks and the brass cases. The results obtained with different rates of energy supply were discordant. The faster the rate of rise, the lower the value found for the resulting specific heat. These differences were roughly proportional to the duration of the heating; the range being practically the same in all.

The source of this error we traced to the effect of radiation, &c., on the inner walls of the vacuum vessel. This surface received heat by radiation from the block and as it parted with the heat but slowly, its temperature rose with that of the block to an extent dependent on the rate of increase of temperature of the metal.

After the removal of the vacuum vessels, the loss or gain by radiation was dependent on \( \theta - \theta_0 \) only, as the surrounding walls were now those washed by the tank water and remained at a constant temperature. Our anxiety to minimize loss or gain of heat from external sources by the interposition of these flasks had led us, when designing the apparatus, to regard the insertion of the non-conducting walls as important; this precaution, however, was a cause of much loss of time and labour.
Explanation of the Tables.

Column I.—The temperature at which experiments were performed. During our experiments at 0° C. we changed both thermometers and lids; we have, therefore, in this column indicated the thermometers and lid used.

Letters AA and AA', indicate the thermometers referred to in Section III.

Letters L₁, the lighter, and L₂, the heavier lid (see Section VI.).

Where no indication is given, the thermometer used was AA', and lid, L₁.

Column II.—The dates on which the series were performed is given to indicate the results obtained on repetition after lapse of time.

Column III.—The number of transits denotes the number of observations of $\delta \theta / \delta t$ obtained during the experiment.

Column IV.—No. Cd Cells.—The number of standard Weston cells in series, whose E.M.F. was balanced at the ends of the heating coil.

Column V. "Rise Above."—This was determined at the close of each experiment. The line representing an experiment was shifted horizontally by this amount.

Column VI. Tangents (Abscissæ 0°1 Pt.).—The slope of the line $\frac{1}{n^2} \cdot \frac{\delta \theta}{\delta t}$ with temperature $\theta$ as abscissa.

Absolute value = number in Column $\times 10^{-9}$.

Column VII.—The points of intersection of the lines of various rates with the ordinate through $\theta_x$ (see p. 158).

Absolute value = number in Column $\times 10^{-9}$.

Column VIII.—The probable observational error per cent. of the group.

Column IX.—The data required for the reduction. It will be noticed that the mass of the metal block has in some cases changed during the course of the experiment, owing to certain alterations such as enlarging holes, &c., which were found necessary.

$ms$ denotes the capacity for heat of copper case and the group of subsidiary substances, or of the latter only when the block itself is copper.
### Table XI.—Copper.

<table>
<thead>
<tr>
<th>I.</th>
<th>II.</th>
<th>III.</th>
<th>IV.</th>
<th>V.</th>
<th>VI.</th>
<th>VII.</th>
<th>VIII.</th>
<th>IX.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature tank.</td>
<td>Date.</td>
<td>No. transists.</td>
<td>No. Cd cells, (n)</td>
<td>Rise above (^\circ) Pt.</td>
<td>Tangents (abscissae (0^\circ \cdot 1^\circ) C.).</td>
<td>(\frac{1}{w^2} \cdot \frac{d\theta}{dt}).</td>
<td>Error per cent.</td>
<td></td>
</tr>
<tr>
<td>°C</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1/6/12</td>
<td>16</td>
<td>4</td>
<td>0.033</td>
<td>362</td>
<td>38993</td>
<td>R = 20.599</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3/6/12</td>
<td>16</td>
<td>5</td>
<td>0.051</td>
<td>232</td>
<td>38999</td>
<td>E = 1.01848 (15° C.)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2/6/12</td>
<td>16</td>
<td>6</td>
<td>0.073</td>
<td>144</td>
<td>38998</td>
<td>M = 3392.37</td>
<td></td>
</tr>
<tr>
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<td>2/6/12</td>
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<td>7</td>
<td>0.098</td>
<td>118</td>
<td>38989</td>
<td>(ms = 4.898)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2/6/12</td>
<td>16</td>
<td>8</td>
<td>0.125</td>
<td>91</td>
<td>38996</td>
<td>(S = 0.09094)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td>0</td>
<td>25/7/12</td>
<td>16</td>
<td>5</td>
<td>0.039</td>
<td>237</td>
<td>38755</td>
<td>R = 20.609</td>
<td></td>
</tr>
<tr>
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<td>23/7/12</td>
<td>16</td>
<td>6</td>
<td>0.057</td>
<td>165</td>
<td>38754</td>
<td>E = 1.01838 (18° C.)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>24/7/12</td>
<td>16</td>
<td>6</td>
<td>0.057</td>
<td>165</td>
<td>38743</td>
<td>M = 3409.18</td>
<td></td>
</tr>
<tr>
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<td>24/7/12</td>
<td>16</td>
<td>7</td>
<td>0.075</td>
<td>121</td>
<td>38737</td>
<td>(ms = 5.577)</td>
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<td>24/7/12</td>
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<td>8</td>
<td>0.097</td>
<td>93</td>
<td>38711 mean of two</td>
<td>(S = 0.09079)</td>
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<tr>
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<td>4</td>
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<td>449</td>
<td>38631</td>
<td>R = 20.620</td>
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</tr>
<tr>
<td>R</td>
<td>3/12/12</td>
<td>16</td>
<td>5</td>
<td>0.058</td>
<td>244</td>
<td>38682</td>
<td>E = 1.01842 (17° C.)</td>
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</tr>
<tr>
<td>R</td>
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<td>6</td>
<td>0.081</td>
<td>169</td>
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<td>7</td>
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<td>124</td>
<td>38642</td>
<td>(ms = 4.799)</td>
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<td>5/12/12</td>
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<td>5</td>
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<tr>
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<td>16</td>
<td>6</td>
<td>0.085</td>
<td>164</td>
<td>38964</td>
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<td></td>
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<tr>
<td>L</td>
<td>5/12/12</td>
<td>12</td>
<td>7</td>
<td>0.114</td>
<td>120</td>
<td>38931</td>
<td>E = 1.01842 (17° C.)</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>5/12/12</td>
<td>10</td>
<td>4</td>
<td>0.040</td>
<td>367</td>
<td>38969 mean of two</td>
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<td>5/12/12</td>
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<td>4</td>
<td>0.040</td>
<td>367</td>
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<td>(ms = 5.819)</td>
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<td>16</td>
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<td>2/8/12</td>
<td>16</td>
<td>7</td>
<td>0.071</td>
<td>138</td>
<td>37877</td>
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<td></td>
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<tr>
<td></td>
<td>2/8/12</td>
<td>10</td>
<td>8</td>
<td>0.091</td>
<td>105</td>
<td>37846</td>
<td>E = 1.01847 (15° C.)</td>
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</tr>
<tr>
<td>63•5</td>
<td>1/9/12</td>
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<td>36763</td>
<td>(S = 0.09365)</td>
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<td></td>
<td></td>
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<td></td>
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<tr>
<td>67•32</td>
<td>13/9/12</td>
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<td>36632</td>
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<td>13/9/12</td>
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<td>6</td>
<td>0.064</td>
<td>191</td>
<td>36655</td>
<td>R = 20.635</td>
<td></td>
</tr>
<tr>
<td></td>
<td>16/9/12</td>
<td>20</td>
<td>7</td>
<td>0.086</td>
<td>139</td>
<td>36630 mean 7</td>
<td>E = 1.0185 (14°.5 C.)</td>
<td></td>
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<tr>
<td></td>
<td>16/9/12</td>
<td>10</td>
<td>8</td>
<td>0.110</td>
<td>108</td>
<td>36629 mean 8</td>
<td>M = 3409.10</td>
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</tr>
<tr>
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<td>13/9/12</td>
<td>13</td>
<td>7</td>
<td>0.084</td>
<td>139</td>
<td></td>
<td>(ms = 6.313)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>13/9/12</td>
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<td>8</td>
<td>0.114</td>
<td>108</td>
<td>36636</td>
<td></td>
<td></td>
</tr>
<tr>
<td>97•4</td>
<td>11/10/12</td>
<td>16</td>
<td>5</td>
<td>0.042</td>
<td>330</td>
<td>35802</td>
<td>(S = 0.09520)</td>
<td></td>
</tr>
<tr>
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<td>11/10/12</td>
<td>16</td>
<td>6</td>
<td>0.057</td>
<td>229</td>
<td>35774</td>
<td>R = 20.621</td>
<td></td>
</tr>
<tr>
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<td>11/10/12</td>
<td>12</td>
<td>7</td>
<td>0.077</td>
<td>151</td>
<td>35825</td>
<td>E = 1.01840 (17°.5 C.)</td>
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</tr>
<tr>
<td></td>
<td>11/10/12</td>
<td>8</td>
<td>8</td>
<td>0.103</td>
<td>100</td>
<td>35783</td>
<td>M = 3409.05</td>
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</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Set of three experiments of little value owing to leakage of tank-heating circuit affecting galvanometer.
<table>
<thead>
<tr>
<th>I.</th>
<th>II.</th>
<th>III.</th>
<th>IV.</th>
<th>V.</th>
<th>VI.</th>
<th>VII.</th>
<th>VIII.</th>
<th>IX.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature</td>
<td>Date.</td>
<td>No.</td>
<td>No. Cid</td>
<td>Rise above</td>
<td>Tangents</td>
<td>$\frac{d\theta}{n^2 \frac{dt}{d}}$</td>
<td>Error</td>
<td></td>
</tr>
<tr>
<td>tank.</td>
<td></td>
<td>transits.</td>
<td>cells, n.</td>
<td>° Pt.</td>
<td>(abscissa)</td>
<td>0° 1 C.).</td>
<td>per cent.</td>
<td></td>
</tr>
<tr>
<td>° C.</td>
<td>30/6/12</td>
<td>19</td>
<td>6</td>
<td>0.111</td>
<td>244</td>
<td>54722</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>19</td>
<td>7</td>
<td>0.148</td>
<td>178</td>
<td>54747</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>AA</td>
<td></td>
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<td>8</td>
<td>0.191</td>
<td>137</td>
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CAPACITY FOR HEAT OF METALS AT DIFFERENT TEMPERATURES.

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CAPACITY FOR HEAT OF METALS AT DIFFERENT TEMPERATURES.

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VOL. CCXIII.—A. Z
SECTION XII.

Summary of Results. Copper.

The validity of our methods was rigorously tested by the determinations of the specific heat of copper at 0° C.

We have already, in the previous sections, discussed the various changes made during the course of these experiments, and the table below summarises the results.

### TABLE XIX.

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<th>No. of experiments</th>
<th>Observed error per cent.</th>
<th>Remarks</th>
<th>Specific heat</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;Total heat&quot; ............</td>
<td>12</td>
<td>0.03</td>
<td>Thermometers AB, CD (see Table II.)</td>
<td>0.09084</td>
</tr>
<tr>
<td>&quot;Total heat&quot; ............</td>
<td>3</td>
<td>0.03</td>
<td>Thermometers AA, BB ......</td>
<td>0.09095</td>
</tr>
<tr>
<td>&quot;Intersection method&quot;</td>
<td>4</td>
<td>0.05</td>
<td>Thermometers AA', BB'; heavy lid; December 3, 1912</td>
<td>0.09098</td>
</tr>
<tr>
<td>&quot;Intersection method&quot;</td>
<td>5</td>
<td>0.08</td>
<td>Thermometers AA', BB'; light lid; December 5, 1912</td>
<td>0.09088</td>
</tr>
<tr>
<td>&quot;Intersection method&quot;</td>
<td>6</td>
<td>0.03</td>
<td>Thermometers AA', BB'; heavy lid; July 23–25</td>
<td>0.09079</td>
</tr>
<tr>
<td>&quot;Intersection method&quot;</td>
<td>5</td>
<td>0.01</td>
<td>Thermometers AA, BB; light lid; June 3</td>
<td>0.09094</td>
</tr>
<tr>
<td>&quot;Intersection method&quot;</td>
<td>12</td>
<td>0.04</td>
<td>Thermometers AB, CD. Intersection method applied to 1st group</td>
<td>0.09081</td>
</tr>
</tbody>
</table>

Giving equal weight to each group, we have

\[ S_0 = 0.09088 \pm 0.000047, \textit{i.e., probable error} = 0.05 \text{ per cent.} \]

In Tables XX. to XXVII. we summarise our final conclusions.

Messrs. Johnson and Matthey state that the previous treatment of all the metals, except copper and iron, was as follows:

"The cylinders in every instance were cast, and then allowed to cool, subsequently being turned in a lathe, they were not annealed."
The data supplied by the manufacturers indicate that the physical condition of the iron is probably distinct from that of the other metals, and this may to some extent account for the marked difference in the rate of change of its $S$ and $\theta$ curve over the range $0^\circ$ C. to $100^\circ$ C., as compared with the remaining curves.

We are desirous of maintaining the iron in its present condition until we have investigated its behaviour at low temperatures, but we hope eventually to ascertain the effect of careful annealing on this specimen.

**Table XX.**—Copper.

<table>
<thead>
<tr>
<th>Weight, 3392 grms.</th>
<th>Density, 8.922.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Temperature</strong></td>
<td><strong>0° C.</strong></td>
</tr>
<tr>
<td>S (experimental value)</td>
<td>0·09088</td>
</tr>
<tr>
<td>Difference from curve per cent.</td>
<td>0</td>
</tr>
</tbody>
</table>

$S_i = 0.09088 \left(1 + 0.0005341t - 0.00000048t^2\right)$.

This copper was electrolytically deposited.

Mr. C. T. Heycock writes as follows:

"Cu = 99·95 per cent. Remaining 0·05 per cent. consists of Pb, Fe, and a very little SiO$_2$. You will be correct in stating that it is of high purity."

**Table XXI.**—Aluminium.

<table>
<thead>
<tr>
<th>Weight, 954 grms.</th>
<th>Density, 2·704.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Temperature</strong></td>
<td><strong>0° C.</strong></td>
</tr>
<tr>
<td>S (experimental value)</td>
<td>0·20957</td>
</tr>
<tr>
<td>Difference from curve per cent.</td>
<td>0</td>
</tr>
</tbody>
</table>

$S_i = 0.20957 \left(1 + 0.0009161t - 0.0000017t^2\right)$.

Messrs. Johnson and Matthey state:

"Aluminium we have reason to believe to be exceptionally pure, say 99·90 per cent., with traces of iron."

With the exception of one group of three at $0^\circ$ C., these experiments were extremely satisfactory, so much so that the fifth figure appears to have some real significance. The perfect agreement of the experimental and the curve values is very noticeable.
TABLE XXII.—Iron (Ingot).
Weight, 2798 grms. Density, 7.858.

<table>
<thead>
<tr>
<th>Temperature . . . .</th>
<th>0° C.</th>
<th>10°-0 C. and 9°-9 C. (mean)</th>
<th>20°-5 C.</th>
<th>21°-5 C.</th>
<th>24°-5 C.</th>
<th>50°-3 C.</th>
<th>66°-3 C.</th>
<th>97°-5 C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>S (experimental value)</td>
<td>0·1045</td>
<td>0·1060</td>
<td>0·1078</td>
<td>0·1077</td>
<td>0·1080</td>
<td>0·1105</td>
<td>0·11124</td>
<td>0·1137</td>
</tr>
<tr>
<td>Difference from curve per cent. . . . .</td>
<td>0</td>
<td>0</td>
<td>+0·22</td>
<td>+0·10</td>
<td>0</td>
<td>-0·22</td>
<td>-0·099</td>
<td>+0·10</td>
</tr>
</tbody>
</table>

\[ S_1 = 0.1045 (1 + 0.001520t - 0.0000617t^2) \]

This specimen was obtained from the American Rolling Mill Company, who state:

"Material rolled from an ingot into a billet (4 inches by 4 inches), on 'Blooming Mill'; billet forged into round section at blacksmith's shop. Same had no further annealing nor additional heat treatment, other than when rolled and forged."

Specimen turned down to size in laboratory workshop.

"Sample from which material was taken and forged shows following analysis:

\[ S = 0.021 \text{ per cent.} \]
\[ Cu = 0.040 \text{ per cent.} \]
\[ P = 0.005 \]
\[ O = 0.015 \]
\[ C = 0.012 \]
\[ N = 0.0026 \]
\[ Mn = 0.036 \]
\[ H = 0.0005 \]

"Silicon, trace; Fe (by diff.), 99.87."

Our sincere thanks are due to the American Rolling Mill Company, Middletown, Ohio, U.S.A., for presenting us with this sample.

TABLE XXIII.—Zinc.
Weight, 2538 grms. Density, 7.141.

<table>
<thead>
<tr>
<th>Temperature . . . .</th>
<th>0° C.</th>
<th>21°-5 C.</th>
<th>50°-5 C.</th>
<th>97°-4 C.</th>
<th>123°-4 C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>S (experimental value)</td>
<td>0·09176</td>
<td>0·09265</td>
<td>0·09412</td>
<td>0·09521</td>
<td>0·09570</td>
</tr>
<tr>
<td>Difference from curve per cent. . . . .</td>
<td>0</td>
<td>-0·14</td>
<td>+0·19</td>
<td>-0·01</td>
<td>+0·08</td>
</tr>
</tbody>
</table>

\[ S_1 = 0.09176 (1 + 0.0005605t - 0.0000178t^2) \]
Messrs. Johnson and Matthey state:

"Approximately, 99'95 per cent. Zn."

The agreement between the results on repetition at the same temperature was less satisfactory than usual, the extreme difference from the adopted value at 0° C. being 0'3 per cent. (see Table XIV.).

**Table XXIV.**—Silver.

<table>
<thead>
<tr>
<th>Weight, 3733 grms.</th>
<th>Density, 10'456.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature</td>
<td>0° C.</td>
</tr>
<tr>
<td>S (experimental value)</td>
<td>0'05560</td>
</tr>
<tr>
<td>Difference from curve per cent.</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ S_t = 0'05560 \left(1 + 0'00003396t - 0'000000141t^2\right). \]

Messrs. Johnson and Matthey state:

"Better than 999'9 fine."

At 0° C. two series of experiments by "total heat" method were performed—

With thermometers AB, CD, ten experiments; probable error,

\[ \pm 0'05 \text{ per cent.} ; S = 0'05551. \]

With thermometers AA, BB, six experiments; probable error,

\[ \pm 0'04 \text{ per cent.} ; S = 0'05575. \]

**Table XXV.**—Cadmium.

<table>
<thead>
<tr>
<th>Weight, 3070 grms.</th>
<th>Density, 8'652.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature</td>
<td>0° C.</td>
</tr>
<tr>
<td>S (experimental value)</td>
<td>0'05475</td>
</tr>
<tr>
<td>Difference from curve per cent.</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ S_t = 0'05475 \left(1 + 0'0000520t - 0'000000725t^2\right). \]
Messrs. Johnson and Matthey state:

"Fully 99.75 per cent. pure, with very 'slight traces of iron and zinc.'"

Series of four total heat experiments at 0° C., with probable error of ±0.08 per cent. gave $S = 0.05468$.

**Table XXVI.—Tin.**

<table>
<thead>
<tr>
<th>Temperature . . . . . . . .</th>
<th>0° C.</th>
<th>28°·4 C.</th>
<th>53°·9 C.</th>
<th>97°·6 C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>S (experimental value) . .</td>
<td>0·05363</td>
<td>0·05465</td>
<td>0·05549</td>
<td>0·05690</td>
</tr>
<tr>
<td>Difference from curve per cent. .</td>
<td>0</td>
<td>+0·02</td>
<td>−0·02</td>
<td>+0·02</td>
</tr>
</tbody>
</table>

$S_i = 0.05363 (1 + 0.00006704t - 0.000000458t^2)$.

Messrs. Johnson and Matthey state:

"Probably analyse to 99.80 per cent., with trifling quantities of arsenic, lead, and iron."

**Table XXVII.—Lead.**

<table>
<thead>
<tr>
<th>Temperature . . . . . . . .</th>
<th>0° C.</th>
<th>28°·38 C.</th>
<th>51°·0 C.</th>
<th>67°·4 C.</th>
<th>97°·45 C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>S (experimental value) . .</td>
<td>0·030196</td>
<td>0·03053</td>
<td>0·03073</td>
<td>0·03102</td>
<td>0·03127</td>
</tr>
<tr>
<td>Difference from curve per cent. .</td>
<td>0</td>
<td>0</td>
<td>−0·16</td>
<td>+0·19</td>
<td>−0·03</td>
</tr>
</tbody>
</table>

$S_i = 0.030196 (1 + 0.00004000t - 0.000000036t^2)$.

Messrs. Johnson and Matthey state:

"Approximate to 99.90 per cent., with inappreciable traces of arsenic and bismuth."

The "probable error" of the various groups with this metal is higher than in the case of other metals. This is probably an effect of the low conductivity of lead and the consequent steepness of the thermal gradient within the cylinder.

Fig. 8 represents the increments in specific heat over the range 0° C. to 100° C., on the assumption that the specific heat at 0° C. for each metal is represented by unity.

We had hoped to present curves showing the actual values of the specific heat over this range, but the scale required was so large, that we found that, if reduced to the size necessary for reproduction, they were of little value.
A noticeable decrease in the increment of the specific heat of zinc is observable at temperatures above 50°C, which may have some connection with its change in physical properties, as zinc becomes malleable about 120°C. For this reason we pushed our examination of this metal up to a temperature of 123°C.

Our thanks are due to the Court of the University of Wales for a grant towards the purchase of the specimens of metals.

We are greatly indebted to Mr. Edgar A. Griffiths, of this College, for his help in the construction of apparatus and in the conduct of the experiments.

SECTION XIII.

Discussion of the Results of Nernst’s Observations at Low Temperatures.

[After the preceding paper was written, it was suggested to us that we should discuss the relation between our experiments over the range 0°C to 100°C and those of Nernst at lower temperatures. We feel, however, that a discussion of this kind would carry more weight after the completion of our own work at temperatures below 0°C.]
From the brief description of the method published\* by Nernst, it is impossible to estimate the magnitude of any errors arising from the neglect of the loss or gain by radiation, &c. It must be remembered that boiling liquid air is not at a steady temperature and therefore the metal block suspended within the envelope could not settle to the temperature of the surroundings; hence, observations of the temperature after switching off the heating current, afford little information concerning losses or gain by radiation.

Nernst's experiments, however, had one great advantage over those of other observers at lower temperatures, inasmuch as the ranges of temperature employed were small, e.g., 27°C. When we consider the curvature of the specific heat curve, it is evident that changes of temperature of the order of 100°C and upwards can give little accurate information as to the value at the centre of such ranges. Two metals, only, appear to have been examined by Nernst, namely, lead and silver. For lead he obtained the values of the atomic heat given in column II. below; column III. gives the values calculated from the modified Einstein's formula

\[
C = 3R \frac{e^{-a/T} (a^2/T)}{(e^{-a/T} - 1)^2} + b T^{3/2},
\]

where \( R \) is the gas constant, equal to 1.985 gr.-calories.

For lead

\( a = 58, \quad b = 7.8 \times 10^{-5}. \)

In column IV. we give values obtained by extrapolation of the parabolic formula representing the locus of our specific heat curve 0°C. to 100°C. (see p. 174 supra).

**Atomic Heat.—Lead.**

<table>
<thead>
<tr>
<th>I. Absolute temperature.</th>
<th>II. Nernst's observed value.</th>
<th>III. Calculated from formula (A).</th>
<th>IV. Calculated from Griffiths' parabolic formula.</th>
</tr>
</thead>
<tbody>
<tr>
<td>62</td>
<td>5.63</td>
<td>5.58</td>
<td>5.62</td>
</tr>
<tr>
<td>66</td>
<td>5.68</td>
<td>5.63</td>
<td>5.64</td>
</tr>
<tr>
<td>79</td>
<td>5.69</td>
<td>5.75</td>
<td>5.68</td>
</tr>
<tr>
<td>93</td>
<td>5.76</td>
<td>5.84</td>
<td>5.73</td>
</tr>
<tr>
<td>Mean ... 75</td>
<td>5.69</td>
<td>5.70</td>
<td>5.67</td>
</tr>
</tbody>
</table>

\* 'Journal de Physique,' tome ix., 1910, p. 721.
CAPACITY FOR HEAT OF METALS AT DIFFERENT TEMPERATURES.

It will be seen that for the purpose of representing the experimental results, there is little to choose between the two formulae, the greatest difference from our parabolic formula being less than 1 per cent. which Nernst states to be the probable experimental error in his observations.

The greatest divergence between Nernst's results and the modified Einstein's formula amounts to 1'4 per cent., and it must be remembered that the empirical term $bT^{3/2}$ in that formula was added as a consequence of these experimental numbers.

Thus it appears that, in the case of lead, the simple parabolic formula holds over the range 62° C. to 373° C. absolute.

In the case of silver, Nernst records five observations (column II., infra).

**Atomic Heat. — Silver.**

<table>
<thead>
<tr>
<th>I. Absolute temperature.</th>
<th>II. Nernst's observed value.</th>
<th>III. Calculated from formula (A).</th>
<th>IV. Calculated from Griffiths' parabolic formula.</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>3.72</td>
<td>3.61</td>
<td>—</td>
</tr>
<tr>
<td>84</td>
<td>4.43</td>
<td>4.44</td>
<td>—</td>
</tr>
<tr>
<td>86</td>
<td>4.50</td>
<td>4.50</td>
<td>—</td>
</tr>
<tr>
<td>200</td>
<td>5.73</td>
<td>5.78</td>
<td>5.84</td>
</tr>
<tr>
<td>208</td>
<td>5.92</td>
<td>5.81</td>
<td>5.86</td>
</tr>
</tbody>
</table>

If we consider the group about 200° C., we have the following results:

At 204° C. absolute—

Mean observed value ........................................ 5.83

Calculated (Einstein's modified formula) ............... 5.80

(Griffiths' parabolic formula) ........................ 5.84

Here, again, the conclusions of the different observers are in close agreement.

At the still lower temperatures, the decrease in the observed values is so marked that, assuming the validity of Nernst's values, the parabolic formula cannot possibly hold good, and we can only conclude that some marked change takes place in the nature of the curve below 200° C. absolute.

We hope to investigate the values of the capacity for heat of silver at some intermediate points in the large gap between the groups determined by Nernst.

In conclusion, it is notable that, with the exception of three observations upon silver taken at closely adjacent temperatures, all the values obtained by Nernst fall (within the margin of probable experimental error) upon the loci of the parabolas which express our experimental results at higher temperatures.]
The hypothesis of Dulong and Petit has undoubtedly been of great service to chemists; nevertheless, it is acknowledged that, at best, it is but approximately true and that whatever value of the constant is assumed, the number of exceptions at ordinary temperatures, especially in the case of elements of small atomic weights, entitles us to regard it as an indication of a probability rather than as a valid generalization.

Let us consider the values it would yield, at 0°C, if we apply it to the metals whose specific heats we have dealt with in this communication, arranged in order of their atomic weights, assuming that

\[
\text{Atomic weight} \times \text{specific heat} = 6.25.
\]

**Table XXVIII.**

<table>
<thead>
<tr>
<th>I.</th>
<th>II. 6.25/atomic weight</th>
<th>III. Our value at 0°C</th>
<th>IV. Column II. – Column III.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>0.2306</td>
<td>0.2096</td>
<td>+0.02100</td>
</tr>
<tr>
<td>Fe</td>
<td>0.1119</td>
<td>0.1045</td>
<td>+0.00740</td>
</tr>
<tr>
<td>Cu</td>
<td>0.09832</td>
<td>0.09088</td>
<td>+0.00744</td>
</tr>
<tr>
<td>Zn</td>
<td>0.09561</td>
<td>0.09176</td>
<td>+0.00385</td>
</tr>
<tr>
<td>Ag</td>
<td>0.05794</td>
<td>0.05560</td>
<td>+0.00234</td>
</tr>
<tr>
<td>Cd</td>
<td>0.05560</td>
<td>0.05475</td>
<td>+0.00085</td>
</tr>
<tr>
<td>Sn</td>
<td>0.05252</td>
<td>0.05363</td>
<td>−0.00111</td>
</tr>
<tr>
<td>Pb</td>
<td>0.03018</td>
<td>0.03020</td>
<td>−0.00002</td>
</tr>
</tbody>
</table>

The increase in the numbers in column IV., as the atomic weights diminish, is very noticeable.

If we plot the experimental values (column III., supra) as ordinates and the atomic weights as abscissae, the points lie very evenly about a smooth curve of an exponential type; Cu being rather low, Zn rather high, and Sn decidedly high.

In order to obtain an expression for the curve, assume (column III.) the following values:—

\[
\text{Al} = 0.2096; \quad \text{mean of Cu and Zn} = 0.09132; \quad \text{and Pb} = 0.03020.
\]

Then the curve drawn through these three points will be found to follow closely a mean path through the above experimental values.

The expression for this curve is

\[
S = 4.804 \times a^{-0.96}.
\]
<table>
<thead>
<tr>
<th>I.</th>
<th>II.</th>
<th>III. S, experimental values at 0° C.</th>
<th>Observer and data indicating how values in Column III. were obtained.</th>
<th>I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>4.804</td>
<td>2.402</td>
<td>JOLY, at constant volume</td>
<td>H</td>
</tr>
<tr>
<td>He</td>
<td>1.299</td>
<td>0.762</td>
<td>Deduced from (1) $C_p - C_v = \frac{R}{T}$; (2) $C_p/\rho = 1.652$</td>
<td>He</td>
</tr>
<tr>
<td>Li</td>
<td>0.763</td>
<td>0.778</td>
<td>BERNNINI, 0° to 19° = 0.837; 0° to 100° = 1.093</td>
<td>Li</td>
</tr>
<tr>
<td>B</td>
<td>0.492</td>
<td>0.251</td>
<td>Mean of values deduced from KOPP and MOIS-SAN AND GAUTIER (amorphous)</td>
<td>B</td>
</tr>
<tr>
<td>C</td>
<td>0.453</td>
<td>0.113</td>
<td>As diamond, WEBER, at 11°</td>
<td>C</td>
</tr>
<tr>
<td>N</td>
<td>0.391</td>
<td>0.175</td>
<td>PIER, at constant volume</td>
<td>N</td>
</tr>
<tr>
<td>O</td>
<td>0.344</td>
<td>0.170</td>
<td>HOLBORN and AUSTIN (Reichsanstalt)</td>
<td>O</td>
</tr>
<tr>
<td>Na</td>
<td>0.244</td>
<td>0.291</td>
<td>BERNNINI, at 10° = 0.297; at 138° = 0.333</td>
<td>Na</td>
</tr>
<tr>
<td>Mg</td>
<td>0.232</td>
<td>0.234</td>
<td>VOIGT, 18° to 99° = 0.246; STÜCKER at 225° = 0.281</td>
<td>Mg</td>
</tr>
<tr>
<td>Al</td>
<td>0.2092</td>
<td>0.2096</td>
<td>GRIFFITHS'</td>
<td>Al</td>
</tr>
<tr>
<td>Si</td>
<td>0.201</td>
<td>0.177</td>
<td>WEBER, at 57° = 0.183; at 232° = 0.203</td>
<td>Si</td>
</tr>
<tr>
<td>P</td>
<td>0.184</td>
<td>0.188</td>
<td>(Yellow), REGNAULT, -78° to +10° = 0.17;</td>
<td>P</td>
</tr>
<tr>
<td>Cl</td>
<td>0.162</td>
<td>0.0859</td>
<td>STRECKER gives $C_p = 0.1155$ over 16° to 343° and $\gamma = 1.322$; $C_p$ reduced to 0°</td>
<td>Cl</td>
</tr>
<tr>
<td>K</td>
<td>0.148</td>
<td>0.167</td>
<td>Data uncertain. SCHÜZ, 0° to 166° at -27°;</td>
<td>K</td>
</tr>
<tr>
<td>A</td>
<td>0.145</td>
<td>0.0738</td>
<td>REGNAULT, 0° to 165° at -39°</td>
<td>A</td>
</tr>
<tr>
<td>Fe</td>
<td>0.1052</td>
<td>0.1047</td>
<td>DITZENBERGER, $C_p = 0.123$; NIEMEYER, $\gamma = 1.667$</td>
<td>Fe</td>
</tr>
<tr>
<td>Ni</td>
<td>0.1003</td>
<td>0.1004</td>
<td>GRIFFITHS'</td>
<td>Ni</td>
</tr>
<tr>
<td>Co</td>
<td>0.1000</td>
<td>0.099</td>
<td>BEHN, -186° to +18° = 0.086; +18° to +100° = 109; TILDEN at 0° = 0° to 1007</td>
<td>Co</td>
</tr>
<tr>
<td>Cu</td>
<td>0.0929</td>
<td>0.0909</td>
<td>TILDEN, -182° to +15° = 0.082; +15° to 100° = 0.103; 15° to 630° = 0.123</td>
<td>Cu</td>
</tr>
<tr>
<td>Zn</td>
<td>0.0905</td>
<td>0.0917</td>
<td>GRIFFITHS'</td>
<td>Zn</td>
</tr>
<tr>
<td>As</td>
<td>0.0795</td>
<td>0.0778</td>
<td>BETTENDORFF (Cryst.), 21° to 68° = 0.0830; amorphous, 21° to 65° = 0.076</td>
<td>As</td>
</tr>
<tr>
<td>Kr</td>
<td>0.0723</td>
<td>0.0359</td>
<td>$C_p - C_v = \frac{R}{J} \frac{C_p}{\rho} = 1.666$</td>
<td>Kr</td>
</tr>
<tr>
<td>Pd</td>
<td>0.0589</td>
<td>0.057</td>
<td>BEHN, -186° to +18° = 0.053; +18° to 100° = 0.059</td>
<td>Pd</td>
</tr>
<tr>
<td>Ag</td>
<td>0.0563</td>
<td>0.0556</td>
<td>GRIFFITHS'</td>
<td>Ag</td>
</tr>
<tr>
<td>Cd</td>
<td>0.0541</td>
<td>0.0547</td>
<td>&quot;</td>
<td>Cd</td>
</tr>
<tr>
<td>Sn</td>
<td>0.0512</td>
<td>0.0536</td>
<td>&quot;</td>
<td>Sn</td>
</tr>
<tr>
<td>Sb</td>
<td>0.0508</td>
<td>0.0499</td>
<td>&quot;</td>
<td>Sb</td>
</tr>
<tr>
<td>Cs</td>
<td>0.0462</td>
<td>0.0482</td>
<td>&quot;</td>
<td>Cs</td>
</tr>
<tr>
<td>Pt</td>
<td>0.0319</td>
<td>0.0314</td>
<td>&quot;</td>
<td>Pt</td>
</tr>
<tr>
<td>Hg</td>
<td>0.0313</td>
<td>0.0355-0.0308</td>
<td>Liquid by BARNES; for solid see WATSON</td>
<td>Hg</td>
</tr>
<tr>
<td>Tl</td>
<td>0.0307</td>
<td>0.0308</td>
<td>SCHMITZ, -192° to +20° = 0.0300; +20° to 100° = 0.0326</td>
<td>Tl</td>
</tr>
<tr>
<td>Pb</td>
<td>0.0303</td>
<td>0.0302</td>
<td>GRIFFITHS'</td>
<td>Pb</td>
</tr>
<tr>
<td>Bi</td>
<td>0.0302</td>
<td>0.0300</td>
<td>&quot;</td>
<td>Bi</td>
</tr>
<tr>
<td>U</td>
<td>0.0265</td>
<td>0.0274</td>
<td>BLÜMCHEN, at 49° (0° to 98°). Assume decrease like Pb</td>
<td>U</td>
</tr>
</tbody>
</table>
Hence, if \( \alpha = 1 \), we obtain \( S = 4'804 \), that is, just twice the value found by Joly for the specific heat of hydrogen at constant volume.

We have endeavoured to ascertain how nearly the values obtained from this expression are in harmony with the conclusions of other observers in the case of elements not included in our list. It is difficult, however, in regard to the majority of the elements, to consider any conclusion thus arrived at as decisive. The determinations in the case of the rarer elements have been made with such small quantities that the results are open to suspicion, and, but few investigators have so arranged their temperature ranges as to include \( 0^\circ \text{C} \). Where values of \( S \) for different values of \( \theta \) have been given, we have, on the assumption that the changes are of a linear order, deduced the probable values at \( 0^\circ \text{C} \), and in Table XXIX., p. 179, we have indicated the authority and the temperature ranges from which those values were deduced. Where no data for such a reduction can be found, we have inserted any values which fall near \( 0^\circ \text{C} \), together with the mid-temperature and the experimental range. For example, Cs \( 13^\circ \text{C} \) (Eckardt, \( 0^\circ \text{C} \) to \( 26^\circ \text{C} \)). We have given all the information we have

### Table XXX.

<table>
<thead>
<tr>
<th>Column I</th>
<th>Column II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Per cent. differences.</td>
</tr>
<tr>
<td>H</td>
<td>0</td>
</tr>
<tr>
<td>Li</td>
<td>+2·0</td>
</tr>
<tr>
<td>O</td>
<td>-1·2</td>
</tr>
<tr>
<td>Mg</td>
<td>+0·9</td>
</tr>
<tr>
<td>Al</td>
<td>0</td>
</tr>
<tr>
<td>P</td>
<td>+2·2</td>
</tr>
<tr>
<td>A</td>
<td>+1·9</td>
</tr>
<tr>
<td>Fe</td>
<td>-0·5</td>
</tr>
<tr>
<td>Ni</td>
<td>0</td>
</tr>
<tr>
<td>Co</td>
<td>-1·0</td>
</tr>
<tr>
<td>Ca</td>
<td>-2·2</td>
</tr>
<tr>
<td>Zn</td>
<td>+1·3</td>
</tr>
<tr>
<td>As</td>
<td>-2·3</td>
</tr>
<tr>
<td>Kr</td>
<td>-0·7</td>
</tr>
<tr>
<td>Pd</td>
<td>0</td>
</tr>
<tr>
<td>Ag</td>
<td>-1·2</td>
</tr>
<tr>
<td>Cd</td>
<td>+1·1</td>
</tr>
<tr>
<td>Sb</td>
<td>-1·6</td>
</tr>
<tr>
<td>Pt</td>
<td>-1·6</td>
</tr>
<tr>
<td>Tl</td>
<td>+0·3</td>
</tr>
<tr>
<td>Pb</td>
<td>-0·3</td>
</tr>
<tr>
<td>Bi</td>
<td>-0·6</td>
</tr>
<tr>
<td>U</td>
<td>+3·0</td>
</tr>
<tr>
<td></td>
<td>Sum of per cent differences ( ) ( ) ( ) ( ) ( ) = -0·5</td>
</tr>
</tbody>
</table>

Column III.

<table>
<thead>
<tr>
<th>Calculated.</th>
<th>Experimental values.</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>0·492</td>
</tr>
<tr>
<td>C</td>
<td>0·452</td>
</tr>
<tr>
<td>Si</td>
<td>0·201</td>
</tr>
<tr>
<td>Hg</td>
<td>0·0313</td>
</tr>
<tr>
<td>Hg</td>
<td>0·0314 (solid)</td>
</tr>
</tbody>
</table>
been able to gather, concerning the specific heats of the elements, which appeared to us to carry sufficient weight to render the deduced values of any service in such an enquiry; many of those included should, for the reasons given in the introduction to this paper, be regarded as rough approximations only. No element has been omitted in connection with which any satisfactory evidence concerning the specific heat at 0° C. was obtainable.

In column I. of Table XXX., p. 180, we enumerate those elements in which the agreement between the calculated and the experimental values may be regarded as close (i.e., within 3 per cent.), and we have in each case indicated the percentage difference and its sign.

In column II, we place those in which the differences vary from 3 to 16 per cent., including some in which the probable error may be of like dimensions; in column III., those experimental results which differ so greatly from the calculated as to exclude the possibility of agreement. In the case of gases, the experimental values are multiplied by 2.

Remarks on Columns.

Column I.—The sum of the differences (−0'5) and the distribution of the signs show that the experimental values are very evenly distributed about the locus of the curve.

Column II.—The experimental values of N, Na, and K do not appear to be sufficiently established to lend much weight to the results.

The experimental value of the specific heat of tin at 0° C., as compared with that of other metals examined by us, is high. It is a significant fact that tin, at temperatures below 0° C., tends to revert into the grey powder form.

[Since the above was written, we have made determinations of the specific heat of sodium at 0° C. Two different samples were used, and the results were in close agreement, giving the value 0'2863 for the specific heat.

The few experiments at higher temperatures (50° C.) indicate that the increase in specific heat with temperature is considerably greater in the case of sodium than in the other metals examined by us, and is of the order of 0'11 per cent. per 1° C.

In this connection it should be remembered that sodium has the lowest melting-point of all the metals considered in the above table.]

Column III.—Two curious coincidences present themselves. The calculated value of C is almost exactly four times that of the diamond.

The mean experimental value for amorphous B is closely half of the calculated one.

It has been shown (see, for example, Al and Pb supra) that the rate of change of $S/\theta$ as $\theta$ changes, varies markedly for different elements; hence, any relation such as that denoted by the equation $S = 4'804 \times a^{-\theta}$, which holds true for any given temperature, cannot be valid at other temperatures. There are, therefore, serious difficulties in the way of accepting any definite connection between “S” and “a” at an arbitrary
temperature such as 0° C., although it is probable that a large majority of elements are in a stable condition at that temperature.

It is, however, evident that the curve \( S^* = 4.804 \times a^{-0.35} \) yields throughout the whole range of atomic weights values of \( S \) (of 2S in the case of gases) which, in the large majority of cases, are within 2 per cent. of the most probable values.

We prefer to postpone any expression of our views on this matter until we are able to ascertain the results of our experiments at low temperatures.

*The Relation between \( S \) and \( \theta \).*

The curves given in fig. 8 show that the curvature from 0° C. to 100° C. is far more marked in the case of Fe and Al than in any of the remaining metals, with, perhaps, the exception of Zn at the higher temperatures. If we produce backwards the parabolas which have been found to represent the mean paths over the above range, it is found that the curves of Al and Fe (if they continue of the same character) must cross those of the remaining metals before the temperature falls to absolute zero.

If we venture to extrapolate, in order to ascertain the values of \( S \) given by the respective parabolic equations at \(-273^\circ\) C., we obtain the numbers given in column II., Table XXXI.

**Table XXXI.**

<table>
<thead>
<tr>
<th>Element</th>
<th>( S ) at (-273^\circ) C.</th>
<th>Atomic heat at (-273^\circ) C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>0.1306</td>
<td>3.540</td>
</tr>
<tr>
<td>Fe</td>
<td>0.0131</td>
<td>0.730</td>
</tr>
<tr>
<td>Cu</td>
<td>0.07438</td>
<td></td>
</tr>
<tr>
<td>Zn†</td>
<td>0.06554</td>
<td>4.728</td>
</tr>
<tr>
<td>Ag</td>
<td>0.04986</td>
<td>4.294</td>
</tr>
<tr>
<td>Cd</td>
<td>0.04402</td>
<td>5.378</td>
</tr>
<tr>
<td>Sn</td>
<td>0.04199</td>
<td>4.948</td>
</tr>
<tr>
<td>Pb</td>
<td>0.02186</td>
<td>4.997</td>
</tr>
<tr>
<td>Mean of all but Al and Fe</td>
<td></td>
<td>= 4.813</td>
</tr>
</tbody>
</table>

It appears possible that the values of the specific heats of the last six metals may continue to follow the parabolic paths as the temperature falls to \(-273^\circ\) C., for an

* The expression, atomic heat = \( 4.804 \times a^{-0.35} \), is obviously an alternative manner of expressing the same relation.

† In the case of Zn, the equation is deduced from values found from 0° C. to 50°.5 C., for reasons given.
exceedingly small error in their coefficients over the range 0° C. to 100° C. would account for considerable discrepancies in the values of their Atomic Heats at absolute zero, the more especially as the resulting values of S are multiplied by factors ranging from 63 to 200.

Experimental errors, however, could not account for such divergent results as those given by Al and Fe, hence either the atomic-heat curves of these two metals undergo change at low temperatures, or their values at absolute zero must be lower than that of the others in the above list.

If we assume the continuity of the paths of the six metals above referred to and deduce their respective specific heats at −273° C. from the mean atomic heat (4.813), and for the other two points on the parabola employ the values at 0° C. and 100° C., we obtain the following equations (t being expressed in the absolute scale)*:

\[ \begin{align*}
\text{Cu} & : S = 0.0758 \left( 1 + 0.0008352t - 0.00000039t^2 \right), \\
\text{Zn} & : S = 0.07374 \left( 1 + 0.0011155t - 0.000000807t^2 \right), \\
\text{Ag} & : S = 0.0447 \left( 1 + 0.00122t - 0.00000122t^2 \right), \\
\text{Cd} & : S = 0.0429 \left( 1 + 0.0013356t - 0.000001186t^2 \right), \\
\text{Sn} & : S = 0.0405 \left( 1 + 0.0014514t - 0.0000009665t^2 \right), \\
\text{Pb} & : S = 0.02327 \left( 1 + 0.001544t - 0.00000166t^2 \right).
\end{align*} \]

If the values of S at the various temperatures at which it was determined by us are now deduced from these equations, it will be found that the differences between the experimental and the calculated values are very small, in no instance exceeding 0.3 per cent., and in most cases much less.

The remarkable approximation between the hypothetical value of the atomic heat at 0° C. (4.804) of a body with atomic weight 1, and the likewise hypothetical value of the atomic heat of this group of metals at absolute zero (4.813), is probably a coincidence, but may possibly be of some significance.

**APPENDIX II.**

An inspection of the atomic heats of the metals investigated by us indicates that those of low melting-points have high atomic heats. This is true throughout the range 0° C. to 100° C., if the values at any given temperature within that range are

* Many equations of an exponential nature, and also of the forms suggested by Professor Perry ("Phil. Trans.," vol. 194, pp. 250–255) have been investigated, but none of them fitted the experimental results so closely as the parabola.
considered. It thus appears as if there was some relation between the temperature of the melting-points and the atomic heats.

In fig. 9 the atomic heats at 50° C. have been plotted as ordinates, and the melting-points as abscissae.

This temperature was selected for comparison as the most reliable data given by other observers have been obtained over temperature ranges including 50° C. as a mean.

![Graph showing atomic heats and melting points.](image)

**Fig. 9.**

[Determinations made by us since the communication of this paper to the Society give the value of the atomic heat of sodium at 50° C. as 7.01, in place of the value 7.37 shown in the diagram, this latter number having been based on the values of NORDMEYER and BERNOULLI between −185° C. and +20° C. (5.38), and BERNINI's at +10° C (6.83) and +128° C. (7.66).]

**APPENDIX III.**

**Soldering Glass to Metal.**

The process is identical for glass, quartz and, no doubt, for porcelain.

The end of the glass or quartz tube is painted with a solution of platinum chloride in a volatile oil. (Solution is sold under the trade name of Liquid Platinum, No. 1.)

The coating is very gently heated at first, and the temperature slowly increased, until all the volatile matter has been driven off and a brilliant film of platinum obtained. The higher the temperature to which the tube is raised, the better the adherence of the film. The tube should glow with a dull red light, before being
allowed to cool. If a thick film is desired, additional coatings can be given. Care should be taken to prevent contamination by flame-gases; if this occurs the surface should be brightened by means of ordinary metal polish.

The next step is to "tin" the surface, and this requires care. The tube is gently heated and rubbed with a lump of resin; the solder melted on with a clean soldering iron which should be only sufficiently hot to just melt the solder. With care the entire platinised surface may be coated with an irregular coating of solder. Vigorous rubbing of the surface with the soldering iron should be avoided, as it would probably tear the film away from the glass.

The tube is then ready to be soldered into the metal ferrule which should be "tinned" on the inside.

By R. V. Southwell, B.A., Fellow of Trinity College, Cambridge.

Communicated by Prof. A. E. H. Love, F.R.S.

Received January 4,—Read January 30, 1913.

**Introduction and Summary of Paper**

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**Introduction and Summary of Paper.**

Problems which deal with the stability of bodies in equilibrium under stress are so distinct from the ordinary applications of the theory of elasticity that it is legitimate to regard them as forming a special branch of the subject. In every other case we

Published separately, August 6, 1913.
are concerned with the integration of certain differential equations, fundamentally
the same for all problems, and the satisfaction of certain boundary conditions; and
by a theorem due to Kirchhoff* we are entitled to assume that any solution which we
may discover is unique. In these problems we are confronted with the possibility of
two or more configurations of equilibrium, and we have to determine the conditions
which must be satisfied in order that the equilibrium of any given configuration may
be stable.

The development of both branches has proceeded upon similar lines. That is to
say, the earliest discussions were concerned with the solution of isolated examples
rather than with the formulation of general ideas. In the case of elastic stability, a
comprehensive theory was not propounded until the problem of the straight strut had
been investigated by Euler,† that of the circular ring under radial pressure by
M. Lévy,‡ and G. H. Halphen,§ and A. G. Greenhill had discussed the stability of
a straight rod in equilibrium under its own weight,|| under twisting couples, and when
rotating.¶

In a paper which has become the foundation of the theory in its existing form,**
G. H. Bryan has brought these isolated problems for the first time within the range
of a single generalization. Examining the conditions under which Kirchhoff's
theorem of determinacy may fail, he was led to the conclusion that instability is only
possible in the case of such bodies as thin rods, plates, or shells, and in these only
when types of distortion can occur which do not involve extension of the central line
or middle surface, so that it is legitimate to discuss any problem in elastic stability
by methods which have been devised for the approximate treatment of such
bodies. He showed, moreover, that the stability of the equilibrium of any given
configuration depends upon the condition that the potential energy shall be a
minimum in that configuration.

A closer examination of Bryan's theory suggests that some of the conclusions
which have been drawn from it are scarcely warranted. The contention that no
closed shell can fail by instability, because any distortion would involve extension of
the middle surface, will be discussed later.†† For our present purpose it is sufficient
to remark that the whole theory is based upon the assumption that the strains
occurring previously to collapse must be kept to the extremely narrow limits within
which, in the case of ordinary materials, Hooke's Law is satisfied. This assumption,
of course, expresses a restriction necessarily imposed upon the range of practical

‡ 'Liouville's Journal,' X. (1884), p. 5.
§ 'Comptes Rendus,' XCVIII. (1884), p. 422.
†† Cf. pp. 222, 236.
problems which can be treated by the ordinary theory of elasticity; but it is not legitimate to conclude that instability is only possible, even if its conditions were only calculable, in the case of materials which obey Hooke’s Law, and there is no warrant for the employment of “crushing formulae” in the design of short struts and thick boiler flues.*

A more serious weakness in the existing theory of elastic stability, when regarded from the mathematical standpoint, is the fact that the methods which it employs are admittedly only approximate. The higher the elastic limit† of the material under consideration, the less adequate are these methods to deal with the whole range of problems which should come within its scope. In fact, we are faced with the anomaly that, while in its ordinary applications the theory of elasticity is not concerned with the conception of an elastic limit, in questions of stability the existence of finite limits is an essential condition for the adequacy of its results. In an ideal material, possessing perfect elasticity combined with unlimited strength, types of instability could occur with which existing methods would be quite insufficient to deal.

The theory of elastic stability is thus in much the same position as that of the ordinary theory of elasticity before the discovery of the general equations, and one aim of the present paper is to remedy its defects by the investigation of general equations, which may be termed “Equations of Neutral Equilibrium,” and which express the condition that a given configuration may be one of limiting equilibrium. These equations are universally applicable only to ideal material of indefinite strength, and the possibility of elastic break-down must receive separate investigation; but they are also applicable, even with materials of finite strength, to any problem which comes within the restrictions imposed by Bryan’s discussion, and therefore enable us to test the accuracy of his treatment of problems, such as that of the boiler flue, for which the ordinary Theory of Thin Shells has been thought insufficiently rigorous.‡

In every problem of this paper it is found that the Theory of Thin Shells gives a solution which is correct as a first approximation, and the practical advantages of the new method of investigation are, therefore, not immediately apparent. But it must be remembered that the approximate theory of thin plates and shells has not as yet been rigorously established, and that much work has recently been undertaken with the object of testing it by comparison with accurate solutions of isolated problems.§

Now in finding conditions for the neutrality of the equilibrium of any given configuration we are at the same time obtaining the solution of a statistical problem; for a configuration of slight distortion from the equilibrium position will also be one

---

† By “elastic limit” is intended, here and throughout this paper, the limit of linear elasticity.
‡ Cf. pp. 210, 224.
§ Love, op. cit., Introduction, p. 29, and Chapter XXII.
of equilibrium. Hence every solution which we can obtain will add to the number of these "test cases," which has not hitherto included solutions for any but plane plates.

A far more important advantage of the new method, from the practical point of view, is the accuracy with which it follows the actual "stress history" in a body which fails by instability under a gradually increasing stress. In cases where instability precedes elastic break-down this difference of method is not important; but for the discussion of instability in overstrained material, where the stress-strain relations are intimately dependent upon the previous stress history, its introduction is absolutely necessary.

The extension of Euler's theory to struts of practical dimensions and materials, which forms the conclusion of this paper, suggests a large and new field for investigation. The number of similar cases which can be treated, in the existing state of our knowledge of plastic strain, is very small, and indications are given below of the questions which still require an answer; there is reason to believe that the requisite experimental research would not present insuperable difficulties, and that we may hope in the future to obtain an adequate theory of experimental results which are at present very little understood.

**Equations of Neutral Equilibrium in Rectangular Co-ordinates.**

*Method of Derivation.*

The question of stability arises in regard to any system in which there is a possibility of slight displacement from the configuration of equilibrium. This possibility may be afforded either by a more or less limited degree of mechanical freedom—in which case the problem is one of statical stability, and practically unaffected by the tendency, which any actual body displays, to distort under the influence of applied forces; or it may be due, more or less entirely, to this tendency. In the latter case the problem is one of elastic stability, and must be treated by distinct methods. There is, however, no essential difference between the two types of
instability, and a general discussion of the elastic type may be very conveniently illustrated by reference to a mechanical example.

In this connection we may consider the system illustrated by fig. 1, in which a uniform heavy sphere rests in equilibrium within a hemispherical bowl, under the action of its own weight and of the pressure exerted by a pointed plunger, which is free to move in a vertical line through the centre of the bowl. This system has been chosen for the illustration which it affords of collapse under a definite “critical loading.” In this it bears an unusual resemblance to examples of elastic instability—the stability of most mechanical systems being dependent solely upon the relative dimensions of their members. In the absence of friction, we find that the equilibrium will become unstable as the load on the plunger is increased through a critical value given by

$$P = \frac{Wr}{R-2r}$$

where

- $W$ is the weight of the sphere,
- $r$ is the radius of the sphere,
- $R$ is the radius of the bowl.

The above solution rests upon the assumption that the sphere, bowl and plunger are absolutely smooth and rigid, and the possibility of slight displacement is afforded by the freedom of the sphere to take up any position of contact with the bowl. To discuss the equilibrium of the sphere in the position illustrated we must consider the forces which act upon it in a position of slight displacement. These include two systems, one tending to restore the initial conditions, the other tending to increase the distortion, and stability depends upon the relative magnitude of the two effects. We may investigate the problem by three methods, fundamentally equivalent, which are described below:

1. **The Energy Method.**—We may derive expressions for the potential energy of the system in a position of slight displacement from the equilibrium position. The condition of stability requires that the expression for the potential energy shall have a minimum value in the equilibrium position.

2. **The Method of Vibrations.**—We assume that the slight displacement has been effected by any cause, and investigate the types of vibration possible to the system when this cause is removed. The condition of stability requires that all such types shall have real periods.

3. **The Statical Method.**—We confine our attention to the special case in which the stability of the equilibrium position is neutral. In this case there must exist some type of displacement for which the collapsing and restoring effects, discussed above, are exactly balanced, so that it may be maintained by the original system of applied forces. We have, therefore, to find conditions for the equilibrium of a configuration of small displacement, under the given system of applied forces.
Any of these methods is valid for the investigation of elastic stability, and all have in fact been employed, the displacement considered being that of the central-line or middle-surface of the rod or shell, and the resultant actions over cross-sections being derived in terms of this displacement, by the approximate theory first suggested by KIRCHHOFF. The third method is generally found to be preferable, and is the basis of the investigation to be described below, but the actual procedure will be found to possess one or two novel features.

In the first place, an endeavour will be made to dispense with the assumption that elastic break-down occurs at very small values of the strains; instead, we shall deal with an ideal material possessing perfect elasticity combined with unlimited strength. Such a material could not fail, unless by instability, and our problems will no longer be confined to thin rods, plates, or shells. It follows that we can only obtain sufficient accuracy in our conditions for neutral stability by deriving them with reference to a volume-element of the material.

Further, since instability will in some cases not occur until the strains in the material have reached finite values, we shall have to introduce an unusual precision into our ideas of stress and strain. The discussion of finite strain is merely a problem in kinematics, and has been worked out with some completeness*; but the corresponding stress-strain relations in our ideal material are necessarily less certain, since they must be based upon experiments in which only small strains are permissible.

For example, if we assume that Hooke's Law is satisfied at all stresses, we must decide whether our definition of stress is to be

\[
\text{Lt.} \left( \frac{\text{Total action over an element of surface}}{\text{Original area of that surface}} \right)
\]

or

\[
\text{Lt.} \left( \frac{\text{Total action over the surface}}{\text{Area of that surface after distortion}} \right).
\]

For the ordinary purposes of elastic theory the two definitions may be regarded as equivalent, and the distinction is too fine to be settled experimentally. In the absence of any generally-accepted molecular theory which might indicate the correct result, it seems legitimate to make the simplest possible assumptions which do not involve self-contradictions, and which yield the usual results when the strains are very small.

It may be shown† that in a distortion of any magnitude three orthogonal linear elements issue from any point after distortion, which were also orthogonal in the unstrained configuration, and that these linear elements undergo stationary (maximum or minimum or minimax) extension. Hence an elementary parallelopiped constructed at the point, with sides parallel to these linear elements, undergoes no change of angle in the distortion. It is clear that only normal stresses will act upon its faces

* For a discussion of the theory, with references, see Love, op. cit., Appendix to Chapter I.
† Love, op. cit., §§ 26, 27.
after distortion, and that if these stresses be expressed in terms of the extensions of the sides we have complete relations between stress and strain.

We shall therefore assume that these principal stresses and principal strains, whatever their magnitude, are connected by the ordinary equations of Hooke's Law; that is to say, if the extensions in the principal directions are \( e_1, e_2, e_3 \), and the corresponding stresses are \( R_1, R_2, R_3 \), then

\[
e_1 = \frac{1}{E} \left[ R_1 - \frac{1}{m} (R_2 + R_3) \right], \quad \ldots, \quad \&c.,
\]

where \( E \) is Young's Modulus, and \( \frac{1}{m} \) is Poisson's ratio for the material under consideration.

These relations may be written in the form

\[
R_i = \frac{mE}{(m+1)(m-2)} \left[ (m-1)e_1 + e_2 + e_3 \right]
\]

\[
= \frac{2C}{m-2} \left[ (m-1)e_1 + e_2 + e_3 \right], \quad \ldots, \quad \&c., \quad (2)
\]

where \( C \) is the Modulus of Rigidity.

In these relations the measure of extension is assumed to be

\[
\frac{\text{Increase in length of linear element}}{\text{Length of the element before strain}},
\]

and of stress* the measure of total action over an element of surface is

\[
\frac{\text{Total action over an element of surface}}{\text{Area of the element before strain}}.
\]

We have then the usual expression† for the energy of strain, per unit volume of the unstrained material, in terms of the principal extensions, viz.:

\[
W = \frac{m-1}{m-2} C (e_1 + e_2 + e_3)^2 - 2C (e_1 e_2 + e_2 e_3 + e_3 e_1). \quad \ldots \quad (3)
\]

The above assumptions yield sufficient data for the calculation of the stress system in any configuration of equilibrium, even when the strains are not small. Assuming that the calculation has been effected, we have to show how conditions for the stability of the system may be obtained.

We must distinguish three configurations: the unstrained configuration, in which the co-ordinates of any point are given by \( x, y, z \); the configuration of equilibrium under the stress-system, the stability of which we are investigating; and a configuration open to the objection that it would render possible the compression of a material to zero volume by means of a finite stress. It will not, however, introduce any serious error, and has the advantage, which more probable assumptions do not possess, of leading to a definite energy-function. The definitions of stress and strain given above are generally employed in the construction of "stress-strain diagrams" from a tension test, the extensions of the specimen being taken as abscissæ, and the total loads as ordinates of the plotted curve.

* This assumption is open to the objection that it would render possible the compression of a material to zero volume by means of a finite stress. It will not, however, introduce any serious error, and has the advantage, which more probable assumptions do not possess, of leading to a definite energy-function.

† Love, op. cit., § 68.
ration of slight distortion from the equilibrium position, which can be maintained without the introduction of additional stress at the boundaries, if the equilibrium of the second configuration is neutral. We shall consider first a stress-system which is such that the principal stresses in the second configuration have the same magnitudes and directions throughout the body; and we shall take these directions as axes of \( x, y \) and \( z \). We may then define the second and third configurations by saying that in them the co-ordinates of the point \( (x, y, z) \) become

\[
x(1+e_1), \quad y(1+e_2), \quad z(1+e_3),
\]

and

\[
x(1+e_1) + u', \quad y(1+e_2) + v', \quad z(1+e_3) + w',
\]

respectively. We shall not limit the values of \( e_1, e_2, e_3 \), although in practical cases they must be small: \( u', v', w' \) are infinitesimal. In the second configuration the axes \( Ox, Oy, Oz \) are directions of principal stress, and the stresses are

\[
X_x = \frac{2C}{m-2} [(m-1) e_1 + e_2 + e_3], \ldots, &c.
\]

In the third configuration we shall find that lines which in the first configuration were slightly inclined to \( Ox, Oy, Oz \) become directions of principal stress and strain. The final extension of a line which originally had direction-cosines \( l, m, n \) is

\[
e' = -1 + \sqrt{\left[l \left(1+e_1 + \frac{\partial u'}{\partial x}\right) + m \frac{\partial u'}{\partial y} + n \frac{\partial u'}{\partial z}\right]^2 + \left[l \frac{\partial v'}{\partial x} + m \left(1+e_2 + \frac{\partial v'}{\partial y}\right) + n \frac{\partial v'}{\partial z}\right]^2 + \left[l \frac{\partial w'}{\partial x} + m \left(1+e_3 + \frac{\partial w'}{\partial y}\right) + n \left(1+e_2 + \frac{\partial w'}{\partial z}\right)^2\right]}.
\]

It may be shown that \( e' \) has a stationary value when

\[
l = 1,
\]

\[
m = m_1 = \frac{(1+e_1) \frac{\partial u'}{\partial y} + (1+e_2) \frac{\partial u'}{\partial x}}{(1+e_1)^2 - (1+e_2)^2},
\]

and

\[
n = n_1 = \frac{(1+e_1) \frac{\partial v'}{\partial z} + (1+e_2) \frac{\partial v'}{\partial x}}{(1+e_1)^2 - (1+e_2)^2},
\]

to terms of the first order in \( u', v', w' \).†

* In some cases, such as Greenhill’s problem of the stability of a heavy vertical rod (p. 188, footnote), it is necessary to allow for variation in one or more of the principal stresses; the necessary alterations are easily made, and as they are not required for the examples of this paper their consideration would involve unnecessary complexity.

† Added May 1.—The approximation of these expressions is insufficient if any two of the principal strains \( (e_1, e_2, e_3) \) in the second configuration are equal; in this case additional terms must be retained in the denominators. The equilibrium under hydrostatic stress \( (e_1 = e_2 = e_3) \) is necessarily and obviously stable.]
Thus the line initially given by the direction-cosines

\[ l, \quad m_1, \quad n_1, \]

becomes a direction of principal stress in the final configuration. Its direction-cosines (referred to Ox, Oy, and Oz) are then

\[
\frac{\partial v' + m_1(1+e_2)}{1+e_1}, \quad \frac{\partial w' + n_1(1+e_3)}{1+e_1},
\]

or

\[
1, \quad \frac{(1+e_2)\frac{\partial u'}{\partial y} + (1+e_1)\frac{\partial v'}{\partial x}}{(1+e_1)^2-(1+e_2)^2}, \quad \frac{(1+e_3)\frac{\partial w'}{\partial z} + (1+e_1)\frac{\partial w'}{\partial x}}{(1+e_1)^2-(1+e_3)^2}, \quad \ldots \ldots (6)
\]

which we shall write as \( l', m', n' \); and its final extension, to terms of the first order in \( u', v', w' \), is

\[ e'_1 = e_1 + \frac{\partial u'}{\partial x}. \quad \ldots \ldots \quad (7) \]

In the same way we find that the other directions of principal strain in the final configuration are given by the direction-cosines

\[
\frac{(1+e_2)\frac{\partial v'}{\partial y} + (1+e_1)\frac{\partial u'}{\partial x}}{(1+e_1)^2-(1+e_2)^2}, \quad 1, \quad \frac{(1+e_3)\frac{\partial w'}{\partial z} + (1+e_1)\frac{\partial w'}{\partial x}}{(1+e_1)^2-(1+e_3)^2},
\]

and

\[
\frac{(1+e_1)\frac{\partial w'}{\partial x} + (1+e_3)\frac{\partial u'}{\partial z}}{(1+e_1)^2-(1+e_3)^2}, \quad \frac{(1+e_1)\frac{\partial w'}{\partial y} + (1+e_3)\frac{\partial v'}{\partial z}}{(1+e_1)^2-(1+e_3)^2}, \quad 1,
\]

(which we may write as \(-m', 1, n'_2\), and \(-n', -n'_2, 1\), and that the final extensions in these directions are

\[ e'_2 = e_2 + \frac{\partial u'}{\partial y}, \quad \ldots \ldots \quad (9) \]

and

\[ e'_3 = e_3 + \frac{\partial w'}{\partial z}. \]

The stresses in these directions, which we shall call the directions of \( x', y', \) and \( z' \), referred to the original areas of the faces on which they act, \( * \) are therefore

\[ X'_x = X_x + \frac{2C}{m-2} \left[ (m-1)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right], \quad \ldots \ldots \quad (10) \]

\* cf. the assumption of p. 193.
Referred to the new areas of the faces on which they act, they are

\[
\begin{align*}
\bar{X}'_x &= \frac{X'_x}{(1+e'_2)(1+e'_3)}, & \bar{Y}'_y &= \frac{Y'_y}{(1+e'_2)(1+e'_3)}, & \bar{Z}'_z &= \frac{Z'_z}{(1+e'_2)(1+e'_3)},
\end{align*}
\]

and to the required degree of approximation we may write

\[
\bar{X}'_x = \frac{X_x}{(1+e_2)(1+e_3)} \left[ 1 - \frac{1}{1+e_2} \frac{\partial v'}{\partial y} - \frac{1}{1+e_3} \frac{\partial w'}{\partial z} \right] + \frac{2C}{(m-2)(1+e_2)(1+e_3)} \left[ (m-1) \frac{\partial v'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right], \ldots, \&c. \ldots (11)
\]

Then if \( \bar{x}, \bar{y}, \bar{z} \) denote the co-ordinates in the final configuration, referred to the original axes, of the point which was originally at \((x, y, z)\), so that

\[
\bar{x} = x(1+e_1)+u', \ldots, \&c.,
\]

we may find the stress components in the third configuration, referred to the original axes, and to the strained areas of the faces upon which they act, by the scheme of transformation

<table>
<thead>
<tr>
<th></th>
<th>( x' )</th>
<th>( y' )</th>
<th>( z' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{x} )</td>
<td>1</td>
<td>(-m'_1)</td>
<td>(-n'_1)</td>
</tr>
<tr>
<td>( \bar{y} )</td>
<td>( m'_1)</td>
<td>1</td>
<td>(-n'_2)</td>
</tr>
<tr>
<td>( \bar{z} )</td>
<td>( n'_1)</td>
<td>( n'_2)</td>
<td>1</td>
</tr>
</tbody>
</table>

The following expressions are thus obtained (to the required order of approximation):

\[
\begin{align*}
\bar{X}_x &= \bar{X}'_x, & \bar{Y}_y &= \bar{Y}'_y, & \bar{Z}_z &= \bar{Z}'_z, \\
\bar{X}_y &= m'_1(\bar{X}'_x-\bar{Y}'_y), \\
&= m'_1 \left\{ \frac{(1+e_1)}{(1+e_2)(1+e_3)} X_x - \frac{(1+e_2)}{(1+e_1)(1+e_3)} Y_y \right\}, \ldots, \&c. \ldots (12)
\end{align*}
\]

Now the stress-components (12) must satisfy the ordinary equations of equilibrium, which are three of the type

\[
\frac{\partial \bar{X}_x}{\partial x} + \frac{\partial \bar{X}_y}{\partial y} + \frac{\partial \bar{X}_z}{\partial z} = 0, \ldots, \ldots (13)
\]
and since the co-ordinates of the point which ultimately goes to \((x + \delta x, y, z)\) were originally

\[
x + \frac{\delta x}{1 + e_1 + \frac{\partial u'}{\partial x}} \quad y - \frac{\partial u'}{\partial x} \frac{\delta x}{1 + e_1 + \frac{\partial u'}{\partial x}} \quad z - \frac{\partial w'}{\partial x} \frac{\delta x}{1 + e_1 + \frac{\partial u'}{\partial x}}
\]

we have

\[
\frac{\partial}{\partial x} = \frac{1}{1 + e_1 + \frac{\partial u'}{\partial x}} \frac{\partial}{\partial x} - \frac{\partial u'}{\partial x} \frac{\partial}{\partial y} - \frac{\partial w'}{\partial x} \frac{\partial}{\partial z}, \ldots, &c.
\]

It follows that (13) may be written (to our approximation) as follows:

\[
\frac{1}{(1 + e_1)(1 + e_2)(1 + e_3)} \left[ -X_x \left\{ \frac{1}{1 + e_2} \frac{\partial^3 u'}{\partial x^2 \partial y} + \frac{1}{1 + e_3} \frac{\partial^3 w'}{\partial x \partial z} \right\} 
+ \frac{2C}{m-2} \left[ (m-1) \frac{\partial^3 u'}{\partial x^2 \partial y} + \frac{\partial^3 u'}{\partial x \partial y^2} + \frac{\partial^3 w'}{\partial z \partial x} \right] 
+ \frac{(1 + e_1) X_x - (1 + e_2) Y_x \frac{\partial n'_1}{\partial y} + (1 + e_3) X_z - (1 + e_3) Z_z \frac{\partial n'_1}{\partial z}}{1 + e_3} \right] = 0. \ldots (14)
\]

Substituting for \(m', n', \) we have finally

\[
2 \frac{m-1}{m-2} \frac{\partial^3 u'}{\partial x^2 \partial y} + \frac{\partial^3 u'}{\partial y^2 \partial z} + \frac{m}{m-2} \left( \frac{\partial^3 u'}{\partial x \partial y} + \frac{\partial^3 w'}{\partial y \partial z} \right) 
+ \frac{X_x + Y_x}{4C} \frac{\partial^3 u'}{\partial x^2 \partial y} + \frac{X_x + Z_x}{4C} \frac{\partial^3 u'}{\partial z^2 \partial x} + \frac{X_y + Z_y}{4C} \frac{\partial^3 w'}{\partial y^2 \partial z} + \frac{X_z + Z_z}{4C} \frac{\partial^3 w'}{\partial x \partial z} = 0, \ldots (15)
\]

and two similar equations. In any ordinary problem we may neglect \(e_1 \frac{X_x}{C} \ldots \) in comparison with \(\frac{X_x}{C} \ldots \)

The equations thus obtained may also be written (with Lamé's notation for the elastic constants) as follows:

\[
(\lambda + \mu) \frac{\partial \Delta'}{\partial x} + \mu \nabla^2 u' - \frac{X_x + Y_x}{2 \left(1 + e_1 + e_2\right)} \frac{\partial \omega'_x}{\partial y} + \frac{X_x + Z_x}{2 \left(1 + e_1 + e_3\right)} \frac{\partial \omega'_x}{\partial z} = 0, \ldots, &c., (16)
\]

where

\[
\Delta' = \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}, \\
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},
\]

\[
2 \omega'_x = \frac{\partial w'}{\partial x} + \frac{\partial w'}{\partial y}, \quad 2 \omega'_y = \frac{\partial u'}{\partial x} - \frac{\partial w'}{\partial x}, \quad 2 \omega'_z = \frac{\partial v'}{\partial x} - \frac{\partial w'}{\partial y},
\]
and in this form they may be conveniently compared with the ordinary equations of
elasticity.

The three equations of the type (15) we shall term Equations of Neutral Equilibrium. The equilibrium of the stress-system \( X, Y, Z \) will be neutral, provided that solutions for \( u', v', w' \) exist which satisfy certain boundary conditions. These boundary conditions are peculiar to each problem, but usually express the condition that the additional stresses involved by \( u', v', w' \) shall vanish on certain boundary surfaces. They never determine the magnitude of \( u', v', w' \), so that our solution gives the form only of the distortion which tends to occur in the body under consideration when its equilibrium becomes unstable. It gives a definite relation between the stress-system \( X, \ldots \) and the dimensions of the body, which must be satisfied in order that any distortion may be permanent; but if this relation be satisfied, no limits are imposed by the equations upon the magnitude of the distortion which may occur.†

Example in Rectangular Co-ordinates. Stability of Thin Plating under Edge Thrust.

It seems advisable, before we employ a new method on problems which have not as yet received satisfactory treatment, in some degree to test its validity by the result to which it leads in a more familiar example. For this purpose we may consider the stability of an infinite strip of flat plating under edge thrusts in its plane. The accepted formula‡ for the thrust necessary to produce instability, per unit length of edge, is

\[
B = \frac{2t}{m^2 - 1} E \pi^2 \frac{l^3}{4}, \quad \ldots \ldots \ldots \ldots \ldots \ldots (17)
\]

where

\[
2t = \text{thickness of plate},
\]

\[
l = \text{breadth of plate},
\]

and the opposite edges are simply supported. If the edges are built in, the thrust required has four times this value.

To investigate this problem by the new method we take axes \( Ox \) and \( Oz \) in the middle surface of the plate, in the direction of its breadth and length respectively, and \( Oy \) perpendicular to the middle surface. The initial stress-system is then given by

\[
\begin{align*}
X_x &= \text{const.} = G \text{ (say)}, \\
Y_y &= Z_z = 0 ;
\end{align*}
\]

* Love, op. cit., § 91, equation (19).
† The equations are, however, rigorous only in the case of infinitesimal displacements; cf. footnote, p. 240.
‡ Cf. Love, op. cit., § 337 (a), whence the above expression may be obtained.
and we may assume that the system of strain which is introduced at collapse will be two-dimensional, so that
\[
\frac{\partial u'}{\partial z} = \frac{\partial v'}{\partial z} = 0, \quad \frac{w'}{z} = \text{const.} \tag{19}
\]

The third equation of neutral stability (for the direction Oz) is then satisfied identically, and the other two equations become (if we neglect terms of order \(\frac{G^2}{G^4} u'\))
\[
\begin{align*}
2 \frac{m-1}{m-2} \frac{\partial^2 u'}{\partial x^2} &+ \frac{\partial^2 u'}{\partial y^2} + \frac{m}{m-2} \frac{\partial^2 v'}{\partial x \partial y} \frac{G}{4C} \frac{\partial}{\partial y} \left( \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \right) = 0, \\
\frac{\partial^4 u'}{\partial x^4} &+ 2 \frac{m-1}{m-2} \frac{\partial^2 u'}{\partial y^2} + \frac{m}{m-2} \frac{\partial^2 v'}{\partial x \partial y} \frac{G}{4C} \frac{\partial}{\partial x} \left( \frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x} \right) = 0.
\end{align*}
\tag{20}
\]

Let us assume a solution of the form
\[
\begin{align*}
u' &= \Sigma [U_* \sin \alpha (x + x_0)] \\
v' &= \Sigma [V_* \cos \alpha (x + x_0)]
\end{align*}
\tag{21}
\]
where \(U_*\) and \(V_*\) are functions of \(y\) only. It is easy to show that this assumption as to the phase-relation of \(u'\) and \(v'\) is justified. We have then
\[
\begin{align*}
-2 \frac{m-1}{m-2} \alpha^2 U_* + \left(1 + \frac{G}{4C}\right) \frac{d^2 U_*}{dy^2} - \left(\frac{m}{m-2} - \frac{G}{4C}\right) \alpha \frac{dV_*}{dy} &= 0, \\
\left(1 + \frac{G}{4C}\right) \alpha^2 V_* + 2 \frac{m-1}{m-2} \frac{d^2 V_*}{dy^2} + \left(\frac{m}{m-2} - \frac{G}{4C}\right) \alpha \frac{dU_*}{dy} &= 0.
\end{align*}
\tag{22}
\]

The solution of these equations is of the form
\[
\begin{align*}
U_* &= (Py + Q) \sinh \alpha y + (Ry + S) \cosh \alpha y, \\
V_* &= - \left\{ Ry + S - \left(\frac{3m-4}{m-2} + \frac{G}{4C}\right)_P \alpha \right\} \sinh \alpha y \\
&\quad - \left\{ Py + Q - \left(\frac{3m-4}{m-2} + \frac{G}{4C}\right)_R \alpha \right\} \cosh \alpha y,
\end{align*}
\tag{23}
\]
where \(P, Q, R, S\) are constants.

The boundary conditions now demand attention. It is clear that the stresses introduced by \(u'\), \(v'\), \(w'\) must vanish at the surfaces of the plate. Hence these surfaces will still be planes of principal stress, and, moreover, the normal stress upon
them must vanish. But, as we have already seen, the line which becomes a direction
of principal stress has initially the direction-cosines

\[-m_1, 1, n_z:\]

it follows that at the surfaces of the plate the expression for \( m \) and \( n_z \) must vanish
identically; moreover, at these surfaces, \( Y'_y \) must vanish. These conditions may be
written in the form

\[
\begin{align*}
(1 + e_2) \frac{\partial v'}{\partial x} + (1 + e_3) \frac{\partial w'}{\partial y} &= 0, \\
(1 + e_3) \frac{\partial u'}{\partial y} + (1 + e_2) \frac{\partial v'}{\partial x} &= 0, \\
(m - 1) \frac{\partial v'}{\partial z} + \frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} &= 0,
\end{align*}
\]

identically, when \( y = \pm t \). . . . (24)

The first condition is already satisfied. The other two give (if we neglect terms of
order \( G^2 \alpha' \) ...)

\[
\begin{align*}
(1 + e_1) \frac{dU_s}{dy} - (1 + e_2) \alpha V_s &= 0, \\
(m - 1) \frac{dV_s}{dy} + \alpha U_s &= 0,
\end{align*}
\]

and

when \( y = \pm t \), . . . . . . . . (25)

or

\[
P \left[ (1 + \frac{m-1}{m+1} \cdot \frac{G}{4C}) \alpha y \cosh \alpha y - \left( 1 - \frac{2m-1}{(m+1)(m-2)} \cdot \frac{G}{2C} \right) \frac{\sinh \alpha y}{m} \right]
\]

\[
+ Q \left( 1 + \frac{m-1}{m+1} \cdot \frac{G}{4C} \right) \alpha \cosh \alpha y
\]

\[
+ R \left[ (1 + \frac{m-1}{m+1} \cdot \frac{G}{4C}) \alpha y \sinh \alpha y - \left( 1 - \frac{2m-1}{(m+1)(m-2)} \cdot \frac{G}{2C} \right) \cosh \alpha y \right]
\]

\[
+ S \left( 1 + \frac{m-1}{m+1} \cdot \frac{G}{4C} \right) \alpha \sinh \alpha y = 0, . . . . . . . . . (26)
\]

and

\[
P \left[ (m-2) \alpha y \sinh \alpha y - (m-1) \frac{1 + \frac{G}{4C}}{m} \cosh \alpha y \right]
\]

\[
+ Q (m-2) \alpha \sinh \alpha y
\]

\[
+ R \left[ (m-2) \alpha y \cosh \alpha y - (m-1) \frac{1 + \frac{G}{4C}}{m} \sinh \alpha y \right]
\]

\[
+ S (m-2) \alpha \cosh \alpha y = 0
\]

when

\( y = \pm t \) . . . . . . . . . . . . . . . (27)
Thus we obtain

\[ aS = P \left[ \frac{1 - \frac{2m-1}{(m+1)(m-2)} \cdot \frac{G}{2C}}{\frac{m}{m-2} \cdot \frac{G}{4C} \left( 1 + \frac{m-1}{m+1} \cdot \frac{G}{4C} \right)} - \alpha \coth \alpha \right] \]

\[ aQ = R \left[ \frac{1 - \frac{2m-1}{(m+1)(m-2)} \cdot \frac{G}{2C}}{\frac{m}{m-2} \cdot \frac{G}{4C} \left( 1 + \frac{m-1}{m+1} \cdot \frac{G}{4C} \right)} - \alpha \tanh \alpha \right] \]

\[ aS = P \left[ \frac{2m-1}{m-2} \left( \frac{1 + \frac{G}{4C}}{\frac{m}{m-2} - \frac{G}{4C}} \right) - \alpha \tanh \alpha \right] \]

\[ aQ = R \left[ \frac{2m-1}{m-2} \left( \frac{1 + \frac{G}{4C}}{\frac{m}{m-2} - \frac{G}{4C}} \right) - \alpha \coth \alpha \right] \]

There are two solutions of the equations (28). Either

\[ \frac{1 - \frac{2m-1}{(m+1)(m-2)} \cdot \frac{G}{2C}}{\frac{m}{m-2} \cdot \frac{G}{4C} \left( 1 + \frac{m-1}{m+1} \cdot \frac{G}{4C} \right)} - \alpha \tanh \alpha = \frac{aQ}{R} = 2 \frac{m-1}{m-2} \left( \frac{1 + \frac{G}{4C}}{\frac{m}{m-2} - \frac{G}{4C}} \right) - \alpha \coth \alpha, \]

and

\[ P = S = 0, \quad \ldots \quad \ldots \quad \ldots \quad (29) \]

or

\[ \frac{1 - \frac{2m-1}{(m+1)(m-2)} \cdot \frac{G}{2C}}{\frac{m}{m-2} \cdot \frac{G}{4C} \left( 1 + \frac{m-1}{m+1} \cdot \frac{G}{4C} \right)} - \alpha \coth \alpha = \frac{aS}{P} = 2 \frac{m-1}{m-2} \left( \frac{1 + \frac{G}{4C}}{\frac{m}{m-2} - \frac{G}{4C}} \right) - \alpha \tanh \alpha, \]

and

\[ Q = R = 0. \quad \ldots \quad \ldots \quad \ldots \quad (30) \]

The criterion for neutral stability is in the first case

\[ \alpha (\coth \alpha - \tanh \alpha) = \left( 1 + \frac{2m^2-1}{2m (m+1)} \cdot \frac{G}{C} \right), \]

and in the second case

\[ \alpha (\tanh \alpha - \coth \alpha) = \left( 1 + \frac{2m^2-1}{2m (m+1)} \cdot \frac{G}{C} \right), \]
so that the values of \( G \), for which collapse by instability may be expected to occur, are given by

\[
\frac{G}{2m(m+1)C} = \frac{1-2at\cosech 2at}{2m^2-1-2at\cosech 2at} \quad \ldots \ldots (31)
\]

and

\[
\frac{G}{2m(m+1)C} = \frac{1+2at\cosech 2at}{2m^2-1+2at\cosech 2at} \quad \ldots \ldots (32)
\]

respectively, the total thrust, per unit length of edge, being

\[
\mathfrak{B} = -2tG \quad \ldots \ldots \ldots \ldots \ldots (33)
\]

The first approximations to a solution, in terms of \( t \), are

\[
G = -\frac{3}{\alpha} \cdot \frac{mC}{m-1} \cdot \alpha^2 t^2 \quad \ldots \ldots (34)
\]

and

\[
G = -2 \frac{m+1}{m} C = -E \quad \ldots \ldots (35)
\]

respectively. Since the complete wave-length of the corrugations into which the plate distorts is

\[
\lambda = \frac{2\pi}{\alpha}, \quad \ldots \ldots \ldots \ldots (36)
\]

we see that (34) is equivalent to (17), and that the latter formula is therefore supported by our investigation as a first approximation. The second solution (35) is without practical interest, owing to the magnitude of the thrust required to produce collapse. It refers to a type of distortion, theoretically possible for an ideal material without limits of elasticity, which is approximately realized in actual specimens of ductile material, when tested to failure under compressive stress. Since \( Q = R = 0 \), we see from (23) that in this type the middle surface remains plane. In the first type of failure, where \( P = S = 0 \), we find that \( U_z = 0 \) when \( y = 0 \), so that the middle surface undergoes no change of extension in the distortion given by \( \nu', \nu', \nu' \).

**Equations of Neutral Equilibrium in Cylindrical Co-ordinates.**

*Derivation of the Equations.*

The equations (15) of neutral equilibrium are expressed in a form which is unsuitable for the investigation of problems concerned with the stability of thin tubes, and we have next to obtain the corresponding equations in cylindrical

* Besides the harmonic solutions to (20) we may have

\[
\nu' = gx, \quad \nu' = hy, \quad \nu' = kz;
\]

but \( g, h, \) and \( k \) vanish in virtue of the boundary conditions.
co-ordinates. We shall limit our discussion to stress-systems which produce a displacement symmetrical about an axis, up to the instant at which the equilibrium becomes unstable and distortion occurs: in Pearson's notation, the principal stresses in the equilibrium configuration are \( \sigma_r \), \( \sigma_\theta \), and \( \sigma_z \), and these quantities are functions of \( r \) only.

The new equations are derived by a method very similar to that which has already been explained. The co-ordinates of a point in the unstrained configuration are

\[
r, \ \theta, \ z;
\]
in the second configuration (of equilibrium) they are

\[
r+u, \ \theta, \ z+w,
\]
and in the third configuration (of slight distortion from the position of equilibrium) they are

\[
r+u+u', \ \theta+\frac{\theta'}{r+u}, \ z+w+w',
\]
(the radial, tangential, and axial displacements \( u' \), \( v' \), \( w' \) being ultimately taken as infinitesimal).

The extension of a line-element joining the point \((r, \theta, z)\) to the point \((r+\delta r, \theta+\delta \theta, \ z+\delta z)\) is

\[
\epsilon' = -1 + \sqrt{\left[ \left( \frac{1}{1+m^2+n^2} \right) \left[ \left( 1 + \frac{\partial u}{\partial r} + \frac{\partial u'}{\partial r} + \frac{1}{r} \frac{\partial u'}{\partial \theta} - \frac{v'}{r} \right) + n \frac{\partial u'}{\partial z} \right]^2 \\
+ \left( \frac{\partial \theta}{\partial r} + m \left( 1 + \frac{u}{r} + \frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta} \right) + n \frac{\partial v'}{\partial z} \right)^2 \\
+ \left( \frac{\partial w}{\partial r} + m \frac{\partial w'}{\partial r} + n \left( 1 + \frac{\partial w}{\partial z} + \frac{\partial w'}{\partial z} \right) \right]^2 \right], \quad \cdots \quad (37)
\]

where

\[
m = r \frac{\delta \theta}{\delta r}, \quad \text{and} \quad n = \frac{\delta z}{\delta r};
\]

and this has a stationary value for a line very slightly inclined to the radius, given by

\[
m_1 = \frac{(1+e_1) \left( \frac{1}{r} \frac{\partial u'}{\partial \theta} - \frac{v'}{r} \right) + (1+e_2) \frac{\partial v'}{\partial r}}{(1+e_1)^2 - (1+e_2)^2}, \quad \cdots \quad (38)
\]

\[
n_1 = \frac{\frac{\partial u'}{\partial z} + (1+e_3) \frac{\partial w'}{\partial z}}{(1+e_1)^2 - (1+e_2)^2},
\]

where \( e_1, e_2, e_3 \) are written for \( \partial u/\partial r \), \( u/r \), and \( \partial w/\partial z \) respectively.
The extension of this line-element in the third configuration is

\[ e_1' = e_1 + \frac{\partial w'}{\partial r}, \]

and its inclination to lines issuing from the point (in its final position) in the radial, tangential, and axial directions is given by the direction cosines

\[
\begin{align*}
m_1' &= m_1 + \frac{\partial w'}{\partial \theta} - \frac{v'}{r} + (1 + e_1) \frac{\partial v'}{\partial r} - \frac{v'}{r} - \frac{e_1}{1 + e_2} \\
n_1' &= \frac{1}{(1 + e_1)^2 - (1 + e_2)^2} \left[ 1 + \frac{\partial w'}{\partial \theta} + (1 + e_1) \frac{\partial v'}{\partial r} \right]
\end{align*}
\]  

We find also that the other directions of principal stress in the final configuration are initially inclined to radial, tangential, and axial lines through \((r, \theta, z)\) at angles whose direction cosines are

\[ m_1, 1, n_2, \]

and

\[ -m_1, -n_2, 1, \]

and that in the third configuration they have corresponding inclinations to radial, tangential, and axial lines through \((r, \theta, z)\)—in its final position—which are given by

\[ m_2', 1, n_2', \]

and

\[ -m_2', -n_2', 1, \]

where

\[
\begin{align*}
n_2 &= \frac{(1 + e_2) \frac{\partial w'}{\partial \theta} + (1 + e_3) \frac{1}{r} \frac{\partial w'}{\partial \theta}}{(1 + e_2)^2 - (1 + e_3)^2}, \\
n_2' &= \frac{(1 + e_2) \frac{\partial w'}{\partial \theta} + (1 + e_3) \frac{1}{r} \frac{\partial w'}{\partial \theta}}{(1 + e_2)^2 - (1 + e_3)^2}.
\end{align*}
\]

The extensions of the corresponding line-elements in the final configuration are respectively

\[ e_2' = e_2 + \frac{w'}{r} + \frac{1}{r} \frac{\partial w'}{\partial \theta}, \]

and

\[ e_2' = e_2 + \frac{\partial w'}{\partial z}, \]
so that the principal stresses in the third configuration, referred to the final areas of the faces on which they act, are

\[
\begin{align*}
\hat{r}'' &= \frac{\hat{r}_{rr}}{(1+e_2)(1+e_3)} \left[ 1 - \frac{\hat{u}''}{r} + \frac{1}{r} \frac{\partial \hat{v}''}{\partial \theta} - \frac{\partial \hat{w}''}{\partial z} \right], \\
\hat{\theta}'' &= \frac{\hat{\theta}''}{(1+e_2)(1+e_1)} \left[ 1 - \frac{\partial \hat{u}''}{\partial z} - \frac{\partial \hat{v}''}{\partial r} \right] + \frac{2C}{(1+e_2)(1+e_3)} \left[ (m-1) \left( \frac{\hat{u}'}{r} + \frac{1}{r} \frac{\partial \hat{v}'}{\partial \theta} \right) + \frac{\partial \hat{w}'}{\partial z} + \frac{\partial \hat{v}'}{\partial r} \right], \\
\hat{z}'' &= \frac{\hat{z}''}{(1+e_1)(1+e_2)} \left[ 1 - \frac{\partial \hat{u}'}{\partial z} - \frac{\partial \hat{v}'}{\partial r} \right] + \frac{2C}{(1+e_1)(1+e_2)} \left[ (m-1) \left( \frac{\hat{u}'}{r} + \frac{1}{r} \frac{\partial \hat{v}'}{\partial \theta} \right) + \frac{\partial \hat{w}'}{\partial z} + \frac{\partial \hat{v}'}{\partial r} \right].
\end{align*}
\]

Then by the scheme of transformation

\[
\begin{array}{c|ccc}
& r' & \theta' & z' \\
\hline
\hat{r} & 1 & -m_1' & -n_1' \\
\hat{\theta} & m_1' & 1 & -n_2' \\
\hat{z} & n_1' & n_2' & 1 \\
\end{array}
\]

we find for the stress components in the third configuration, at the point which was initially at \((r, \theta, z)\)—referred to axes in the radial, tangential, and axial directions through the final position of the point, and to the final areas of the faces upon which they act—the expressions

\[
\begin{align*}
\hat{r}'' &= \hat{r}', \\
\hat{\theta}'' &= \hat{\theta}', \\
\hat{z}'' &= \hat{z}', \\
\hat{r}'' &= m_1' (\hat{r}' - \hat{\theta}'), \\
\hat{\theta}'' &= \frac{m_1'}{(1+e_1)(1+e_3)} \left[ (1+e_1) \hat{r}' - (1+e_3) \hat{\theta}' \right] \text{(to the required approximation), &c.}
\end{align*}
\]

(44)
The equations of equilibrium, to be satisfied by the stress components (44), are

\[
\begin{align*}
\frac{\partial \tau}{\partial r} + \frac{1}{r} \frac{\partial \tau}{\partial \theta} + \frac{\partial \tau}{\partial z} + \frac{\partial r}{\partial \tau} - \frac{\partial \theta}{\partial r} &= 0, \\
\frac{\partial \tau}{\partial r} + \frac{1}{r} \frac{\partial \tau}{\partial \theta} + \frac{\partial \tau}{\partial z} + \frac{2}{r} \frac{\partial \theta}{\partial r} &= 0, \\
\frac{\partial \tau}{\partial r} + \frac{1}{r} \frac{\partial \tau}{\partial \theta} + \frac{\partial \tau}{\partial z} + \frac{2}{r} \frac{\partial r}{\partial r} &= 0,
\end{align*}
\]

where

\[r = r + u + u',\]
\[\theta = \theta + \frac{v'}{r + u},\]

and

\[z = z + w + w'.\]

Moreover, since

\[
\frac{\partial}{\partial r} = \frac{1}{1 + e_1 + \frac{\partial u}{\partial r}}, \quad \frac{\partial}{\partial \theta} = \frac{1 + e_1}{1 + e_2 + \frac{\partial u}{\partial \theta}}, \quad \frac{\partial}{\partial z} = \frac{1 + e_1}{1 + e_2 + \frac{\partial u}{\partial z}},
\]

the equations (45) may be expressed in differentials with respect to \(r\), \(\theta\), \(z\). The terms which do not involve \(u', v', w'\) vanish in virtue of the equilibrium conditions for the second configuration, and only terms of the first order in these quantities need be retained.

In general, \(e_1, e_2, e_3\) may all be functions of \(r\), but in this paper we shall only consider problems in which \(e_3\) and \(zz\) have constant values. Moreover, in all problems of practical importance we may neglect terms of order \(\frac{r^2}{C} u'\) in comparison with

*Love, op. cit., § 59 (i).
terms of order \( \frac{r^2}{C} u' \). We then obtain, as the equations of neutral equilibrium in cylindrical co-ordinates,

\[
\frac{m-1}{m-2} \left( \frac{\partial^2 u'}{r \partial r} - \frac{u'}{r} \right) + \frac{1}{r^2} \frac{\partial^2 u'}{\partial \theta^2} + \frac{\partial^2 u'}{\partial z^2} - \frac{3m-4}{m-2} \cdot \frac{r^2}{r^2} \frac{\partial v'}{\partial \theta} + \frac{m}{m-2} \left( \frac{1}{r} \frac{\partial^2 v'}{\partial r \partial \theta} + \frac{\partial^2 w'}{\partial z \partial r} \right) \\
+ \left( \frac{rr + \theta z}{4C} \right) \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial v'}{\partial r} - \frac{v'}{r} \right) + \left( \frac{rr + \theta z}{4C} \right) \frac{\partial}{\partial z} \left( \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial z} \right) = 0, \quad \ldots \quad (46)
\]

\[
\frac{m-1}{m-2} \frac{\partial^2 u'}{r \partial r} + \frac{m-4}{m-2} \frac{\partial^2 v'}{r \partial r} + \frac{\partial^2 u'}{\partial r^2} + \frac{1}{r} \frac{\partial v'}{\partial r} - \frac{v'}{r} + 2 \frac{m-1}{m-2} \left( \frac{1}{r^2} \frac{\partial^2 v'}{\partial \theta^2} + \frac{\partial^2 w'}{\partial z^2} + \frac{m}{m-2} \frac{\partial^2 w'}{r \partial \theta \partial z} \right) \\
+ \left( \frac{\theta z + \theta z}{4C} \right) \frac{\partial}{\partial \theta} \left( \frac{\partial v'}{\partial z} - \frac{1}{r} \frac{\partial u'}{\partial z} \right) + \frac{\partial}{\partial r} \left[ \frac{rr + \theta z}{4C} \left( \frac{\partial v'}{\partial r} + \frac{v'}{r} - \frac{1}{r} \frac{\partial u'}{\partial r} \right) \right] = 0, \quad \ldots \quad (47)
\]

and

\[
\frac{m-1}{m-2} \frac{\partial^2 u'}{r \partial r} + \frac{1}{r} \frac{\partial u'}{\partial z} + \frac{\partial^2 v'}{r \partial \theta} + \frac{\partial^2 w'}{r \partial r} + \frac{1}{r} \frac{\partial v'}{\partial r} + \frac{1}{r} \frac{\partial w'}{r \partial r} + 2 \frac{m-1}{m-2} \frac{\partial^2 w'}{r \partial z^2} \\
+ \left( \frac{\theta z + \theta z}{4C} \right) \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial u'}{\partial z} - \frac{\partial v'}{\partial z} \right) + \frac{\partial}{\partial r} \left[ \frac{rr + \theta z}{4C} \left( \frac{\partial w'}{\partial r} - \frac{\partial u'}{\partial r} \right) \right] = 0, \quad \ldots \quad (48)
\]

Equations (46–48) represent the conditions for neutral stability in the equilibrium of a body subjected to a stress-system \( \tilde{r}, \tilde{\theta}, \tilde{z} \), where \( \tilde{z} \) is constant, and \( \tilde{r} \) and \( \tilde{\theta} \) are functions of \( r \) only, which satisfy the condition of equilibrium

\[
\frac{\partial \tilde{r}}{\partial r} + \frac{\tilde{r} - \tilde{\theta}}{r} = 0. \quad \ldots \quad (49)
\]

For comparison with the ordinary equations of equilibrium in cylindrical co-ordinates they may be written (with LAME's notation for the elastic constants) in the forms\(^*\)

\[
(\lambda + 2\mu) \frac{\partial \Delta'}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \theta} \left[ \left( 2\mu + \frac{\tilde{r}r + \tilde{\theta}}{2} \right) \omega_r^2 \right] + \frac{1}{r} \frac{\partial}{\partial r} \left[ \left( 2\mu + \frac{\tilde{r}r + \tilde{\theta}}{2} \right) \omega_r^2 \right] = 0,
\]

\[
(\lambda + 2\mu) \frac{\partial \Delta'}{r \partial \theta} - \frac{1}{r} \frac{\partial}{\partial z} \left[ \left( 2\mu + \frac{\tilde{r}r + \tilde{\theta}}{2} \right) \omega_r^2 \right] + \frac{1}{r} \frac{\partial}{\partial r} \left[ \left( 2\mu + \frac{\tilde{r}r + \tilde{\theta}}{2} \right) \omega_r^2 \right] = 0,
\]

and

\[
(\lambda + 2\mu) \frac{\partial \Delta'}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{r^2 \omega_r^2 \left( 2\mu + \frac{\tilde{r}r + \tilde{\theta}}{2} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left[ \left( 2\mu + \frac{\tilde{r}r + \tilde{\theta}}{2} \right) \omega_r^2 \right] }{r} = 0,
\]

where

\[
\Delta' = \frac{1}{r} \frac{\partial}{\partial r} (ru') + \frac{1}{r} \frac{\partial v'}{\partial \theta} + \frac{\partial w'}{\partial z},
\]

\(^*\) Cf. Love, op. cit., § 199.
and

$$2\pi r = \frac{1}{r} \frac{\partial u'}{\partial \theta} - \frac{\partial u'}{\partial z}, \quad 2\pi' = \frac{\partial u'}{\partial z} - \frac{\partial u'}{\partial r}, \quad 2\pi'' = \frac{1}{r} \left( \frac{\partial}{\partial r} (rv') - \frac{\partial u'}{\partial \theta} \right),$$

so that

$$\frac{1}{r} \frac{\partial}{\partial r} (rv') + \frac{1}{r} \frac{\partial u'}{\partial \theta} + \frac{\partial u'}{\partial z} = 0.$$

**Examples in Cylindrical Co-ordinates. Stability of Boiler Flues and Tubular Struts.**

The Equations of Neutral Equilibrium in Cylindrical Co-ordinates enable us to deal successfully with some difficult problems connected with the stability of cylindrical tubes. Two examples of considerable importance will be discussed in this paper—the collapse of boiler flues and the strength of tubular struts. It should be noticed that neither of these problems has been quite satisfactorily treated by the ordinary theory of thin shells, which requires the assumptions that the middle surface of the shell is unextended, and the inner and outer surfaces free from applied tractions*; hence their solution is a problem of considerable interest, even apart from practical considerations, and has attracted a great deal of attention. It will be convenient at this point to review the work which has already been done.

The question of the stability of tubular struts is important, owing to the frequency of their employment in practice. In economy of material the cylindrical tube possesses an advantage over struts of solid cross-section, and both the theory of

Euler* and Lagrange† and the more practical formula of Rankine suggest that this advantage increases without limit as the thickness of the tube is reduced. Such a conclusion is, however, inaccurate, for types of distortion are possible in the case of a tube which do not involve flexure of the axis, and when the tube is thin these types, of which some practical examples are shown in fig. 2, may be maintained by a smaller thrust than would be required to produce failure of the kind discussed by Euler. Moreover, the natural wave-length, for these symmetrical types of distortion, is in general small, so that distortion can occur without hindrance in quite short tubes. Hence, for a considerable range of length the strength of a tube to resist end-thrust is practically constant, and is not given by any of the usual formulae for struts.

The determination of the strength of tubes to resist these symmetrical types of distortion is obviously a problem of the highest practical importance, and has attracted a great deal of attention in recent years. Illustrations of collapsed tubes, showing symmetrical types of distortion, have been published by A. Mallock‡ and R. Lorenz,§ and a great deal of experimental work has been carried out by W. E. Lilly.|| Theoretical discussions, by approximate methods, have been proposed by A. Gros,* W. E. Lilly,** S. Timoschenko†† and R. Lorenz.‡‡

The problem of the boiler flue seems first to have been suggested by the experiments commenced by Fairbairn in 1858. §§ These showed that the collapse of tubes under external pressure was in some degree analogous to that of straight columns under end-thrust, and a discussion of the phenomenon, based on Euler’s theory of struts, was given by W. C. Unwin, ||| who assisted Fairbairn in his research. The similar problem of a circular wire ring subjected to radial pressure has been discussed by M. Bresse,* and M. Lévy,** and rational theories of the boiler-flue problem have been given by G. H. Bryan,††† A. Möpfl, ‡‡‡ and P. Forchheimer.§§§

‡ 'Comptes Rendus,' CXXXIV. (1902), p. 1041.
‡‡‡ 'Cours de Mécanique Appliquée,' I. Partie, Paris, 1859.
*** 'Liouville’s Journal,' X. (1884), p. 5.
‡‡‡ 'Résistance des Matériaux' (1901), p. 286.
§§§ 'Zeitschrift des Oesterreichischen Ingenieur- und Architekten-Vereines,' 1904.
and R. Lorenz.* W. E. Lilly† has indicated the correct form of the result for 
an infinitely long flue, and A. E. H. Love‡ has discussed the strengthening effects of 
constraints which keep the tube circular at its ends.

A. B. Basset§ has given a very clear exposition of the difficulties which are 
encountered in an attempt to construct a theory of flue collapse by usual methods. 
To obtain sufficient equations we must assume that the middle surface undergoes no 
extension; and the existence of pressure on one or both surfaces of the tube not only 
makes this assumption very improbable, but violates an essential condition upon 
which the theory of thin shells is based. When one surface only is subjected to 
pressure, there is reason to believe that Bryan's solution is substantially correct; 
but no treatment can be looked upon as rigorous which neglects the cross-stresses in 
the material.

The experimental researches of A. P. Carmann|| and R. T. Stewart+++ have revived 
interest in this problem, since they offer the first information which has been obtained 
as to the behaviour under practical conditions of tubes which in circularity, uniformity 
of thickness and homogeneity are fair approximations to the ideal tube of theoretical 
analysis.**

We commence our discussion by considering the stability of a thin cylindrical tube, 
subjected to the combined action of end and surface pressures. We shall thus be 
able to derive the required solutions for the thin tubular strut, and for a boiler flue 
without end-thrust, as particular cases, and from the general solution we may obtain 
indications of the way in which end-thrust tends to promote the collapse of a 
boiler flue.

In the most general form of the boiler-flue problem, as enunciated by Basset,†† pressures are acting on both surfaces of the tube, and we shall therefore investigate 
conditions for neutral stability in a tube subjected to the following system of 
stresses:—

(i.) An end-thrust of total amount $S$, uniformly distributed;
(ii.) An external hydrostatic pressure, of intensity $P_1$; and
(iii.) An internal hydrostatic pressure, of intensity $P_2$.

§ 'Phil. Mag.', XXXIV. (1892), p. 221.
1906, p. 730.
** The experiments of Fairbairn were restricted to tubes which were constructed from sheet metal, 
with brazed and riveted seams.
†† Loc. cit., p. 223.
We shall consider a tube of indefinite length, of which the inner and outer radii are 

\[ \alpha \pm t \] 

(so that the thickness is \( 2t \)), and we shall write 

\[ \tau \] 

for the ratio \( \frac{t}{\alpha} \).

The corresponding stress-system, for the position of equilibrium, is easily obtained.*

We have

\[
\begin{align*}
\hat{rr} &= -\frac{1}{4\tau} \left[ \mathcal{P}_1 (1 + \tau)^2 - \mathcal{P}_2 (1 - \tau)^2 - \frac{\alpha^2}{r^2} (\mathcal{P}_1 - \mathcal{P}_2) (1 - \tau)^2 \right], \\
\hat{\theta \theta} &= -\frac{1}{4\tau} \left[ \mathcal{P}_1 (1 + \tau)^2 - \mathcal{P}_2 (1 - \tau)^2 + \frac{\alpha^2}{r^2} (\mathcal{P}_1 - \mathcal{P}_2) (1 - \tau)^2 \right], \\
\hat{zz} &= -\frac{S}{4\pi at}.
\end{align*}
\]

and

\[
\begin{align*}
\frac{1}{r^2} \frac{\partial^2 u'}{\partial r^2} + \frac{1}{r} \frac{\partial u'}{\partial r} - \frac{u'}{r^2} + \left( 1 + \frac{A}{2} \right) \frac{1}{r^3} \frac{\partial^2 u'}{\partial \theta^2} + \left[ 1 + \frac{A}{4} \left( 1 + \frac{\sigma \alpha^2}{r^2} \right) - \frac{B}{4} \right] \frac{\partial^2 u'}{\partial z^2} \\
+ \left( \frac{m}{m-2} - \frac{A}{2} \right) \frac{1}{r} \frac{\partial^2 u'}{\partial \theta \partial r} + \left( \frac{m}{m-2} \right) \frac{1}{r^2} \frac{\partial^2 u'}{\partial \theta^2} + \left( \frac{m}{m-2} + \frac{A}{2} \right) \frac{1}{r^3} \frac{\partial^2 u'}{\partial \theta^2} \\
+ \left\{ 1 + \frac{A}{4} \left( 1 - \frac{\sigma \alpha^2}{r^2} \right) - \frac{B}{4} \right\} \frac{\partial^2 u'}{\partial z^2} = 0,
\end{align*}
\]

\[ (51) \]

It can also be shown that \( e_3 \) is constant, and equations (46–48) may therefore be taken to express the conditions of neutral equilibrium. The degree of approximation to which these equations have been obtained (p. 206) will be maintained for the rest of this paper, i.e., terms of order \( \frac{r^2}{G} u' \ldots \) will be neglected. They may also be written as follows:

\[
\begin{align*}
2 \frac{m-1}{m-2} \left( \frac{\partial^2 u'}{\partial r^2} + \frac{1}{r} \frac{\partial u'}{\partial r} - \frac{u'}{r^2} \right) + \left( 1 + \frac{A}{2} \right) \frac{1}{r^3} \frac{\partial^2 u'}{\partial \theta^2} + \left[ 1 + \frac{A}{4} \left( 1 + \frac{\sigma \alpha^2}{r^2} \right) - \frac{B}{4} \right] \frac{\partial^2 u'}{\partial z^2} \\
+ \left( \frac{m}{m-2} - \frac{A}{2} \right) \frac{1}{r} \frac{\partial^2 u'}{\partial \theta \partial r} + \left( \frac{m}{m-2} + \frac{A}{2} \right) \frac{1}{r^2} \frac{\partial^2 u'}{\partial \theta^2} + 2 \frac{m-1}{m-2} \frac{1}{r^3} \frac{\partial^2 u'}{\partial \theta^2} \\
+ \left\{ 1 + \frac{A}{4} \left( 1 - \frac{\sigma \alpha^2}{r^2} \right) - \frac{B}{4} \right\} \frac{\partial^2 u'}{\partial z^2} + \left\{ 1 - \frac{A}{4} \left( 1 - \frac{\sigma \alpha^2}{r^2} \right) + \frac{B}{4} \right\} \frac{1}{r} \frac{\partial^2 u'}{\partial \theta \partial z} = 0,
\end{align*}
\]

\[ (52) \]

and

\[
\begin{align*}
\left\{ \frac{m}{m-2} - \frac{A}{4} \left( 1 + \frac{\sigma \alpha^2}{r^2} \right) + \frac{B}{4} \right\} \frac{\partial^2 u'}{\partial z^2} + \left\{ \frac{m}{m-2} - \frac{A}{4} \left( 1 - \frac{\sigma \alpha^2}{r^2} \right) + \frac{B}{4} \right\} \frac{1}{r} \frac{\partial u'}{\partial z} \\
+ \left\{ \frac{m}{m-2} - \frac{A}{4} \left( 1 - \frac{\sigma \alpha^2}{r^2} \right) + \frac{B}{4} \right\} \frac{1}{r} \frac{\partial^2 u'}{\partial \theta \partial z} + \left\{ 1 + \frac{A}{4} \left( 1 + \frac{\sigma \alpha^2}{r^2} \right) - \frac{B}{4} \right\} \frac{\partial^2 u'}{\partial \theta^2} \\
+ \left\{ 1 + \frac{A}{4} \left( 1 - \frac{\sigma \alpha^2}{r^2} \right) - \frac{B}{4} \right\} \frac{1}{r} \frac{\partial u'}{\partial r} + \left\{ 1 + \frac{A}{4} \left( 1 - \frac{\sigma \alpha^2}{r^2} \right) - \frac{B}{4} \right\} \frac{1}{r} \frac{\partial^2 u'}{\partial \theta^2} + 2 \frac{m-1}{m-2} \frac{1}{r} \frac{\partial^2 u'}{\partial z^2} = 0,
\end{align*}
\]

\[ (54) \]

where

\[
\begin{align*}
A &= -\frac{1}{4C'} \left[ \mathcal{P}_1 (1+\tau)^2 - \mathcal{P}_2 (1-\tau)^2 \right], \\
B &= \frac{\mathcal{S}}{4\pi \omega C'},
\end{align*}
\]

and

\[
\sigma = -\frac{(\mathcal{P}_1 - \mathcal{P}_2) (1-\tau^2)}{\mathcal{P}_1 (1+\tau)^2 - \mathcal{P}_2 (1-\tau)^2}.
\]

We may assume a solution for equations (52–54) of the form

\[
\begin{align*}
u' &= \Sigma \left[ U_{k,q} \sin k (\theta+\theta_k) \sin \frac{q}{\alpha} (z+z_q) \right], \\
v' &= \Sigma \left[ V_{k,q} \cos k (\theta+\theta_k) \sin \frac{q}{\alpha} (z+z_q) \right], \\
w' &= \Sigma \left[ W_{k,q} \sin k (\theta+\theta_k) \cos \frac{q}{\alpha} (z+z_q) \right],
\end{align*}
\]

where \( k \) must be integral, and \( U_{k,q}, V_{k,q}, W_{k,q} \) are functions of \( r \) only, which satisfy the differential equations

\[
\begin{align*}
\left[ \frac{2m-1}{m-2} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) - \frac{k^2}{r^2} \left( 1 + \frac{A}{2} \right) - \frac{q^2}{\alpha^2} \left( 1 + \frac{A}{4} \left( 1 + \frac{\sigma \alpha^2}{\pi^2} \right) - \frac{B}{4} \right) \right] U_{k,q} \\
- k \left[ \frac{m}{m-2} - \frac{A}{2} \right] \frac{1}{r} \frac{d}{dr} - \frac{3m-4}{m-2} + \frac{A}{2} \frac{1}{r^2} \right] V_{k,q} \\
- \frac{q}{\alpha} \left[ \frac{m}{m-2} - \frac{A}{4} \left( 1 + \frac{\sigma \alpha^2}{\pi^2} \right) + \frac{B}{4} \right] \frac{1}{r} W_{k,q} = 0, \\
\end{align*}
\]

and

\[
\begin{align*}
\frac{q}{\alpha} \left[ \frac{m}{m-2} - \frac{A}{4} \left( 1 + \frac{\sigma \alpha^2}{\pi^2} \right) + \frac{B}{4} \right] \frac{d}{dr} + \left[ \frac{m}{m-2} - \frac{A}{4} \left( 1 - \frac{\sigma \alpha^2}{\pi^2} \right) + \frac{B}{4} \right] \frac{1}{r} \right] U_{k,q} \\
k \frac{q}{\alpha} \left[ \frac{m}{m-2} - \frac{A}{4} \left( 1 - \frac{\sigma \alpha^2}{\pi^2} \right) + \frac{B}{4} \right] \frac{1}{r} V_{k,q} \\
+ \left[ \frac{1}{r} \frac{d}{dr} \left( 1 + \frac{A}{4} \left( 1 - \frac{\sigma \alpha^2}{\pi^2} \right) - \frac{B}{4} \right) \right] \frac{d}{dr} + \left[ 1 + \frac{A}{4} \left( 1 - \frac{\sigma \alpha^2}{\pi^2} \right) - \frac{B}{4} \right] \frac{1}{r} \frac{d}{dr} - \frac{k^2}{r^2} \right] W_{k,q} = 0.
\end{align*}
\]

It is easy to show that the phase-relations assumed in equations (56) are necessary.
The boundary conditions now require investigation. From the consideration that the cylindrical boundary surfaces of the tube must continue to be tangent to principal planes of stress, in any possible type of distortion, we deduce the conditions

$$m_1 = 0$$
$$n_1 = 0$$

identically, when \( r = a \pm t \). . . . . . (60)

The other boundary conditions are more complex. Since the pressures acting on the surfaces of the tube are hydrostatic, it is clear that the radial stress, as defined on p. 193, is increased at points on the boundary surfaces of the tube where the distortion involves positive extension. In the notation employed above, we have

$$\tilde{rr}' = -\mathcal{P}_1, \quad \text{when } r = a+t,$$
$$= -\mathcal{P}_2, \quad \text{when } r = a-t,$$

and from (43) we deduce the following equations, which must be satisfied identically,*

$$\frac{2}{m-2} \left[ (m-1) \frac{\partial u'}{\partial r} + \frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta} + \frac{\partial w'}{\partial z} \right]$$

$$= \left\{ -\frac{\mathcal{P}_1}{C} \left[ \frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta} + \frac{\partial w'}{\partial z} \right], \quad \text{when } r = a+t, \right\} . . . . (61)

Substituting from (56) in the identities (60) and (61), we finally obtain, as the required boundary conditions in \( U_{k,q}, V_{k,q} \), and \( W_{k,q} \),

$$\frac{k}{r} U_{k,q} + \left\{ \left( 1-A \sigma \frac{a^3}{r^3} \right) \frac{d}{dr} \frac{1}{r} \right\} V_{k,q} = 0,$$

$$\frac{q}{\alpha} U_{k,q} + \left\{ 1 - A \left( 1+\frac{a^3}{r^3} \right) - \frac{B}{2} \right\} \frac{d}{dr} W_{k,q} = 0,$$

and

$$2 \frac{m-1}{m-2} \frac{d}{dr} U_{k,q} + \left\{ \frac{2}{m-2} - A \left( 1+\frac{a^3}{r^3} \right) \left( \frac{1}{r} U_{k,q} - \frac{k}{r} V_{k,q} - \frac{q}{\alpha} W_{k,q} \right) \right\} = 0,$$

when \( r = a \pm t \).

* In obtaining these equations it should be noticed that before distortion occurs - \( \mathcal{P}_1 \) and - \( \mathcal{P}_2 \) are the values at the boundary of

$$\frac{\tilde{rr}}{(1+\epsilon_2)(1+\epsilon_3)}$$

and not of \( \tilde{rr} \), if we retain the significance for \( \tilde{rr} \) which was assumed on p. 193. The distinction is not really needed for the approximation of the following work, but it may lead to confusion if neglected.
The differential relations (57-59), with the boundary conditions (62), are theoretically sufficient for an exact solution of our problem: we shall, however, content ourselves with approximate solutions for \((\mathfrak{B}_1 - \mathfrak{B}_3)\) and \(\mathfrak{S}\), correct to terms in \(\tau^2\). To obtain these, we assume solutions for \(U_{k,q}, V_{k,q}, W_{k,q}\) in series of ascending powers of the quantity \((r - \alpha)\). Thus we write

\[
\begin{align*}
U_{k,q} &= \xi_0 + \xi_1 \frac{h}{\alpha} + \frac{\xi_2}{2!} \left(\frac{h}{\alpha}\right)^2 + \ldots \\
V_{k,q} &= \eta_0 + \eta_1 \frac{h}{\alpha} + \frac{\eta_2}{2!} \left(\frac{h}{\alpha}\right)^2 + \ldots \\
W_{k,q} &= \xi_0 + \xi_1 \frac{h}{\alpha} + \frac{\xi_2}{2!} \left(\frac{h}{\alpha}\right)^2 + \ldots
\end{align*}
\]

where \(r = \alpha + h\).

We may now derive, from equations (57-59), any required number of relations between the undetermined coefficients \(\xi_0, \eta_0, \xi_1, \ldots\), and the boundary conditions (62) take the form of equations in series of ascending powers of the small quantity \(\tau\), in which the sums of the odd and of the even powers must vanish separately. If we neglect in these equations terms of order higher than some definite power of \(\tau\), we may obtain corresponding approximations to the values of \(A\) and \(B\), by the elimination of the undetermined coefficients.

The approximate boundary conditions, correct to terms in \(\tau^2\), are*

\[
\begin{align*}
k \xi_0 + k \tau^2 \xi_2 - \eta_0 + \{1 - A (\sigma - \tau^2)\} \eta_1 + \left(\frac{1}{2} - A\right) \tau^2 \eta_2 + (1 + A) \tau^2 \eta_3 &= 0, \ldots \ldots \ldots \ \ \ \ \ (64) \\
k \xi_1 + k \tau^2 \xi_3 + A (\sigma - \tau^2) \eta_0 + \{1 - A (\sigma - \tau^2)\} \eta_1 + \left(\frac{1}{2} - A\right) \tau^2 \eta_2 + (1 + A) \tau^2 \eta_3 &= 0, \ldots \ldots \ldots \ \ (65) \\
q \xi_0 + q \tau^2 \xi_2 + \left\{- \frac{A (1 + \sigma) + B}{2} + \frac{3}{2} A \tau^2\right\} \xi_2 - A \tau^2 \xi_3 + \left(1 - \frac{B}{2}\right) \tau^2 \xi_3 &= 0, \ldots \ldots \ldots \ (66) \\
q \xi_1 + q \tau^2 \xi_3 + A (\sigma - 2 \tau^2) \xi_1 + \left\{- \frac{A (1 + \sigma) + B}{2} + \frac{3}{2} A \tau^2\right\} \xi_3 - \frac{1}{2} A \tau^2 \xi_3 + \left(1 - \frac{B}{2}\right) \tau^2 \xi_3 &= 0, \ldots \ldots \ldots \ (67)
\end{align*}
\]

* In deriving these boundary conditions it is to be noticed that \(\sigma\) is to a first approximation equal to \(-1\), so that to our approximation \(\tau^2\) may be written for \(\sigma \tau^2\).
and

\[
(m-2) A \left(\sigma-2\tau^2\right) \xi_0 + \left\{m - \frac{m-2}{2} A \left(1+\sigma-3\tau^2\right)\right\} \xi_1 + \left\{m - 1 - \frac{m-2}{2} A \tau^2\right\} \xi_2 \\
+ (3m-2) \frac{\tau^2}{6} \xi_3 + (m-1) \frac{\tau^2}{6} \xi_4 - (m-2) k A \left(\sigma-2\tau^2\right) \eta_0 - k \left\{1 - \frac{m-2}{2} A \left(1+\sigma-3\tau^2\right)\right\} \eta_1 \\
+ (m-2) k A \frac{\tau^2}{2} \eta_0 - k \frac{\tau^2}{6} \eta_0 - q \left\{1 - \frac{m-2}{2} A \left(1+\sigma-\tau^2\right)\right\} \xi_0 \\
- q \left\{1 - \frac{m-2}{2} A \left(1+\sigma-\tau^2\right)\right\} \xi_1 - q \left\{1 - (m-2) A \right\} \frac{\tau^2}{2} \xi_0 - q \frac{\tau^2}{6} \xi_1 = 0, \ldots (69)
\]

and these equations involve the fifteen coefficients 

\[
\xi_0 \ldots \xi_4, \quad \eta_0 \ldots \eta_1, \quad \xi_0 \ldots \xi_4
\]

From equations (57–59) we may obtain nine other relations between these coefficients, as follows:

\[
- \left[2 \frac{m-1}{m-2} + k^2 \left(1 + \frac{A}{2}\right) + q^2 \left(1 + \frac{A-B}{4} + \frac{\sigma A}{4}\right)\right] \xi_0 + 2 \frac{m-1}{m-2} \xi_1 + 2 \frac{m-1}{m-2} \xi_2 \\
+ k \left[\frac{3m-4}{m-2} + \frac{A}{2}\right] \eta_0 - k \left[\frac{m}{m-2} - \frac{A}{2}\right] \eta_1 \\
- q \left[\frac{m}{m-2} - \frac{A-B}{4} - \frac{\sigma A}{4}\right] \xi_1 = 0, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (70)
\]

\[
2 \left[2 \frac{m-1}{m-2} + k^2 \left(1 + \frac{A}{2}\right) + q^2 \frac{A}{4}\right] \xi_0 - \left[4 \frac{m-1}{m-2} + k^2 \left(1 + \frac{A}{2}\right) + q^2 \left(1 + \frac{A-B}{4} + \frac{\sigma A}{4}\right)\right] \xi_1 \\
+ 2 \frac{m-1}{m-2} \xi_2 + 2 \frac{m-1}{m-2} \xi_3 - 2k \left[\frac{3m-4}{m-2} + \frac{A}{2}\right] \eta_0 + q \frac{m-1}{m-2} k \eta_1 - k \left[\frac{m}{m-2} - \frac{A}{2}\right] \eta_2 \\
- q \frac{A}{2} \xi_1 - q \left[\frac{m}{m-2} - \frac{A-B}{4} - \frac{\sigma A}{4}\right] \xi_3 = 0, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (71)
\]

\[
-3 \left[2 \frac{m-1}{m-2} + k^2 \left(1 + \frac{A}{2}\right) + q^2 \frac{A}{4}\right] \xi_0 + 2 \left[3 \frac{m-1}{m-2} + k^2 \left(1 + \frac{A}{2}\right) + q^2 \frac{A}{4}\right] \xi_1 \\
- \frac{1}{2} \left[6 \frac{m-1}{m-2} + k^2 \left(1 + \frac{A}{2}\right) + q^2 \left(1 + \frac{A-B}{4} + \frac{\sigma A}{4}\right)\right] \xi_2 + \frac{m-1}{m-2} \xi_3 + \frac{m-1}{m-2} \xi_4 \\
+ 3k \left[\frac{3m-4}{m-2} + \frac{A}{2}\right] \eta_0 - k \left[\frac{7m-8}{m-2} + \frac{A}{2}\right] \eta_1 + \frac{1}{2} k \left[\frac{5m-4}{m-2} - \frac{A}{2}\right] \eta_2 \\
- \frac{1}{2} k \left[\frac{m}{m-2} - \frac{A}{2}\right] \eta_3 + \frac{3}{4} q \sigma A \xi_1 - q \frac{A}{2} \xi_2 - \frac{1}{2} q \left[\frac{m}{m-2} - \frac{A-B}{4} - \frac{\sigma A}{4}\right] \xi_3 = 0, \ldots (72)
\]
k \left[ \frac{3m-4}{m-2} + \frac{A}{2} \right] \xi_0 + k \left[ \frac{m}{m-2} - \frac{A}{2} \right] \xi_1 \\
- \left[ 1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 + q^2 \left( 1 + \frac{A-B}{A} - \frac{\sigma A}{4} \right) \right] \eta_0 + \left[ 1 + \frac{A}{2} \right] \eta_1 + \left[ 1 + \frac{A}{2} \right] \eta_2 \\
-kq \left[ \frac{m}{m-2} - \frac{A-B}{4} + \frac{\sigma A}{4} \right] \xi_0 = 0, \ldots \ldots \ldots \ldots (73)

-k \left[ \frac{3m-4}{m-2} + \frac{A}{2} \right] \xi_0 + k \left[ \frac{3m-4}{m-2} + \frac{A}{2} \right] \xi_1 + k \left[ \frac{m}{m-2} - \frac{A}{2} \right] \xi_2 \\
+ \left[ 1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 - q^2 \left( 1 + \frac{A-B}{4} + \frac{\sigma A}{4} \right) \right] \eta_0 \\
- \left[ 1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 + q^2 \left( 1 + \frac{A-B}{4} - \frac{\sigma A}{4} \right) \right] \eta_1 \\
+ 2 \left[ 1 + \frac{A}{2} \right] \eta_2 + \left[ 1 + \frac{A}{2} \right] \eta_3 + kq \sigma \frac{A}{2} \xi_0 - kq \left[ \frac{m}{m-2} - \frac{A-B}{4} + \frac{\sigma A}{4} \right] \xi_1 = 0, \ldots \ldots \ldots \ldots (74)

k \left[ \frac{3m-4}{m-2} + \frac{A}{2} \right] \xi_0 - k \left[ \frac{3m-4}{m-2} + \frac{A}{2} \right] \xi_1 + \frac{1}{2} k \left[ \frac{3m-4}{m-2} + \frac{A}{2} \right] \xi_2 + \frac{1}{2} k \left[ \frac{m}{m-2} - \frac{A}{2} \right] \xi_3 \\
- \left[ 1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 - \frac{\sigma A}{4} q^2 \right] \eta_0 + \left[ 1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 - q^2 \left( 1 + \frac{A-B}{4} + \frac{\sigma A}{4} \right) \right] \eta_1 \\
- \frac{1}{2} \left[ 1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 + q^2 \left( 1 + \frac{A-B}{4} - \frac{\sigma A}{4} \right) \right] \eta_2 + \frac{1}{2} \left[ 1 + \frac{A}{2} \right] \eta_3 + \frac{1}{2} \left[ 1 + \frac{A}{2} \right] \eta_4 \\
- \frac{1}{2} k q \sigma A \xi_0 + \frac{1}{2} kq \sigma A \xi_1 - \frac{1}{2} k q \left[ \frac{m}{m-2} - \frac{A-B}{4} + \frac{\sigma A}{4} \right] \xi_2 = 0, \ldots \ldots \ldots \ldots (75)

q \left[ \frac{m}{m-2} - \frac{A-B}{4} + \frac{\sigma A}{4} \right] \xi_0 + q \left[ \frac{m}{m-2} - \frac{A-B}{4} - \frac{\sigma A}{4} \right] \xi_1 \\
-k q \left[ \frac{m}{m-2} - \frac{A-B}{4} + \frac{\sigma A}{4} \right] \eta_0 \\
- k^2 \left( 1 + \frac{A-B}{4} - \frac{\sigma A}{4} \right) + 2 \frac{m-1}{m-2} q^2 \xi_0 + \left[ 1 + \frac{A-B}{4} - \frac{\sigma A}{4} \right] \xi_1 + \left[ 1 + \frac{A-B}{4} + \frac{\sigma A}{4} \right] \xi_2 = 0, \ldots \ldots \ldots \ldots (76)

- q \frac{A}{2} \xi_0 + 2 q \left[ \frac{m}{m-2} - \frac{A-B}{4} + \frac{\sigma A}{4} \right] \xi_1 + q \left[ \frac{m}{m-2} - \frac{A-B}{4} - \frac{\sigma A}{4} \right] \xi_1 + q \left[ \frac{m}{m-2} - \frac{A-B}{4} + \frac{\sigma A}{4} \right] \xi_3 \\
+ kq \sigma \frac{A}{2} \eta_0 - kq \left[ \frac{m}{m-2} - \frac{A-B}{4} + \frac{\sigma A}{4} \right] \eta_1 \\
+ \left[ k^2 \left( 1 + \frac{A-B}{4} - \frac{\sigma A}{4} \right) - 2 \frac{m-1}{m-2} q^2 \right] \xi_3 + \left[ \frac{\sigma A}{2} - k^2 \left( 1 + \frac{A-B}{4} - \frac{\sigma A}{4} \right) - 2 \frac{m-1}{m-2} q^2 \right] \xi_3 \\
+ 2 \left[ 1 + \frac{A-B}{4} - \frac{\sigma A}{4} \right] \xi_3 + \left[ 1 + \frac{A-B}{4} + \frac{\sigma A}{4} \right] \xi_3 = 0, \ldots \ldots \ldots \ldots (77)
\[ \frac{3}{2} q \sigma A \xi + \frac{3}{2} q A \xi + \frac{3}{2} q \left[ \frac{m}{m-2} - \frac{A-B}{4} + \sigma \frac{A}{4} \right] \xi + \frac{1}{2} q \left[ \frac{m}{m-2} - \frac{A-B}{4} - \sigma \frac{A}{4} \right] \xi = 0. \] 

\[ -\frac{3}{2} kq \sigma A \eta + \frac{3}{2} kq \sigma A \eta = \frac{3}{2} kq \left[ \frac{m}{m-2} - \frac{A-B}{4} + \sigma \frac{A}{4} \right] \eta 

\[ -k^2 \left[ 1 + \frac{A-B}{4} - \frac{3}{2} \sigma A \right] \xi + \left[ k^2 \left( 1 + \frac{A-B}{4} - \frac{3}{2} \sigma A \right) - \frac{3}{2} \sigma A - 2 \frac{m-1}{m-2} q^2 \right] \xi = 0. \] 

\[ + \frac{1}{2} \left[ \frac{3}{2} \sigma A - k^2 \left( 1 + \frac{A-B}{4} - \sigma \frac{A}{4} \right) - 2 \frac{m-1}{m-2} q^2 \right] \xi + \frac{1}{2} \left[ 1 + \frac{A-B}{4} - \sigma \frac{A}{4} \right] \xi = 0. \] 

\[ + \frac{1}{2} \left[ 1 + \frac{A-B}{4} + \sigma \frac{A}{4} \right] \xi = 0. \] 

(78)

We may now eliminate the coefficients from equations (64–78), and obtain a determinantal equation, of fifteen rows, which gives a relation between A, B and the dimensions of the tube. This relation is the condition for neutral equilibrium of the initial stress-system, and is clearly correct to terms in \( \tau^2 \); but by further consideration of the terms involved we may show that the labour which would be required for its complete evaluation is unnecessary, and as the fifteen-row determinant may be written down directly from the above equations it will not be given here.

**Solution for Boiler Flue without End Thrust.**

We shall begin by deriving a sufficiently approximate expression for the difference of pressure required to produce collapse of a tube, when there is no resultant end thrust or tension; and in the first case we shall deal with a form of collapse possible only in the case of tubes of infinite length. That is to say, we make B and q equal to zero in the fifteen-row determinant, which may then be reduced to one of ten rows.

In the latter determinant we may treat A as a quantity of order \( \tau^2 \); for if A be put equal to zero, and the determinant be expanded, the terms which are independent of \( \tau \) vanish identically. Hence \(-A\) may be written for \( \sigma A \), and \( A\tau^2 \) may be neglected.

The ten-row determinant, simplified by these and other obvious modifications, is given on pp. 218 and 219. Expanding it from the top row, with the neglect of terms of order higher than \( \tau^2 \), we obtain

\[ \frac{4m}{m-2} \left( \frac{m-1}{m-2} \right) \left[ \tau^2 \left( m (k^2-1) + m - \frac{m}{3} (k^2+2) \right) + 2 (m-1) A \right] = 0, \]

whence

\[ -A = \frac{\tau^2}{3} \frac{m}{m-1} (k^2-1). \]

(79)
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\[
\begin{align*}
0, & \quad 0, & \quad \frac{\tau^2}{2}, & \quad 0, & \quad 0, \\
0, & \quad 1, & \quad 0, & \quad \frac{\tau^2}{6}, & \quad 0, \\
-1, & \quad m-1, & \quad (2m-1)\frac{\tau^2}{2}, & \quad (m-1)\frac{\tau^2}{2}, & \quad 0, \\
(m-2)A, & \quad m, & \quad m-1, & \quad (3m-2)\frac{\tau^2}{6}, & \quad \frac{\tau^2}{6}, \\
\frac{2m-1}{m-2}, & \quad \frac{2m-1}{m-2}, & \quad 2\frac{m-1}{m-2}, & \quad 0, & \quad 0, \\
-4\frac{m-1}{m-2}, & \quad -\left[\frac{4m-1}{m-2} + k^2\left(1 + \frac{A}{2}\right)\right], & \quad 2\frac{m-1}{m-2}, & \quad 2\frac{m-1}{m-2}, & \quad 0, \\
6\frac{m-1}{m-2}, & \quad 2\left[\frac{3m-4}{m-2} + k^2\left(1 + \frac{A}{2}\right)\right], & \quad -\frac{1}{2}\left[6\frac{m-1}{m-2} + k^2\left(1 + \frac{A}{2}\right)\right], & \quad \frac{m-1}{m-2}, & \quad \frac{1}{m-2}, \\
-2\frac{m-1}{m-2}, & \quad \frac{m}{m-2} - \frac{A}{2}, & \quad 0, & \quad 0, & \quad 0, \\
2\frac{m-1}{m-2}, & \quad \frac{3m-4}{m-2} + \frac{A}{2}, & \quad \frac{m}{m-2} - \frac{A}{2}, & \quad 0, & \quad 0, \\
-2\frac{m-1}{m-2}, & \quad -\left(\frac{3m-4}{m-2} + \frac{A}{2}\right), & \quad \frac{1}{2}\left(\frac{3m-4}{m-2} + \frac{A}{2}\right), & \quad \frac{1}{2}\left(\frac{m}{m-2} - \frac{A}{2}\right), & \quad 0, \\
1, & \quad 0, & \quad 0, & \quad 0, & \quad -1, \\
0, & \quad 0, & \quad 1, & \quad 0, & \quad 0, \\
1, & \quad 0, & \quad 0, & \quad 0, & \quad 0, \\
0, & \quad 0, & \quad 1, & \quad 0, & \quad 0, \\
1 - \frac{m-2}{2}A\left(1 + \sigma\right), & \quad m-1, & \quad 0, & \quad -k^2\left[1 - \frac{m-2}{2}A\left(1 + \sigma\right)\right], \\
(m-2)A\sigma, & \quad m - \frac{m-2}{2}A\left(1 + \sigma\right), & \quad 1, & \quad -k^2\left(m-2\right)A\sigma, \\
-\left[\frac{2m-1}{m-2} + k^2\left(1 + \frac{A}{2}\right) + q^2\left\{1 + \frac{A}{4}\left(1 + \sigma\right)\right\}\right], & \quad 2\frac{m-1}{m-2}, & \quad 2\frac{m-1}{m-2}, & \quad k^2\left(\frac{3m-4}{m-2} + \frac{A}{2}\right), \\
\frac{3m-4}{m-2} + \frac{A}{2}, & \quad \frac{m}{m-2} - \frac{A}{2}, & \quad 0, & \quad -\left[1 + \frac{A}{2} + 2\frac{m-1}{m-2}k^2 + q^2\left\{1 + \frac{A}{4}\left(1 - \sigma\right)\right\}\right], \\
\frac{m}{m-2} - \frac{A}{4}\left(1 - \sigma\right), & \quad \frac{m}{m-2} - \frac{A}{4}\left(1 + \sigma\right), & \quad 0, & \quad -k^2\left[\frac{m}{m-2} - \frac{A}{4}\left(1 - \sigma\right)\right],
\end{align*}
\]
\[
\begin{array}{cccccc}
-1, & \frac{A}{k^2}, & \frac{\tau^2}{2}, & \frac{\tau^2}{2}, & 0, \\
0, & -\frac{A}{k^2}, & 1+A, & \frac{\tau^2}{3}, & \frac{\tau^2}{3}, \\
-k^2, & -1, & -k^2 \frac{\tau^2}{2}, & 0, & 0, \\
k^2 (m-2) A, & -1+(m-2) A, & 0, & -k^2 \frac{\tau^2}{6}, & 0, \\
k^2 \left(\frac{3m-4}{m-2} + \frac{A}{\frac{A}{2}}\right), & 2+A, & 0, & 0, & 0, \\
-2k^2 \left(\frac{3m-4}{m-2} + \frac{A}{\frac{A}{2}}\right), & -(2+A), & -k^2 \left(\frac{m}{m-2} - \frac{A}{\frac{A}{2}}\right), & 0, & 0, \\
3k^2 \left(\frac{3m-4}{m-2} + \frac{A}{\frac{A}{2}}\right), & 2+A, & \frac{1}{4} k^2 \left(\frac{5m-4}{m-2} - \frac{A}{\frac{A}{2}}\right), & -\frac{1}{4} k^2 \left(\frac{m}{m-2} - \frac{A}{\frac{A}{2}}\right), & 0, \\
-\left(1+\frac{A}{2} + 2 \frac{m-1}{m-2} k^2\right), & -\frac{2m-1}{m-2}, & 1+\frac{A}{2}, & 0, & 0, \\
1+\frac{A}{2} + 2 \frac{m-1}{m-2} k^2, & 0, & 2 \left(1+\frac{A}{2}\right), & 1+\frac{A}{2}, & 0, \\
-\left(1+\frac{A}{2} + 2 \frac{m-1}{m-2} k^2\right), & 0, & -\frac{1}{2} \left(1+\frac{A}{2} + 2 \frac{m-1}{m-2} k^2\right), & \frac{1}{2} \left(1+\frac{A}{2}\right), & 1+\frac{A}{2}, \\
1-\Delta \sigma, & 0, & 0, & 0, & 0, \\
\Delta \sigma, & 1-\Delta \sigma, & 0, & 0, & 0, \\
0, & 0, & 0, & 1-\frac{1}{2} \Delta (1+\sigma), & 0, \\
0, & 0, & 0, & \Delta \sigma, & 1-\frac{1}{2} \Delta (1+\sigma), \\
0, & 0, & -q^2 \left[1-\frac{m-2}{2} A (1+\sigma)\right], & 0, & 0, \\
-k^2 \left[1-\frac{m-2}{2} A (1+\sigma)\right], & 0, & -q^2 \left[1-\frac{m-2}{2} A (1-\sigma)\right], & -q^2 \left[1-\frac{m-2}{2} A (1+\sigma)\right], & 0, \\
-k^2 \left(\frac{m}{m-2} - \frac{A}{\frac{A}{2}}\right), & 0, & 0, & -q^2 \left(\frac{m}{m-2} - \frac{A}{4} (1+\sigma)\right], & 0, \\
1+\frac{A}{2}, & 1+\frac{A}{2}, & -q^2 \left[\frac{m}{m-2} - \frac{A}{4} (1-\sigma)\right], & 0, & 0, \\
0, & 0, & -k^2 \left[1+\frac{A}{4} (1-\sigma)\right] + 2 \frac{m-1}{m-2} q^2, & 1+\frac{A}{4} (1-\sigma), & 1+\frac{A}{4} (1+\sigma) \\
\end{array}
\]
But to a first approximation

\[ A = -\frac{1}{4C_T} (P_1 - P_2), \]

so that we have

\[ P_1 - P_2 = \frac{3}{2} \cdot \frac{m}{m-1} C_T^2 (k^2 - 1) \]
\[ = \frac{3}{2} \cdot \frac{m^2}{m^2-1} E \left( k^2 - 1 \right) \left( \frac{t}{\alpha} \right) \]

which agrees with Bryan’s result.*

To complete our discussion of this problem we must consider types of distortion in which the axial wave-length is finite, and thus obtain a theoretical estimate of the strength of short flues with fixed ends. A solution giving \( A \) correctly to terms in \( \tau^2 \) may be derived from the complete fifteen-row determinant; but we may show that for practical purposes the labour which this evaluation would entail is quite unnecessary.

We find first of all that those terms in the expression for \( A \) which are independent of \( \tau \) contain \( q^4 \) as a factor. Now \( 2AC \) being approximately equal to the mean hoop stress in the tube before collapse, it is clear that \( A \) must in all cases of practical importance be a very small quantity. It follows that in the expanded equation the terms in \( A \) are of primary importance, and \( A^2 \) and higher powers may be neglected; further, since \( q \) must also be small, that terms in \( q^4 \) and higher powers of \( q \) may be neglected in comparison with terms in \( q^4 \), and that of the terms in \( \tau^2 \) those which involve \( q \) are negligible in comparison with the terms already found.

In accordance with these principles we may derive the terms which are required to complete our solution from a nine-row determinant, obtained by omitting terms in \( \tau^2 \) from the general determinant. This simplified determinant is given on pp. 218 and 219. Further, we may neglect \( A^2 \) in the expansion, and in the coefficient of \( A \) retain only those terms which do not involve \( q \); we thus obtain the equation

\[ \sigma A = \frac{m+1}{m} \frac{q^4}{k^4 (k^2 - 1)} \]

But, by equations (55),

\[ \sigma A = \frac{(P_1 - P_2)}{4C_T} (1 - \tau^2)^2, \]

and therefore, to the approximation of equation (81),

\[ P_1 - P_2 = 4 \frac{m+1}{m} C \frac{q^4}{k^4 (k^2 - 1)} \cdot \frac{t}{\alpha} = 2E \frac{q^4}{k^4 (k^2 - 1)} \frac{t}{\alpha}. \]

* Cf. footnote, p. 209.
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Combining this result with (80) we have, as our final expression for the pressure-difference which can produce collapse of the flue,

$$\mathbf{P}_1 - \mathbf{P}_2 = 2E \frac{t}{a} \left[ \frac{q^2}{k^3} \left(k^3 - 1\right) + \frac{m^2}{m^2 - 1} \left(k^3 - 1\right) \frac{t^2}{\alpha^2} \right]. \quad \ldots \quad (83)$$

In this equation $t/\alpha$ is the ratio of the thickness to the diameter of the tube, and $k$ is the number of lobes in the distorted form of its cross-section. The quantity $q$ is connected with the axial wave-length $\lambda$ of the distortion by the relation

$$q\lambda = 2\pi a. \quad \ldots \quad \ldots \quad (84)$$

We may imagine a flue subjected at its ends to constraints which merely keep the ends circular, without imposing any other restrictions upon the type of distortion.* In this case the end conditions may be written in the form

$$u' = 0 \quad \text{when} \quad z = \left\{ \begin{array}{c} 0 \\ l \end{array} \right\}, \quad \ldots \quad \ldots \quad (85)$$

and from (56) it is clear that $l$, the length of the flue, is equal to $\pi a/q$.

* Added June 8.—Thin circular discs, inserted into the tube at its ends, but not fixed to it, would approximately realize these conditions.]"
In practice, the end constraints will also tend to maintain the cylindrical form at the ends of the flue, and this effect will strengthen the tube, by an amount which is not easy to determine exactly. In any case we may say that
\[
\frac{l}{a} \approx \frac{1}{q},
\]
and we may illustrate the way in which the end effects die out by plotting the pressure differences \((p_1 - p_2)\) against the quantity \(q^{-1}\). To do this we must take some definite value of the ratio \(t/a\), and plot different curves for the values 2, 3, ..., \&c., of \(k\). The result is shown by fig. 3, in which the following values have been assumed for the constants:—

\[
E = 3 \times 10^7 \text{ pounds per sq. inch},
\]
\[
= 2.07 \times 10^{12} \text{ dynes per sq. cm}.
\]
\[
m = \frac{1}{3}, \quad \frac{l}{a} = \frac{1}{10}.
\]

From an inspection of the different curves we see that long tubes will always tend to collapse into the two-lobed form, since the curve for \(k = 2\) then gives the least value for the collapsing pressure, but that at a length corresponding to the point A the three-lobed distortion becomes natural to the tube, and for shorter lengths still, of which the point B gives the upper limit, the four-lobed form requires least pressure for its maintenance. Thus the true curve connecting pressure and length is the discontinuous curve CBAE, shown in the diagram by a thickened line.

Whatever be the relation between \(q\) and the length of the flue, it is clear that instability is theoretically possible in cases where the distortion involved is not even approximately "inextensional." For if \(\tau\) is sufficiently small, the collapsing pressure, as given by (83), need not involve elastic break-down in the position of equilibrium, even though the first (or "extensional") term in (83) be equal to, or even greater than, the second. Of course, elastic break-down will occur by reason of the extension very soon after the commencement of the distortion. Nevertheless, failure in such a case must be regarded as due entirely to instability; for if this source of weakness were removed, effective resistance could be offered for an indefinite period to pressures which actually result in collapse.

*Comparison with Experimental Results.*

Although, as we have just remarked, it is theoretically possible for failure to occur by true elastic instability in comparatively short tubes, yet the relative dimensions of the tubes must be such as it would be quite impossible to test experimentally. In any practical case, instability will not occur until the properties of the material have been altered by overstrain, and the value of the pressure at collapse is therefore very much less than the foregoing theory would suggest.
It is, however, of interest to compare the general shape of the theoretical curve CBAE (fig. 3) with the results of experiment, and fig. 4 has been constructed for this purpose. It represents a number of tests conducted by the author upon seamless steel tube (0.028 inches thick and 1 inch in external diameter), and shows the relative amounts of resistance to external pressure offered by different lengths of tube. In these experiments (selected for fig. 4 from a more comprehensive series which is still in progress) the ends of the tube were gripped by means of slightly conical plugs and sockets, the interior being kept in free connection with the atmosphere, and no attempt was made to balance the axial thrust due to hydrostatic pressure on the plugged end of the tube. Other experiments have shown that the existence of this thrust is not seriously important.

It will be seen that the general shape of the theoretical curve is well reproduced, as well as the changes in the number of lobes which characterize the distorted cross-section. Similar results to those of fig. 4 have been obtained by Carman,* but his experiments were not sufficiently numerous for a satisfactory comparison with the theoretical curve of fig. 3, his object in conducting them being merely to discover what is the limit of length beyond which the strength of a tube may be taken as

sensibly the same for all lengths. The main interest both of Carman and of Stewart* was confined to tubes in excess of this limit, experiments on which may fairly be compared with the theoretical formula (80); their results showed that this formula gives a satisfactory estimate of the strength of very thin brass and steel tubes, but must not be taken as a basis for design throughout the whole range of dimensions employed in practice.

The experiments of Fairbairn,† on the other hand, were restricted to tubes of such relatively small length that he failed to realize the existence of a definite minimum below which the strength of a tube, however long, will not fall. He also neglected the possibility of discontinuities in the curve of collapsing pressure at points where there is a change in the form of the distorted cross-section. In the light of these facts, figs. 3 and 4 help to explain his well-known formula, by which the collapsing pressure is given as inversely proportional to the length of the flue; for a curve of hyperbolic form will represent as well as any other single curve the scattered points of fig. 4, and trial shows that the hyperbola

\[
 \frac{p_1 - p_2}{q} = 464 \quad \cdots \quad (86)
\]

is very closely an envelope of the discontinuous curve CBAE in fig. 3, in each case down to the point of least collapsing pressure.

**Validity of Investigation by the Theory of Thin Shells.**

One important result of our investigation, which is apparently new, is shown by equation (83). It may be seen that collapse is practically dependent upon the pressure-difference alone, and that the absolute values of the pressures are immaterial. In view of this result, the objections raised by Basset against Bryan's treatment of the problem‡ require further consideration.

These objections are: first, that the ordinary expressions for the stress-couples in a plate or shell, in terms of the curvature of its middle surface, are not valid when the surfaces are subject to pressure; and secondly, that it is not legitimate to assume, as we must if sufficient equations are to be obtained, that the middle surface is unextended in a configuration of slight distortion. Hence the theory of thin shells is not applicable to this problem.

The above difficulties may be almost entirely overcome by a change in the method of investigation which is employed. It is customary to derive equations for the equilibrium of the distorted shell directly, and without reference to the position of equilibrium. Such procedure renders it necessary to make Bryan's assumptions, that the middle surface is unextended, and that the usual expressions for the stress-couples

† Cf. footnote, p. 209.
are valid. But we may also proceed, as in the foregoing discussion, by first determining the stress-system for the equilibrium position, and then deriving equations for an infinitesimal displacement. The stress-couples which appear in these equations will be due to the additional stresses introduced by the distortion, and since these, to a first approximation at least, vanish at the surfaces of the tube, they will be given with sufficient accuracy by the usual expressions. Moreover, when the distortion is two-dimensional (as in Basset's problem), the change in the "hoop" stress-resultant will be of an order which is negligible, so that the middle surface may be regarded as undergoing no extension relatively to the equilibrium position, even though its area may be sensibly changed in comparison with the unstrained configuration.*

The method of investigation just described, which follows the actual sequence of occurrences in the material, is suggested as in every way preferable to existing methods, for the investigation of any problem in elastic stability. For the present example, in particular, it leads to the same results as the more rigorous methods of this paper.

Comparison with Existing Formulae.

Previous discussion of the boiler-flue problem by analytical methods have, without exception, dealt with a tube subjected to pressure on one surface only, and almost all of them have been restricted to the case of an indefinitely long flue. Their results have, therefore, to be compared with our equation (80), when $P_2$ is zero. It will be found that this equation agrees with the formula obtained by Bryan† and Basset;‡ Föppl's formula† omits the factor $\frac{m^2}{m^2-1}$, which measures the increased resistance to flexure of a long tube as compared with a circular ring.

The more general formula may be compared with that of Lorenz,‡ if $P_2$ be put equal to zero. It will be found that there is a serious want of agreement in regard to both terms in the expression (83). In support of the latter result, it may be urged that Lorenz' solution gives for the indefinitely long flue a result which does not agree with equation (80) (and, as we have just noticed, this is supported by previous investigations), and which vanishes, not when $k = 1$, but when $k = 0$. Now the value 1, in the case of an infinitely long flue, corresponds to translation of the tube as a whole, without distortion, and the value 0 to a change in the diameter of the tube, without any departure from circularity. It is clear that the applied pressures can have no tendency to maintain such a form of distortion, so that Lorenz' formula can hardly be correct.

[* Added June 8.—The arguments of this section are more fully developed in a paper by the author "On the Collapse of Tubes by External Pressure," published in the 'Philosophical Magazine' for May, 1913 (pp. 687-698).]
† Cf. footnote, p. 209.

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The "Critical Length."

A. E. H. Love* has investigated the rate at which the strengthening effect of circular ends falls off when the length of a boiler flue is increased. His result suggests that at a distance which is great compared with the quantity \( \sqrt{at} \) the influence of the ends becomes negligible, and the flue collapses under sensibly the same pressure as a tube of infinite length; hence, in order that "collapse rings" may have any appreciable effect, their distance apart must not exceed some experimentally-determined multiple of this quantity.

The greatest length of tube over which the ends exert any appreciable strengthening influence, or the least length for which collapse is possible under a pressure sensibly equal to the critical pressure, has been called by Prof. Love† the "critical length." It is a conception of great importance in experimental work; for, as we have seen,‡ tests on any length of tube in excess of this limit may be taken to give the strength of an infinite length of the same tube, and their results compared with the theoretical formula (80)§: but as a basis for the spacing of "collapse rings" it is superseded by the theory of this paper, which yields an expression for the greatest length of tube consistent with stability, when the thickness and diameter of the flue, and also the collapsing pressure, are given; and Prof. Love has suggested to the author that it would be better now to employ the term "critical length" in this more general significance. As we have seen (p. 222), the length of the tube is some multiple of the quantity \( a/q \), and we may therefore obtain from (83) the following formula:—

\[
\text{Critical length} = \frac{Ma}{k \sqrt{(k^2-1) \left[ \frac{\Pi_1 - \Pi_2}{2E} \left\{ \frac{m^2}{t^2} - \frac{1}{2} \left( \frac{m^2}{t^2} \right)^2 - \frac{1}{2} \left( \frac{m^2}{t^2} - 1 \right) \right\} \right]}, \quad (87)
\]

where \( M \) is a constant, depending upon the type of the collapse ring, and \( k \) has that integral value which gives the least value for the right-hand expression of equation (87).

Before this subject is dismissed, it should be noticed that the theory of this paper does not support Prof. Love's estimate, mentioned above, of the rate of decay of end effects. The term in equation (83) which depends upon the length of the tube may be regarded as negligible, compared with the constant term, when the ratio

\[
\frac{Q^4}{k^4 (k^2-1) \left\{ \frac{m^2}{t^2} - \frac{1}{2} \left( \frac{m^2}{t^2} \right)^2 - \frac{1}{2} \left( \frac{m^2}{t^2} - 1 \right) \right\}}
\]

† 'Theory of Elasticity' (2nd edition), § 337 (b).
‡ Page 224.
§ In this sense the term "critical length" has also been employed by CARMAN, who began his research by investigating the strengthening effects of the end plugs with which he sealed his tubes for test.
has some sufficiently small value; and \( \frac{Q}{a} \) being inversely proportional to the length of the tube, we deduce for the "critical length," *in the original sense of the term*, an equation of the form

\[
L = f' \sqrt[2]{\frac{a^3}{b}},
\]

where \( f' \) is constant. Prof. Love, as has been said, has obtained an equation of the form

\[
L = f'' \sqrt{at},
\]

which is very different; but he has informed the author that in the light of the above investigation (pp. 210–222) he does not regard his method as adequate.*

**Solution for Tubular Strut: Special Case.**

We may obtain another simplification of the general determinant to ten rows by taking a zero value for \( k \). This corresponds to a type of distortion, possible in the case of a tubular strut, in which the axis remains straight and the cross-sections circular, the diameter varying in a sinusoidal manner.

The ten-row determinant for this case is given on pp. 228 and 229; the factor \(-q^2\) has been cancelled from the sixth column, and terms in \( A \) have been omitted, so as to yield a result for tubes collapsed by end pressure alone. The expansion is only correct to terms of order \( \tau^2 \), and for a first approximation we may also neglect the square and higher powers of \( B \), which must be small in any case of practical importance. Investigating first the terms which are independent of \( \tau^2 \), we obtain

\[
B = 2 \frac{m+1}{m} \cdot \frac{1}{q^2} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (88)
\]

We may now employ the substitution

\[
B = 2 \frac{m+1}{m} \cdot \frac{1}{q^2} + B' \tau^2
\]

in the determinant, and expand it from the top row, neglecting terms of higher order than \( \tau^2 \).

A considerable amount of unnecessary labour may be avoided by a preliminary examination of the relative importance of the various terms involved. It will be

[* Added May 4.—An argument in favour of the new formula may be drawn from physical considerations. The resistance offered by a tube to any given form of distortion is due partly to the extension and partly to the flexure which such distortion entails; and it is clear that the relative importance of the extensional part increases as the thickness is reduced. Hence, other things being equal, the effects of the ends, which necessitate extension of the middle surface, are more important in a thin than in a thick tube; that is to say, they are sensible over a greater length.]
\[
\begin{array}{cccc}
0, & 0 & 1 - \frac{B}{2}, & 0, \\
0, & 0, & 0, & 1 - \frac{B}{2}, \\
0, & 1, & -q^2 \tau^2, & -q^2 \tau^2, \\
\tau^2 & 6 & 1, & -q^2, \\
0, & 0, & -q^2 \left( \frac{m}{m-2} + \frac{B}{4} \right), & 0, \\
0, & 0, & 0, & -q^2 \left( \frac{m}{m-2} + \frac{B}{4} \right), \\
\frac{1}{m-2} & 0, & 0, & 0, \\
0, & \frac{2m-1}{m-2}, & 1 - \frac{B}{4}, & 1 - \frac{B}{4}, \\
0, & \frac{2m-1}{m-2}, & -2 \frac{m-1}{m-2} q^2, & 2 \left( 1 - \frac{B}{4} \right), \\
0, & 0, & -2 \frac{m-1}{m-2} q^2, & - \frac{m-1}{m-2} q^2, \\
\end{array}
\]
found that $B'$ contains terms in $q^2$ and $\frac{1}{q^2}$, as well as terms independent of $q$. Thus

the complete expression for $B$ is of the form

$$B = \frac{1}{q^2} \left( 2 \frac{m+1}{m} + \alpha^2 \right) + \tau^2 (\beta + \gamma q^2),$$

and it is clear that $B$ has a minimum value when the axial wave-length has a finite value, given by

$$\gamma^2 q^4 = 2 \frac{m+1}{m} + \alpha^2.$$

This minimum value, which alone is of practical importance, is given, to a first approximation in terms of $\tau$, by the equation

$$B_{\text{min.}} = 2\tau \sqrt{2 \frac{m+1}{m}} \gamma, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (89)$$

so that the determination of $\alpha$ and $\beta$ is not required.

By expansion of the determinant we find

$$\gamma = \frac{3}{2} \frac{m}{m-1},$$

and from (55) we deduce, for the minimum thrust required to produce collapse,

$$S_{\text{min.}} = 8\tau E\ell^2 \sqrt{\frac{1}{3} \frac{m^2}{m^2-1}}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (90)$$

This expression is correct to terms in $\ell^2$.

**Validity of Investigation by the Theory of Thin Shells.**

A complete investigation of the tubular strut problem must deal with lobed forms of deformation, since it is possible that one of these may require a smaller end-pressure for its maintenance than the circular form treated above. We have, therefore, to obtain a general expression for $B$ (when $A$ is zero) in terms both of $k$ and $q$.

The derivation of this expression, if we employ the rigorous methods of the present paper, will entail nothing less than the evaluation of the complete fifteen-row determinant; for the existence of a "favourite type of distortion," of finite axial wave-length, which we have noticed in the particular case ($k = 0$), is found by practical experiment to be equally a feature of the lobed forms of distortion, and shows that the terms in $\tau^2$ are important. Now it will be shown that the value of $S_{\text{min.}}$, when $k = 0$, may be obtained, correctly to terms in $\ell^2$, by the ordinary theory of thin shells; and as there is no reason to believe that the latter theory will lead to
less accurate results when \( k \) has a finite value, it does not seem necessary to employ our more rigorous method, with the very laborious calculations which it entails. We shall therefore rely upon the approximate theory for the treatment of the tubular strut problem in its general form. Slight modifications in method will be introduced, as suggested above (pp. 224–225), and only the more important steps will be given here.

**Solution by the Theory of Thin Shells: General Case.**

We consider the stability of an element of the tube, originally bounded by the planes

\[
\theta, \quad \theta + \delta \theta, \quad \text{and} \quad z, \quad z + \delta z,
\]

as shown below—

![Diagram of element](image)

The other dimension of the element is the full thickness of the tube, denoted in this paper by \( 2t \). The radius of the middle surface is \( a \).

The initial stress system is

\[
P_1 = \text{const.} = -\frac{\xi}{2\pi a} = [P_1] \text{ (say)}.
\]

In the distorted position this system produces a radial force on the element, of amount

\[
\frac{1}{R}[P_1] a \delta \theta \delta z,
\]

where \( R \) is the radius of curvature of a section of the distorted element by an axial plane (see fig. 5).

It also produces a tangential force, in the direction of \( \theta \) increasing, of amount

\[
-[P_1] \alpha \psi \delta \theta,
\]

where \( \psi \) (see fig. 5) = \( \frac{1}{a} \frac{\partial}{\partial \theta} (1 + e_{zz}) \delta z \).
The above system of distorting forces must be exactly balanced by the restoring system shown in the upper part of the figure. Hence we obtain the following equations of neutral stability:

\[
\begin{align*}
\frac{[P_1]}{R} + \frac{\partial T_1}{\partial z} + \frac{1}{\alpha} \frac{\partial T_2}{\partial \theta} - \frac{P_2}{\alpha} &= 0, \\
- \frac{[P_1]}{\alpha} \frac{\partial e_{zz}}{\partial \theta} + \frac{T_2}{\alpha} + \frac{1}{\alpha} \frac{\partial P_2}{\partial \theta} + \frac{\partial U_1}{\partial z} &= 0, \\
\frac{\partial P_1}{\partial z} + \frac{1}{\alpha} \frac{\partial U_2}{\partial \theta} &= 0, \\
\frac{\partial G_1}{\partial z} - T_1 - \frac{1}{\alpha} \frac{\partial H}{\partial \theta} &= 0, \\
\frac{1}{\alpha} \frac{\partial G_2}{\partial \theta} + T_2 + \frac{\partial H}{\partial z} &= 0,
\end{align*}
\tag{91}
\]

Now \( R \) and \( e_{zz} \) may be expressed in terms of the displacements of the middle surface, as follows:

\[
\frac{1}{R} = \frac{\partial^2 w'}{\partial z^2}, \quad e_{zz} = \frac{\partial w'}{\partial z}; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (92)
\]

and the restoring system of stress resultants may also be expressed in terms of this system, as follows*:

\[
\begin{align*}
P_1 &= \frac{3D}{\ell^2} \left[ \frac{\partial w'}{\partial z} + \frac{1}{m \alpha} \left( u' + \frac{\partial v'}{\partial \theta} \right) \right], \\
P_2 &= \frac{3D}{\ell^2} \left[ \frac{1}{\alpha} \left( u' + \frac{\partial v'}{\partial \theta} \right) + \frac{1}{m} \frac{\partial w'}{\partial z} \right], \\
U_1 = U_2 &= \frac{3}{2} \frac{m-1}{m} \frac{D}{\ell^2} \left( \frac{\partial^2 w'}{\partial z^2} + \frac{1}{m \alpha \theta} \right), \\
G_1 &= -D \left[ \frac{\partial^2 w'}{\partial z^2} + \frac{1}{m \alpha} \left( u' + \frac{\partial^2 u'}{\partial \theta^2} \right) \right], \\
G_2 &= D \left[ \frac{1}{\alpha} \left( u' + \frac{\partial^2 u'}{\partial \theta^2} \right) + \frac{1}{m} \frac{\partial^2 u'}{\partial z^2} \right], \\
H &= \frac{m-1}{m} D \left( \frac{\partial w'}{\partial \theta} - \frac{\partial v'}{\partial z} \right),
\end{align*}
\tag{93}
\]

where \( D \) is the quantity

\[
\frac{3}{2} \frac{m^2}{m^2 - 1} Ed^2. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (94)
\]

* Cf. Love, 'Mathematical Theory of Elasticity' (second edition), Chap. XXIV. \( u', v', w' \) have the same significance as in the earlier part of this paper, except that they now refer to the middle surface, and are functions only of \( \theta \) and \( z \).
Eliminating $T_1$ and $T_2$ from equations (91), and substituting from (92) and (93), we have

$$
\frac{u'}{\alpha^2} - \Psi \frac{\partial^2 u'}{\partial z^2} + \frac{1}{\alpha^2} \frac{\partial v'}{\partial \theta} + \frac{1}{\alpha} \frac{\partial u'}{\partial \theta} + \frac{1}{ma} \frac{\partial w'}{\partial z} + \frac{t}{3} \left[ \frac{1}{\alpha^4} \frac{\partial^4 u'}{\partial \theta^4} + \frac{1}{\alpha^4} \frac{\partial^4 v'}{\partial \theta^4} + \frac{2}{\alpha^2} \frac{\partial^4 u'}{\partial \theta^2 \partial z^2} + \frac{\partial^4 u'}{\partial \theta \partial z^4} + \frac{1}{ma^2} \frac{\partial^2 u'}{\partial \theta^2} - \frac{2}{m} \frac{\partial^2 u'}{\partial \theta \partial z^2} - \frac{m-1}{m} \frac{\partial^2 v'}{\partial \theta \partial z^2} \right] = 0,
$$

$$
\frac{1}{\alpha^2} \frac{\partial u'}{\partial \theta} + \frac{1}{\alpha^2} \frac{\partial^2 u'}{\partial \theta^2} + \frac{m-1}{2m} \frac{\partial^2 v'}{\partial z^2} + \frac{1}{\alpha} \frac{\partial v'}{\partial \theta} \frac{\alpha^2}{\partial \theta} + \frac{1}{\alpha^2} \frac{\partial v'}{\partial \theta} = 0,
$$

$$
\frac{1}{\alpha^2} \frac{\partial w'}{\partial \theta} + \frac{m+1}{2} \frac{\partial^2 v'}{\partial z^2} + \frac{1}{\alpha^2} \frac{\partial w'}{\partial \theta} + \frac{m-1}{2m} \frac{\partial^2 w'}{\partial \theta^2} + \frac{\partial^2 w'}{\partial z^2} = 0,
$$

where

$$
\Psi = \frac{[P_i] t^2}{3D} = -\frac{m^2-1}{m^2} \frac{\xi}{4\pi at E}.
$$

Assuming a solution of the type (56), we find, as the criterion for neutral stability,

$$
1 + \Psi q^2 + \frac{t^2}{\alpha^2} \left[ (k^2 + q^2)^2 - k^2 + \frac{1}{m} q^2 \right], \quad 1 + \frac{3}{m} \frac{m-1}{m} q^2 \frac{t^2}{\alpha^2}, \quad \frac{1}{m},
$$

$$
\frac{k^2 + \frac{1}{3} k^2 \frac{t^2}{\alpha^2} [k^2 + q^2 - 1]}{m^2}, \quad \frac{k^2 + \frac{m-1}{m} q^2 + \frac{1}{3} \frac{m-1}{m} q^2 \frac{t^2}{\alpha^2}}{m}, \quad k^2 \left( \frac{m+1}{2m} - \Psi \right),
$$

$$
\frac{1}{m} q^2, \quad \frac{m+1}{2m} q^2, \quad \frac{m-1}{2m} q^2 + k^2,
$$

$$
= 0.
$$

This equation, in its expanded form, is

$$
\Psi^2 \left[ \frac{m+1}{2m} k^2 q^2 \right] + \Psi \left[ \frac{m-1}{2m} q^2 \left( (k^2 + q^2)^2 + k^2 \right) + \left( 1 - \frac{1}{m^2} \right) \frac{m-1}{2m} q^2 \right] + \frac{t^2}{\alpha^2} \left\{ \Psi \left[ \frac{m+1}{2m} k^2 \left( (k^2 + q^2)^2 - k^2 \right) + \frac{m^2-7m+4}{2m^2} k^2 q^2 + \frac{m-1}{m} q^2 \right] \right. \left. + \frac{m-1}{2m} \left[ (k^2 + q^2)^2 - q^2 - k^2 \left( 2k^2 + 7k^2 q^2 + \frac{7m^2 + m - 2}{m^2} q^2 \right) \right] \right. \left. + k^2 + 3k^2 q^2 + 2 \frac{m^2-1}{m^2} q^2 \right\} = 0.
$$

Taking first the terms which are independent of $t^2$, and neglecting the square of $\Psi$ (which must be small), we find, as the first term in our solution,

$$
-\Psi = \frac{m^2-1}{m^2} \frac{q^2}{(k^2 + q^2)^2 + k^2}.
$$
When \( k = 0 \), this becomes
\[
-\Psi = \frac{m^2-1}{m^2} \cdot \frac{1}{q^3}, \quad \ldots \quad \ldots \quad \ldots \quad (100)
\]
which shows that \( q \) must be great, if \( \Psi \) has a value possible in practice. Similarly, when \( k > 0 \), we see that \( q \) must be small, and the approximate expression in this case is
\[
-\Psi = \frac{m^2-1}{m^2} \frac{q^2}{k^2 (k^2+1)}, \quad \ldots \quad \ldots \quad \ldots \quad (101)
\]
We may now determine sufficiently approximate expressions for the terms in \( t^2/\alpha^2 \), by treating \( q \) as great when \( k = 0 \), and as small when \( k > 1 \). That is to say, we retain only the highest and the lowest powers of \( q \) in the two cases.*

Thus, when \( k = 0 \), the important terms are
\[
q^2 \Psi + \left[ \frac{m^2-1}{m^2} + \frac{1}{3} q^4 \frac{t^2}{\alpha^2} \right] = 0,
\]
and we have
\[
-\Psi = \frac{m^2-1}{m^2} \cdot \frac{1}{q^2} + \frac{1}{3} q^4 \frac{t^2}{\alpha^2},
\]
or
\[
\mathcal{S} = 4\pi atE \left[ \frac{1}{q^2} + \frac{1}{3} \frac{m^2}{m^2-1} q^4 \frac{t^2}{\alpha^2} \right], \quad \ldots \quad \ldots \quad \ldots \quad (102)
\]
When \( k > 1 \), the important terms are
\[
q^2 \Psi \left[ k^2 (k^2+1) + \frac{1}{3} \frac{t^2}{\alpha^2} \frac{m^2+1}{m^2-1} k^4 (k^2-1) \right]
\]
\[
+ \frac{m^2-1}{m^2} q^4 + \frac{1}{3} \frac{t^2}{\alpha^2} k^4 (k^2-1)^2 = 0, \quad \ldots \quad \ldots \quad (103)
\]
whence, to terms in \( t^2/\alpha^2 \),
\[
-\Psi = \frac{m^2-1}{m^2} \frac{q^2}{k^2 (k^2+1)} \left[ 1 + \frac{1}{3} \frac{t^2}{\alpha^2} \left( \frac{m^2}{m^2-1} k^4 (k^2-1)^2 - \frac{m^2+1}{m^2-1} k^2 (k^2-1) \right) \right],
\]
\[
= \frac{m^2-1}{m^2} \frac{q^2}{k^2 (k^2+1)} + \frac{1}{3} \frac{k^2 (k^2-1)^2 t^2}{q^2 k^2+1} \frac{t^2}{\alpha^2},
\]
with sufficient accuracy, when \( q \) is small.
This leads to the result
\[
\mathcal{S} = \frac{4\pi atE}{k^2 (k^2+1)} \left[ q^2 + \frac{1}{3} \frac{m^2}{m^2-1} k^4 (k^2-1)^2 \frac{t^2}{\alpha^2} \right], \quad \ldots \quad \ldots \quad (104)
\]
For practical purposes only the stationary values of \( \mathcal{S} \) are important. It is readily seen that the minimum value obtained from (102) agrees with (90), and is therefore

* In every case it is legitimate for practical purposes to neglect the term in \( \Psi^2 \).
accurate as far as terms in $t^2$; we shall assume that (104) gives the same approximation, which for practical purposes is quite sufficient. We then find, for values of $k$ other than 0 and 1, the expression

$$ S_{\text{min.}} = 8\pi E t^2 \sqrt{\frac{1}{3} \frac{m^2}{m^2-1} \frac{(k^2-1)}{(k^2+1)}}. \quad (105) $$

When $k = 1$, the axis does not remain straight after distortion of the tube has occurred. This is the type of distortion (sometimes called "primary flexure") which was discussed by Euler, and it is easy to see that his result is identical with that of equation (104), which becomes in this case

$$ S_{k=1} = 2\pi at E q^2. \quad (106) $$

The exact expression for the length of the tube, in terms of $q$, is not a matter of great importance in the present problem, because the wavelength corresponding to a minimum value of the collapsing pressure is in all cases small, and the strength of any strut of ordinary dimensions will therefore be given by equations (90) or (105), into which the length does not enter. As in the case of the boiler-flue problem, we may illustrate the effects of length upon the collapsing thrust by plotting the intensity of stress, or $S/4\pi at$, against $q^{-1}$. For this purpose we must take some definite value of the ratio $t/a$, and draw separate curves for different integral values of $k$. The result is shown by fig. 6, in which the following values are assumed:

$$ \frac{t}{a} = \frac{1}{30}, \quad m = \frac{1}{3}. $$

$$ E = 3 \times 10^7 \text{ pounds per sq. inch.} $$

$$ = 207 \times 10^{12} \text{ dynes per sq. cm.} $$

Fig. 6. Strength of Tubular Struts.
From an inspection of these curves it is easily seen that as the axial wave-length increases the type of distortion which involves the least value for the collapsing thrust (and which the tube therefore tends naturally to assume) changes. For very short lengths we shall expect the circular type \((k = 0)\); then, as the length increases, lobed forms of distortion, in which the value of \(k\) becomes less as the length increases. The limit is reached when \(k = 1\); hence, the tendency of very long tubes is always to collapse in the manner discussed by Euler.

It is also to be noticed that those parts of the different curves which lie to the right of their lowest points have no practical significance. The actual curve, which shows the effect of length upon the value of the collapsing thrust, will approximate to the form shown in thick lines, since the wave-length (which varies as \(q^{-1}\)) will naturally not increase beyond that value which involves the least collapsing thrust.

**Comparison with Existing Formula.**

The formulæ of equations (90) and (104) may be compared with the results of other discussions of this problem. Equation (90) has been obtained by Lorenz,* and Lilly* has given the same result, except that the factor \(\sqrt{\frac{m^2}{m^2 - 1}}\) is omitted.†

The only existing solution for lobed forms of distortion is due to Lorenz,* and this is not in agreement with equation (104). In support of the latter formula it may be urged that Lorenz' formula does not agree with Euler's result when \(k = 1\).

It may also be remarked that the foregoing results for the tubular strut problem contradict Bryan's theorem, that a closed shell cannot fail by instability, because distortion would involve extension of the middle surface; for although the first terms in equations (102) and (104) are due solely to extension of the middle surface, yet the compressive stress at collapse, as given by (90) or (105), may be insufficient to produce elastic breakdown in the position of equilibrium, if the ratio \(\frac{t}{a}\) has a sufficiently low value.

**Stability of Tubes under Combined End and Surface Pressure.**

We shall not treat this case in any detail, but it requires notice in connection with the "localization of collapse" which is observed in experiments conducted upon long tubes tested under hydrostatic pressure, the permanent distortion being generally confined to a portion only of the length of the tube. This result is not predicted by the theoretical formula \((83)\), which suggests a steady fall in the value of the collapsing pressure as the wave-length increases; and a partial explanation may possibly be found in the fact that the method of test has generally left a wholly or partially

* Cf. footnote, p. 209.
† For a similar omission in a solution of the boiler-flue problem cf. p. 225.
unbalanced end-thrust, due to the water pressure acting upon the closed ends of the tube.

It is clear that the expansion of the general fifteen-row determinant will give an equation of the form

\[ a + \beta A + \gamma B + \delta A^2 + \epsilon AB + \xi B^2 + \ldots = 0, \]

where \( a, \beta, \gamma, \ldots \) depend upon the dimensions of the tube and the type of the distortion. But in any practical case, as we have already observed, \( A \) and \( B \) must be very small quantities. It follows that an approximate solution may be obtained from the terms

\[ a + \beta A + \gamma B = 0. \quad \ldots \ldots \ldots \quad (107) \]

Let \( \mathbf{p}_e \) and \( \mathbf{s}_e \) be the values of the external pressure and of the end-thrust, each of which, acting alone, could produce collapse into the assumed type of distortion. Then equation (107) may clearly be written as follows:

\[ \mathbf{p}_i = \left( 1 - \frac{\mathbf{s}_e}{\mathbf{p}_e} \right) \mathbf{p}_i, \quad \ldots \ldots \ldots \quad (108) \]

where \( \mathbf{p}_i \) and \( \mathbf{s} \) are the values of the external pressure and end-thrust which can produce collapse when acting in conjunction.

It may be seen from this equation that \( \mathbf{p}_i \) can have a minimum value for some finite value of the axial wave-length when, and only when, \( \mathbf{s} \) exists. If the end-thrust be entirely unbalanced, we have

\[ \mathbf{s} = \pi a^2 (1 + \tau)^2 \mathbf{p}_i, \quad \ldots \ldots \ldots \quad (109) \]

and the collapsing pressure may, in this case, be determined from equation (108).

**General Theory of Instability in Materials of Finite Strength.**

*The Practical Value of a Theory of Instability.*

In the concluding section of this paper an attempt will be made to estimate the practical value of a theory of elastic instability; to suggest ways in which we may hope to increase this value; and to indicate the questions to which answers must be found in order that further advance may be possible.

The first point which must be noticed is the non-realization in practice of our conception of a "critical loading," owing to imperfections which always exist, and which violate our ideal assumptions. In any actual example the displacement of the system increases continuously with the load, and the system collapses at a smaller value of the load than our theory would dictate. It is necessary to inquire whether serious discrepancies are to be expected.

In some mechanical problems the effects of imperfections may be calculated. We may take, as an example, the system illustrated in fig. 1, and consider any one of the
many imperfections which occur in practice. For simplicity, let us assume that the sphere, bowl, and plunger are still smooth, rigid, and accurately formed, but that the line of thrust of the plunger is eccentric by an amount $\delta$. It is easy to see from fig. 7 that the displacement of the sphere from the line of thrust of the plunger, when the system is in equilibrium under a load $P$, is

$$d = r \sin \theta = \delta + (R - r) \sin \phi,$$

where

$$\frac{P}{W} = \frac{\tan \phi}{\tan \theta - \tan \phi};$$

and these equations enable us to trace the steady increase of the sphere's displacement as the load on the plunger is increased from a zero value.

Thus in fig. 8 curves are drawn to connect $P$ and $d$, for a value 3 of the ratio $R/r$, when the initial displacement $\delta$ has the values $0$, $0'01r$, and $0'1r$ respectively. At the points on these curves for which $P$ has a maximum value, “collapse” will occur, since the equilibrium then becomes unstable. The locus of these points is shown in the figure by a broken line, and a dot-and-dash line shows the connection between $\delta$ and the maximum value of $P$. From the latter curve it is evident that a small initial inaccuracy may cause a material reduction in the “collapsing load”; nevertheless the “critical load” gives a limit which will be more and more nearly attained as our experimental accuracy is improved, and its investigation is by no means useless for practical purposes.

When the problem is one of elastic stability, the discussion of imperfections by analytical methods will, in general, be beyond our power; but it is clear that similar remarks will apply. An “exchange of stabilities” at some “point of bifurcation”* must be regarded as a purely ideal conception, and in practice there will always be a steady increase of distortion as the load is increased, owing principally to practical imperfections of form. A strut, for example, may be very accurately loaded, if suitable methods are employed, but its centre-line will never be quite straight; the initial deflection which characterizes it may be regarded as composed of a series of

---

harmonic terms, and when the load is applied one of these harmonics will be developed very much more than the others, just as one constituent harmonic may be developed by "resonance" in an alternating current wave of irregular shape. In the ordinary strut problem this magnified harmonic is such that one-half wave occupies the length of the strut, but in other problems, such as that of the tubular strut, though there is always a "favourite" or "natural harmonic" which is especially magnified, its relation to the dimensions may be more complicated. In any case the effects of practical imperfections of form might be studied, if the analytical difficulties could be surmounted, by investigating the rate at which the amplitude of this "natural harmonic" increases with the load, when its value in the initial configuration is given; and the results of the investigation might be shown graphically by curves of distortion, similar in character to the curves of fig. 8, in which the abscissae represented the amplitude of the natural harmonic, and the ordinates represented the magnitude of the applied stress-system, or "load."

These "curves of distortion" are of considerable utility for the study of problems in elastic stability, even though their true form can only be guessed. They help us, for example, to explain, and in some degree to remedy, the serious discrepancy existing between Euler's theory and the results of experiments on short struts. The discrepancy has often been attributed to practical imperfections of form; but it should hardly be necessary to point out that practical imperfections are likely to diminish rather than to increase in importance, as the dimensions of an elastic solid become more nearly comparable, so that they will never be more effective as causes of weakness than in struts of great length, which, as a matter of fact, give results in close agreement with Euler's formula.

A more satisfactory explanation of this, and of similar discrepancies in other problems, may be found in the fact that the ordinary theory of elastic stability neglects the possibility of elastic break-down. If we attempt to draw "curves of distortion" for any single problem, we shall find that, apart from the other data of the problem, three possible cases exist, depending upon the elastic limit of the material under consideration:—

(1) The material may be of infinite strength;
(2) Its elastic limit may be so high that the critical load, as determined by the theory of instability, is not sufficient to cause elastic break-down in the configuration of equilibrium;
(3) Elastic break-down may occur, even in the position of equilibrium, at a load less than the critical value.

In the first case (which is, of course, purely ideal), the distortion due to loading will vanish when the loading is removed, and in this sense we may say that the

* In the problem of the tubular strut, the "favourite harmonic" is, of course, defined by that value of $q$ which corresponds to a minimum value of $\mathcal{S}$ in equations (102) or (104).
material will never fail. The "curves of distortion," if we could determine their true shape, would probably be approximately of the form shown in fig. 9. The theoretical methods of this paper enable us to fix the position of A, the "point of bifurcation," but give no information as to the form of AB, beyond the fact that it cuts OA at right angles.* The other curves of the diagram will approach more and more closely the limiting form OAB as the initial value of the amplitude is decreased.

In the second case, we have the additional complication of elastic break-down under finite stress, which reduces the resistance of the material and causes the new "curves of distortion," shown by thick lines in fig. 10, to begin at certain points to fall away from the corresponding curves of fig. 9 (reproduced in fine lines for comparison); these points will lie on some line such as CD, cutting OA at a point above A, and it is clear that to the right of CD the curves of distortion refer to displacements which do not wholly vanish when the load is removed. Total collapse of the system will obviously occur at the points of maximum load on the curves of distortion, and the locus of these points, which is shown on the diagram by the dot-and-dash line EF, may be termed the "line of final collapse."

![Fig. 9](image1)

![Fig. 10](image2)

A knowledge of the true form of EF would enable us, when we are given the initial value of the amplitude, to predict the load at which the system will collapse; and these quantities could be connected by another curve AG, which would show at once whether the resistance of the system to collapse is seriously reduced by practical inaccuracies of form. A complete theory of any problem in elastic stability must yield information on this very important point, as well as an expression for the "critical load"; but in most cases more powerful methods would be needed for its derivation than are at present available. The investigation of the "critical load" is therefore not without utility, for although never realized in practice, this forms a limit which should be fairly closely approached when considerable accuracy is possible.

In our third case the "critical load," as deduced by theoretical methods, is more than sufficient to cause elastic break-down. We may proceed as before to draw hypothetical curves of distortion. The line C'D' (fig. 11), which corresponds to the

* It must not be assumed that AB is a horizontal straight line; in general, since the distorting effect of the applied stress-system, which varies as the deflection, increases less rapidly than the resistance, which varies as the curvature, AB will tend to rise from A.
line CD of fig. 10, will intersect OA at a point below A, and the other curves of
distortion at correspondingly lower points. We have seen that the effect of local
elastic breakdown upon fig. 10 was to deflect the curves of distortion from the forms
which they would have assumed if the material had possessed indefinite strength;
and it is clear that this deflection will begin at lower values of the loads in the
present case. We may therefore expect curves of the type shown in thick lines in
fig. 11, where the curves already obtained are reproduced in fine lines for comparison.
As before, we may draw a line A'F' "of final collapse" through the points of
maximum load on the curves of distortion, and connect the collapsing load with the
initial value of the amplitude by another curve A'G'.

It is clear that the curves of distortion must tend to a limit which is no longer
OAH, but some other curve OA'H', where OA', the critical load under the new
conditions, is more than sufficient to produce elastic breakdown, but less than OA.
We can see further that the curve A'G' is not likely to fall away from A' much more
steeply than AG from A in fig. 10. The great weakness of short struts in practice,
compared with Euler's theoretical estimate, is now explained. Whereas long struts
come within the conditions of fig. 10, the failure of short struts will be repre-
sented by fig. 11, and occurs at comparatively low stresses, not because practical
imperfections have a greater effect upon the strength, but because OA', the true
value of the critical load, is less than OA, the value which Euler's theory would
dictate.*

It is the rule, rather than the exception, that the critical load, as found by the
ordinary theory of elastic stability, is more than sufficient in practice to produce
elastic breakdown. This may be readily seen in reference to any particular example.
In the case of the tubular strut, fig. 10 is only applicable when the ratio of diameter to
thickness is greater than 560 (for an average quality of mild steel), and for thicker
tubes the critical load falls, apparently by a very considerable amount,† below the
theoretical estimate. The determination of the critical load, in cases where this is
more than sufficient to produce elastic breakdown, is thus a problem of great
importance, since it forms a limit which can never, under any circumstances, be
exceeded. In the ordinary strut problem the determination can be effected without
difficulty, and an apparently new field is thus indicated for research. The distin-
guishing feature of its problems is the dependence of the stress-strain relations upon
the past history of the material, rendering absolutely necessary a method which
follows the actual cycle of events up to the occurrence of collapse.

[* Added May II.—Since this paper was written, the author's attention has been drawn to a
dissertation by T. von Karman ("Untersuchungen über Knickfestigkeit," Berlin, 1909), in which the
forms of these "curves of distortion," for solid struts of practical dimensions, are deduced both from theory
and from experiments. Karman also gives a relation equivalent to that of equation (112).]

† Experiments conducted by the author upon seamless steel tubes showed failure under loads which
were in every case little more than sufficient to produce "permanent set."
Stability of Short Struts.

This problem has been discussed elsewhere by the author,* and it will be noticed here only at sufficient length to indicate the directions in which further research is needed. We have to derive an expression for the collapsing load of a straight strut, when this is more than sufficient to cause elastic break-down of the material; and we proceed as before by considering three configurations of the strut: (1) before strain; (2) in a position of neutral equilibrium under uniform end-thrust; and (3) in a position of infinitesimal distortion from the second configuration.

For a first approximation we may say that cross-sections remain plane in the third configuration, so that the diagram of longitudinal compressive strain for any cross-section is as shown in the upper part of fig. 12; the horizontal line $fy$ shows the uniform strain of the second configuration. Then, if fig. 13 be the stress-strain diagram for a compression test of our material, and this uniform strain corresponds to a stress $p$ which is represented by the point B, we see that to the right of the point F in fig. 12 the longitudinal compressive stress in the third configuration must be greater, and to the left less than $p$.

Now it is a well-known property of metals that if at any point B on the stress-strain diagram, beyond the elastic limit, we begin to decrease the load, the diagram is not retraced, but that we obtain a line BC which is parallel to OA.† It follows that the ratio
\[
\frac{\text{decrease of stress}}{\text{decrease of strain}}
\]
is still given by $E$, YOUNG'S Modulus for the material. On the other hand, the diagram shows that if we increase the load beyond $B$ by an infinitesimal amount, the ratio
\[
\frac{\text{increase of stress}}{\text{increase of strain}}
\]
is a smaller quantity $E'$, which may be found from the slope of the diagram at $B$.

* 'Engineering,' August 23, 1912.
† A. MORLEY, 'Strength of Materials,' § 42.
We are considering an infinitesimal distortion in the third position, so that if we represent the increase of strain in fig. 12 by $\lambda z$, the increase of longitudinal compressive stress to the right of F may be taken as $E'\lambda z$, and the decrease of this stress to the left of F as $E\lambda z$. Hence we obtain, for the section under consideration; the diagram of longitudinal stress which is shown in the lower part of fig. 12. The uniform stress of the second configuration is shown by the horizontal line $l_n$, and it is a condition for neutral stability in the second configuration that no increase of thrust shall be required to maintain the distortion. If the cross-section of the strut is rectangular, of dimensions $a \times 2t$, it follows that the triangles $lmk$ and $qmn$ must be equal in area, or

$$\frac{FQ^2}{PF^2} = \frac{E}{E'}.$$

This relation fixes the position of F on the cross-section, and in terms of the dimensions shown in fig. 12 we may write for the moment of resistance about G,

$$M = \int_0^{b+t} E'\alpha \lambda (z-b) \, dz + \int_{b-t}^b E\alpha \lambda (z-b) \, dz = \frac{3}{2}a\lambda Et^2 + \frac{4}{3}a\lambda (E - E')(b^2 - 3b^2 - 2t^2).$$

But if $y$ is the deflection of the strut at the point G, in the infinitesimal distortion, we have, as in the ordinary theory of bending,

$$\lambda = -\frac{d^2 y}{dx^2},$$

where $x$ denotes the distance of the section from one end; and for equilibrium in the third configuration $M$ must be equal to the bending moment due to thrust, or $2atpy$: hence,

$$py + \frac{d^2 y}{dx^2} \times \frac{4}{3}Et^2 \left(1 + \frac{1}{4} \left(1 - \frac{E'}{E}\right) \left(\frac{b^2}{t^2} - 3 \frac{b}{t} - 2\right)\right) = 0. \quad \text{(112)}$$

Equation (112) shows the modification which must be introduced, to take account of elastic break-down, into Euler's equation

$$py + \frac{d^2 y}{dx^2} \times \frac{4}{3}Et^2 = 0. \quad \text{(113)}$$

and it is easy to see that if $l$ is the length, calculated from (113), of a strut which can just support the stress $p$, and $l'$ the length as calculated from (112), then

$$\frac{l^2}{l'^2} = 1 + \frac{1}{4} \left(1 - \frac{E'}{E}\right) \left(\frac{b^2}{t^2} - 3 \frac{b}{t} - 2\right).$$

But, from (111),

$$\frac{t+b}{t-b} = \sqrt{\frac{E}{E'}}.$$
so that, finally,

\[ \frac{\mu}{I} = \frac{2}{1 + \sqrt{\frac{E}{E'}}} \]  

(114)

This result leads to a simple method by which the collapsing loads of short struts may be obtained graphically from the compressive stress-strain diagram. A full explanation is given in the paper to which reference has already been made, and a comparison, showing satisfactory agreement, is made with the results of experiments. One conclusion of some practical importance may be noticed: the curves of collapsing stress show that great ductility of material is by no means desirable in struts, the primary requisite being a high elastic limit.

**Need for Further Research. Conclusion.**

With slight modification, the theory just given for short struts might be applied to the problem of circular rings under radial pressure; but these appear to be the only cases in which we can at present discuss the stability of overstrained material. In any problem dealing with plates or shells distortion from the equilibrium position must introduce new stresses, in directions perpendicular to that of the stress which has caused elastic failure. The circular type of distortion in a tubular strut, for example, will introduce "hoop" stresses, and at present we have no knowledge of the corresponding stress-strain relations when "set" has occurred.

This and many other stability problems may be regarded as special cases of a general problem, viz., the determination of the changes of strain which occur when an infinitesimal stress-system, defined by principal stresses \( q, r, s \), is impressed upon a material already overstrained by a simple stress \( p \). The problem is not simple, and its solution would probably entail much theoretical and experimental work; but this would be justified by the importance, both for theory and practice, of its applications.

In conclusion, the author desires to express his indebtedness to Profs. Love and Hopkinson, for valuable criticism and advice; to Mr. L. S. Palmer, for the photographs reproduced in fig. 2; and to Messrs. H. J. Howard and D. P. Scott, for assistance in the prosecution of the experiments described on p. 223. He also takes this opportunity of thanking Messrs. Stewarts and Lloyds, Ltd., of Glasgow, for gifts of very accurate steel tube for experimental purposes.
VI. Some Phenomena of Sunspots and of Terrestrial Magnetism.—Part II.

By C. Chree, Sc.D., LL.D., F.R.S., Superintendent of Kew Observatory.

Received June 10,—Read June 26, 1913.

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§ 1. In a previous paper, described here for brevity as S.M.,* which referred to sunspots and terrestrial magnetism, I had occasion to enquire into the existence of any relation between the magnetic character of individual days and that of days separated from them by a given interval of time. References to previous work bearing on the subject will be found in S.M., p. 97.

The material of which I made principal use consisted of magnetic "character" figures—on the international scale 0 (quiet), 1 (moderately disturbed), and 2 (highly disturbed)—assigned by myself to all the days of the eleven years 1890 to 1900, from consideration of the Kew magnetic curves.

In each of the 132 months of the eleven years the five days were taken which gave the largest daily range to the magnetic horizontal force. In default of any more satisfactory means of selection, the 660 days thus obtained were taken as representative of disturbed conditions. Regarding any one of the selected days as day n, the "character" figures for the 41 successive days \( n-5 \) to \( n+35 \) were written down in a row. This was done for each of the 660 selected days in succession, so that there were in all 41 columns of figures, each containing 660 entries.

* 'Phil. Trans.,' A, vol. 212, p. 75.
The "character" figures in each column were then added up as if they were purely arithmetical quantities, and an arithmetic mean was taken. This was regarded as a measure of the disturbance existent on the representative day of the column. Thus the means for columns \( n \), \( n-1 \), and \( n+1 \) represented respectively the amount of disturbance on the typical selected disturbed day, and on the days immediately preceding and following it. These mean "character" figures showed in the clearest way the existence of a period somewhat in excess of 27 days, but no shorter period was disclosed. This implied that if any day were considerably more disturbed than the average day of the month, then the day 27 days subsequent to it was likely to be also more disturbed than usual.

The acceptance of the arithmetic mean of a number of "character" figures as itself a measure of magnetic disturbance is open to criticism on several grounds. There is no strict line of demarcation between the three classes of days. There are in reality an infinite variety of grades intermediate between the extremely quiet day, which cannot get less than "0," and the extremely disturbed day which cannot get more than "2." Some days to which "2" is allotted represent disturbances whose energy on any conceivable view must be immensely more than twice—possibly more than twenty times—the energy of disturbance on the average day of character "1." The procedure was suggested by the practice followed at de Bilt, where the "character" figures supplied by the different observatories are dealt with. Supposing data to be supplied by, say, 40 observatories, the 40 figures assigned to any one day are summed and the mean taken to the nearest 0'1, and the result is accepted as an international measure of the amount of magnetic disturbance on the day in question.

§ 2. The "character" figures in S.M. were based on the curves of only one station, Kew; they were assigned by a single individual, myself; and they referred to one period of years, 1890 to 1900. I have thus thought it desirable to repeat the investigation for a second period of years, 1906 to 1911, making use of the international "character" figures published at de Bilt. 1906 was the earliest year for which international figures existed, and 1911 was the latest for which these figures were complete when the present enquiry commenced. As before, five days were selected for each month; but they were selected solely by reference to the international lists, being the five days of highest "character" figures in each month. When, as occasionally happened, there was a possible choice between two or more days for the last place on the monthly list of five, the criterion applied was that the selected days should, if possible, be consecutive. I had had occasion some years ago, before the present enquiry was even thought of, to select the five most disturbed days of each month of the years 1906 to 1909, and had made use of the above criterion. There seemed no reason to discard the old list, or to follow a different principle when dealing with 1910 and 1911. My experience when forming the first list had led me to regard five as a happy choice for the monthly total of disturbed days. A considerably smaller number, such as one or two a month, gave too few days to eliminate
accidental features, unless a much larger number of years were available. On the other hand, if one took as many as ten days, there would in most months be several days competing for the last place on the list, and during magnetically quiet times many of the days occurring in the monthly choice would have represented quiet rather than disturbed conditions.

The present paper is not confined to the period 1906 to 1911, but utilises as well my original data for 1890 to 1900 for the investigation of various points not considered in S.M.

§ 3. The first step was to make sure that the period of approximately 27 days was confirmed by the international “character” figures from 1906 to 1911. The mean results obtained for the individual years from 5 days before to 30 days after the representative day of large disturbance are given in Table I. The entries represent the mean international “character” figure. The last column gives for comparison the mean “character” figure for all days of the year. In the case of 1911, December was excluded, so as to keep all the days dealt with within the six years. The results were really taken out to three decimal places, and these more exact values were used in calculating some of the later results in the paper.

§ 4. Before discussing the main question, some phenomena in Table I. call for remark. The entries in column $n$ and the means from all days show but little variation from year to year, and the natural inference would be that the six years were almost equally disturbed. The phenomena, however, is I believe largely due to another cause. The international data are published quarterly. Thus the man whose duty it is to assign “caractère” figures at any observatory naturally deals with the curves of not more than three months at a time. In most cases, doubtless, he has a desire to maintain something like a uniform standard; but unless his verdict is based on the exact measurement of some definite quantity, such as the daily range, he is inevitably much influenced by the accident of whether the months he is dealing with are quiet or disturbed. One of the leading objects is the discrimination between the days of each individual month, and if “0’s” are given to nearly all the days of a very quiet month, there is no adequate discrimination. The natural tendency is thus to assign a “1” in quiet months to days which in highly-disturbed months would naturally get a “0.”

§ 5. Another point to bear in mind is that highly disturbed conditions are seldom confined to a single day, and not infrequently extend over three or four consecutive days or even more. Not infrequently three or even four of the five most disturbed days of the month were consecutive. In February, 1907, the whole five were consecutive days, and in March and April, 1910, seven of the ten selected disturbed days were consecutive. This explains why the “character” figures for days $n - 1$ and $n + 1$ in Table I. invariably are next in magnitude to those for days $n$. But the next highest figure, it will be seen, occurs on day $n + 26$ (once), $n + 27$ (four times), or $n + 28$ (once).
TABLE I.—Mean "Character" Figures from Selected Disturbed Days and from Previous and Subsequent Days.

<table>
<thead>
<tr>
<th>Year</th>
<th>(n-5)</th>
<th>(n-4)</th>
<th>(n-3)</th>
<th>(n-2)</th>
<th>(n-1)</th>
<th>(n)</th>
<th>(n+1)</th>
<th>(n+2)</th>
<th>(n+3)</th>
<th>(n+4)</th>
<th>(n+5)</th>
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<td>0.63</td>
<td>0.56</td>
<td>0.59</td>
<td>0.66</td>
<td>0.92</td>
<td>1.31</td>
<td>0.99</td>
<td>0.63</td>
<td>0.55</td>
<td>0.57</td>
<td>0.58</td>
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<td>0.72</td>
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</tr>
<tr>
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<td>1.34</td>
<td>1.08</td>
<td>0.81</td>
<td>0.64</td>
<td>0.57</td>
<td>0.57</td>
</tr>
<tr>
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<td>0.61</td>
<td>0.66</td>
<td>0.55</td>
<td>0.66</td>
<td>0.91</td>
<td>1.32</td>
<td>0.99</td>
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</tr>
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<td>0.64</td>
<td>0.95</td>
<td>1.32</td>
<td>1.07</td>
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<tr>
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<td>0.95</td>
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<td>0.81</td>
<td>0.75</td>
<td>0.70</td>
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</table>
Taking the means from the six years, the mean "character" figures for days \( n+27 \) and \( n+28 \) considerably exceed all others, that for day \( n+27 \) being decidedly the larger. The close resemblance to the results for the epoch 1890 to 1900 in S.M. will be readily recognised on consulting fig. 1.

In 1890 to 1900 the mean character figures for day \( n \), for day \( n+27 \), and for the mean day of the period were respectively 1.51, 0.94 and 0.70, so that the excess of the "character" figure for day \( n+27 \) over that for the average day was 30 per cent. of the excess for day \( n \). In 1906 to 1911 the corresponding percentage is 27.

It is unlikely that my personal standard for disturbance when assigning "character" figures to the days of 1890 to 1900 agreed with that of the international list, which represents a compromise of most diverse standards from some forty observatories. Thus the fact that the mean "character" figure for the selected disturbed days of 1906 to 1911 was only 87 per cent. of that for the selected disturbed days of 1890 to 1900 does not necessarily imply that the second epoch was the quieter of the two. Such, however, was actually the case on the whole, though no year of the later period was as quiet as 1900.

The two curves of fig. 1 agree in showing no decided trace of any period shorter than 27 days. Other points of resemblance are that the fall subsequent to the maximum during days \( n+28 \) to \( n+30 \) is decidedly slower than the rise during days \( n+25 \) to \( n+27 \), and that the pulse centering about day \( n+27 \) is spread over more days than the primary pulse centering at day \( n \). The latter phenomenon would obviously tend to happen if the period had not always the same length but oscillated slightly about a mean value.

§ 6. With a view to following up this last idea, I took from the selected disturbed days of the six years all those whose "character" figures were not less than 1.5, the

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group thus representing a specially high grade of disturbance. There were in all 103 of these days, the annual number varying from 15 in 1907 to 20 in 1906. The following were the mean character figures found for the primary day and the subsequent days indicated:—

<table>
<thead>
<tr>
<th>Day</th>
<th>n</th>
<th>n+25</th>
<th>n+26</th>
<th>n+27</th>
<th>n+28</th>
<th>n+29</th>
<th>n+30</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;Character&quot;</td>
<td>1.683</td>
<td>0.564</td>
<td>0.689</td>
<td>0.821</td>
<td>0.871</td>
<td>0.842</td>
<td>0.748</td>
</tr>
</tbody>
</table>

This gives a period if anything in excess of 28 days, and so suggests a slight increase in the length of the 27-day period as the intensity of the primary disturbance is increased; but a considerably larger number of days, and so a considerably longer period of years, would be required to establish the result.

The mean "character" figures given above for days $n+28$ to $n+30$ are distinctly larger than the corresponding figures in Table I., but the excess in these days is relatively less than that on day $n$ itself. Thus the excess in the "character" figure given above for day $n+28$ over the average day of the six years (i.e., 0.871–0.663 ≡ 0.208) is only 20 per cent. of the excess on day $n$ (1.683–0.663 ≡ 1.020), while the corresponding percentage from Table I. was 27.

§ 7. If individual magnetic storms are directly due to individual sunspots, as various writers have suggested, it is, of course, a natural inference that when the sun's rotation has brought a spot round to the position it occupied relative to the earth when a magnetic storm occurred, a second storm will be experienced. This seemingly is what led Harvey and Maunder independently to suggest a 27.5-day period for magnetic storms.

Our previous investigations show a period of about 27 days, which, however, is not confined to what are usually termed "magnetic storms," but belongs equally to moderate disturbances, which are frequent events. If, then, magnetic storms are due to sunspots, equally so it would seem must be the minor disturbances; and if magnetic storms sometimes recur, as Mr. Maunder and the Rev. A. L. Cortie believe, at several reappearance of one and the same sunspot, the same thing is to be expected of minor disturbances. This implies that "character" figures should show a pulse near day $n+54$, as well as near day $n+27$.

This conclusion, however, seems a natural one apart from all theory. The impression left on my own mind after a study of the "character" figures was that a tendency existed for the magnetic conditions, whether disturbed or not, to be in some way related to or—as biometricians would say—correlated with the magnetic conditions prevalent 27 days earlier or later. The days forming columns $n+26$ to $n+30$ in Table I., or in the corresponding table for the years 1890 to 1900, are disturbed sensibly more than the average day, and we should thus expect more than average disturbance on days $n+53$ to $n+57$, with a culmination about days $n+54$ and $n+55,$
as the period seems in excess of 27 days. As the expected effect appeared likely to be small, it seemed best to utilise the data from the longer period of years 1890 to 1900. Calculations in that case had previously extended to day \( n + 35 \), and they were now extended to day \( n + 60 \). “Character” figures were assigned to the earlier days of 1901, so as to utilise all the 660 selected disturbed days of the 11 years. The mean “character” figure from all days of the 11 years was 0.70. The mean “character” figures up to day \( n + 35 \) are given in S.M. (Table XI., p. 101); those for days \( n + 36 \) to \( n + 60 \) are given in Table II.

**Table II.**—Mean “Character” Figures for Days \( n + 36 \) to \( n + 60 \), \( n \) being the Representative Disturbed Day of the 11 Years 1890 to 1900.

<table>
<thead>
<tr>
<th>Day</th>
<th>( n + 36 )</th>
<th>( n + 37 )</th>
<th>( n + 38 )</th>
<th>( n + 39 )</th>
<th>( n + 40 )</th>
<th>( n + 41 )</th>
<th>( n + 42 )</th>
<th>( n + 43 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Character”</td>
<td>0.63</td>
<td>0.68</td>
<td>0.68</td>
<td>0.66</td>
<td>0.66</td>
<td>0.63</td>
<td>0.64</td>
<td>0.66</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Day</th>
<th>( n + 44 )</th>
<th>( n + 45 )</th>
<th>( n + 46 )</th>
<th>( n + 47 )</th>
<th>( n + 48 )</th>
<th>( n + 49 )</th>
<th>( n + 50 )</th>
<th>( n + 51 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Character”</td>
<td>0.65</td>
<td>0.66</td>
<td>0.67</td>
<td>0.64</td>
<td>0.64</td>
<td>0.66</td>
<td>0.63</td>
<td>0.65</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Day</th>
<th>( n + 52 )</th>
<th>( n + 53 )</th>
<th>( n + 54 )</th>
<th>( n + 55 )</th>
<th>( n + 56 )</th>
<th>( n + 57 )</th>
<th>( n + 58 )</th>
<th>( n + 59 )</th>
<th>( n + 60 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Character”</td>
<td>0.72</td>
<td>0.78</td>
<td>0.84</td>
<td>0.85</td>
<td>0.81</td>
<td>0.76</td>
<td>0.71</td>
<td>0.68</td>
<td>0.64</td>
</tr>
</tbody>
</table>

As shown in S.M. (Table XI.), the “character” figure lay between 0.61 and 0.66 from day \( n + 5 \) to day \( n + 24 \), and exceeded 0.70 only from days \( n - 2 \) to \( n + 3 \), and days \( n + 25 \) to \( n + 31 \). There is thus clear evidence in Table II. of a pulse from day \( n + 52 \) to day \( n + 58 \), or possibly \( n + 59 \). The figures for days \( n + 54 \) and \( n + 55 \) distinctly overtop their neighbours, that for day \( n + 55 \) being slightly the higher.

§ 8. Reasoning in the same way as before, we should now expect an excess in the “character” figures for days \( n + 79 \) to \( n + 84 \), and so on. It will probably have been realised ere this that carrying the investigation up to day \( n + 60 \) entailed exceedingly heavy arithmetical labour, and, as the time at my disposal was limited, it was important to economise effort. It was anticipated that the successive pulses would diminish rapidly in magnitude, and that they would spread themselves over an increasing number of days, so that the distinction from neighbouring days would be more and more difficult to establish. Further, there is the possibility that normal conditions at the time, which includes days which follow the selected disturbed days after a long interval, may differ sensibly from normal conditions answering to the selected days themselves.
Eventually a practical and economical plan suggested itself. Before adopting it I had assured myself that the 27-day phenomenon applied to quiet days. It then became clear that if one selected 5 quiet days for each month, and considered the days which followed them after any given interval, as well as the days following the selected disturbed days after the same interval, it was necessary to consider only a comparatively few consecutive days near the date when the pulse was expected to appear. For instance, days from 79 to 84 days subsequent to the 5 selected disturbed days of January, 1906, are practically contemporaneous with days from 79 to 84 days subsequent to the 5 selected quiet days of the same month. If there is an appreciable pulse with crest (or hollow) about 81 days subsequent to the representative disturbed or quiet days, this will be rendered manifest by the differences between the two sets of subsequent days, irrespective of what the appropriate average character figure from all days might be.

By this time I had also discovered that the 27-day period is as clearly recognisable in days which precede as in those which follow selected disturbed days. It was thus decided to consider days before as well as days after the selected days, and to go equally far in both directions. It was also decided to take the later period, 1906 to 1911, so as to have an international basis for the selected days, whether quiet or disturbed. The quiet days were those actually selected at de Bilt.

The final mean results of the investigation are given in Table III., p. 254, and are shown graphically in fig. 2. But for considerations of time, it would have been desirable to take more than six days near the epochs where the pulses were expected.

The columns headed D and Q respectively in Table III., refer to the days associated with the selected disturbed days and to those associated with the selected quiet days. The number of selected days used was always the same for the disturbed and the quiet days, but varied, as shown in the second line, because only parts of the first and last years of the series could be utilised. For example, when dealing with the days which were from 84 to 79 days prior to selected days, April 1906 was the earliest month whose selected days one could employ. For that particular quest the 15 selected days of the first 3 months of 1906 had to be omitted, leaving only 345 selected days. Similarly, as no data subsequent to December 1911 were to be used, the last 15 selected days of 1911 had to be omitted when dealing with the days 79 to 84 days subsequent to selected days. January 1, 1906, was a selected quiet day, and December 31, 1911, a selected disturbed day. Thus the earliest and the latest of the selected days, both quiet and disturbed, were omitted from the central group of days $n-3$ to $n+3$, leaving 358 available.

The "character" figures in the third line of Table III. relate to the periods covered by the corresponding selected days. Thus 0.659 given for the group of days $n-84$ to $n-79$ is the mean for the period commencing April 1, 1906, and ending December 31, 1911. In some ways it would have been better to have replaced this by a mean applicable to the period containing the days which preceded the selected
days by an interval of from 84 to 79 days, but complications would have ensued, because a day 80 days, for instance, prior to a selected April day may fall in January or in February.

A general idea of the phenomena disclosed by Table III will be most easily grasped by consulting fig. 2. The central vertical line in the figure applies to the representative days, disturbed and quiet. Abscissa, measured from this line, represent the interval in days from the representative day, time previous being measured to the left, and time subsequent to the right. The numeral attached to any particular point on a curve signifies the interval in days from the representative day, whether previous or subsequent. The ordinate represents the algebraic excess of the "character" figure over the corresponding normal "character" figure in the third line of Table III.

The representative disturbed day had a "character" 1.321. Its excess, 0.664, over the corresponding normal value (0.657) is represented by the positive ordinate marked 0. The representative quiet day, on the other hand, had a "character" of
Table III.—"Character" Figures on Specified Days preceding or following Selected Disturbed and Quiet Days $n$, of Years 1906 to 1911.

<table>
<thead>
<tr>
<th>Days</th>
<th>$n - 84$ to $n - 79$</th>
<th>$n - 57$ to $n - 52$</th>
<th>$n - 30$ to $n - 25$</th>
<th>$n - 3$ to $n + 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of disturbed or quiet days used</td>
<td>345</td>
<td>350</td>
<td>355</td>
<td>358</td>
</tr>
<tr>
<td>Mean &quot;character&quot; from all days of period</td>
<td>0.659</td>
<td>0.660</td>
<td>0.663</td>
<td>0.657</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Days</th>
<th>$n + 25$ to $n + 30$</th>
<th>$n + 52$ to $n + 57$</th>
<th>$n + 79$ to $n + 84$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of disturbed or quiet days used</td>
<td>355</td>
<td>350</td>
<td>345</td>
</tr>
<tr>
<td>Mean &quot;character&quot; from all days of period</td>
<td>0.663</td>
<td>0.666</td>
<td>0.667</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Days</th>
<th>$n + 25$</th>
<th>$n + 26$</th>
<th>$n + 27$</th>
<th>$n + 28$</th>
<th>$n + 29$</th>
<th>$n + 30$</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;Character.&quot;</td>
<td>D.</td>
<td>Q.</td>
<td>D.</td>
<td>Q.</td>
<td>D.</td>
<td>Q.</td>
</tr>
<tr>
<td>$n + 25$</td>
<td>0.652</td>
<td>0.591</td>
<td>0.652</td>
<td>0.591</td>
<td>0.652</td>
<td>0.591</td>
</tr>
<tr>
<td>$n + 26$</td>
<td>0.748</td>
<td>0.490</td>
<td>0.748</td>
<td>0.490</td>
<td>0.748</td>
<td>0.490</td>
</tr>
<tr>
<td>$n + 27$</td>
<td>0.839</td>
<td>0.486</td>
<td>0.839</td>
<td>0.486</td>
<td>0.839</td>
<td>0.486</td>
</tr>
<tr>
<td>$n + 28$</td>
<td>0.806</td>
<td>0.535</td>
<td>0.806</td>
<td>0.535</td>
<td>0.806</td>
<td>0.535</td>
</tr>
<tr>
<td>$n + 29$</td>
<td>0.746</td>
<td>0.614</td>
<td>0.746</td>
<td>0.614</td>
<td>0.746</td>
<td>0.614</td>
</tr>
<tr>
<td>$n + 30$</td>
<td>0.704</td>
<td>0.661</td>
<td>0.704</td>
<td>0.661</td>
<td>0.704</td>
<td>0.661</td>
</tr>
</tbody>
</table>
only 0'135, and its deficiency, 0'522, is represented by the negative ordinate marked 0. The algebraic difference of these ordinates, 1'186, represents the difference in "character" between the representative disturbed and quiet days.

It will be seen that the day which is three days prior to the representative disturbed day is decidedly quieter than normal, and is less disturbed than the day which precedes by three days the representative quiet day. On the other hand, the day which is three days subsequent to the representative quiet day is decidedly more disturbed than normal, and is less quiet than the day which is three days subsequent to the representative disturbed day. The latter result especially was quite unexpected, in view of the frequent occurrences of sequences of disturbed days, and still more of quiet days. A sequence of five, or even ten, successive 0's in the returns from an individual observatory is not unusual in months of minor disturbance. The natural inference is that the proverb "the calm precedes the storm" has some claim to recognition even in terrestrial magnetism.

It may create surprise that the representative quiet day had so large a "character" figure as 0'135. Days, however, of international "character" 0'0 are very rare. There were only four, for instance, during 1906. The phenomenon is considerably due to a few observatories where 0's are assigned to only exceptionally quiet days. On the other hand, if latitudes over 55 degrees were adequately represented, 0'0's would be still rarer.

A glance at fig. 2 will show that the 27-day period is just as prominent for quiet as for disturbed characteristics, and that it can be traced backwards as readily as forwards. The corresponding patches of curve associated respectively with the disturbed and the quiet days, as it were, repel one another. This would probably serve to prove the existence of pulses considerably beyond the range covered by Table III. and fig. 2.

§ 9. One of the principal objects originally in view was to obtain a more exact estimate of the length of the period by measuring the interval in days between the crests of pulses remote from one another. But even in the 79- to 84-days' pulses—i.e., the third subsequent pulses—the difference between the ordinates answering to successive days has become very small, so that trifling accidental irregularities are prejudicial to accurate time deductions. This difficulty will naturally tend to disappear as the number of years for which international data are available increases, and the power of the method will thus continually develop.

In § 6, it will be remembered, we obtained a result which suggested that the length of the period increased with the amplitude of the selected disturbance. If, however, this were the case, one would expect the interval between successive subsequent pulses associated with the selected disturbed days to gradually diminish, and the intervals derived from pulses associated with quiet days to be shorter than those from pulses associated with disturbed days. These tendencies are not apparent in fig. 2.

§ 10. The fact that the rise in the "character" figure in the two days immediately
preceeding the representative disturbed day exceeds the fall in the two immediately following days has been already noticed. This peculiarity is a prominent feature in all the associated pulses in fig. 2, except the third previous, where the exact day of incidence of the maximum is not clearly indicated. In the case of the selected quiet days, on the other hand, the fall in the "character" figure in the two immediately preceeding days is less rapid than the rise in the two immediately succeeding days, and the same peculiarity is reproduced in the first previous and the first and second subsequent pulses. The second previous pulse shows the opposite phenomenon, but this may arise from the same disturbing cause which has brought the maximum to day \(-55\) instead of day \(-54\). In the third previous and third subsequent pulses the shape of the curve is irregular.

Speaking generally, in the case both of the disturbed and the quiet days, while corresponding pulses respectively to right and left of the central line 00 are very similar, the curves are not images of one another with respect to 00. The character of the primary (i.e., central) pulse seems to be impressed on the associated pulses which precede it, as well as on those which follow it.

The curves for days \(-30\) to \(-25\) and for days \(+25\) to \(+30\) will have a much closer fit if we cut the paper along the line 00, and bring the lines answering to days \(-27\) and \(+27\) over one another by sliding the one half sheet over the other, than if we effect this superposition by folding the paper about the line 00.

If the curves had been images of one another, by adding "character" figures for days \(n+m\) and \(n-m\)—where \(n\) denotes the representative disturbed or quiet day—we might have got as smooth results for day \(m\) as if we had been able to use 12 years' data while confirming ourselves to days following the selected days. The want of symmetry makes the conditions somewhat less favourable for evaluating the length of the period, supposing that not to be an exact number of days. The maxima at days \(-54\), \(-27\) and \(+27\) in the associated disturbed pulses are sufficiently prominent to fairly justify the view that the true maxima lie within half a day of the apparent maxima. This gives for the time of three periods \(81\pm 1\) days, or for one period \(27\pm 0.3\).

The ordinates answering to days \(+54\) and \(+55\) differ but little, while those for days \(+81\) and \(+82\) are practically equal. Thus the values deduced for the period from these summits and that at day \(-54\) are respectively \(108.5/4\), and \(135.5/5\) days, or both approximately 27.1 days.

On the curves associated with the selected quiet days, the maxima at days \(-81\), \(-55\), \(-27\), and \(+54\) are the clearest. From \(-81\) and \(+54\) we get 27'0, and from \(-55\) and \(+54\) we get 27'25 days.

The associated disturbed curve for days \(-30\) to \(-25\) and the associated quiet curve for days \(+25\) to \(+30\) both suggest slightly under 27 days for the period.

§11. If instead of treating the "character" figures from the disturbed and the quiet associated days separately, we combine them, we obtain results of much greater
symmetry. This has been done in Table IV., the entries in which represent the differences of corresponding D and Q results in Table III. To save decimals, the results are expressed in terms of 0'001 "character" unit as unit. As day 0—i.e., what is called day n in Table III.—is neither previous nor subsequent, but fundamental for both previous and subsequent days, it appears in both the first and second lines of Table IV. The entry 1186 ascribed to it represents of course (1'321—0'135) × 1000. The algebraic sign when omitted is plus. The "character" figure for the associated disturbed day was invariably the larger, except for the third days before and after the selected days.

**Table IV.—Differences Disturbed less Quiet Associated Days (Unit = 0'001 of "Character" Unit).**

<table>
<thead>
<tr>
<th></th>
<th>0.</th>
<th>1.</th>
<th>2.</th>
<th>3.</th>
<th>25.</th>
<th>26.</th>
<th>27.</th>
<th>28.</th>
<th>29.</th>
<th>30.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Previous</td>
<td>1186</td>
<td>602</td>
<td>124</td>
<td>—89</td>
<td>94</td>
<td>252</td>
<td>321</td>
<td>217</td>
<td>93</td>
<td>21</td>
</tr>
<tr>
<td>Subsequent</td>
<td>1186</td>
<td>607</td>
<td>103</td>
<td>—87</td>
<td>61</td>
<td>258</td>
<td>353</td>
<td>271</td>
<td>132</td>
<td>43</td>
</tr>
<tr>
<td>Sum</td>
<td>2372</td>
<td>1209</td>
<td>227</td>
<td>—176</td>
<td>155</td>
<td>510</td>
<td>674</td>
<td>488</td>
<td>225</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>52</td>
<td>53</td>
<td>54.</td>
<td>55.</td>
<td>55.</td>
<td>55.</td>
<td>56.</td>
<td>57.</td>
<td>58.</td>
<td>58.</td>
</tr>
<tr>
<td></td>
<td>79.</td>
<td>80.</td>
<td>81.</td>
<td>82.</td>
<td>83.</td>
<td>84.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Previous</td>
<td>92</td>
<td>144</td>
<td>205</td>
<td>206</td>
<td>117</td>
<td>63</td>
<td>56</td>
<td>87</td>
<td>112</td>
<td>91</td>
</tr>
<tr>
<td>Subsequent</td>
<td>90</td>
<td>181</td>
<td><strong>249</strong></td>
<td>197</td>
<td>135</td>
<td>48</td>
<td>48</td>
<td>93</td>
<td>115</td>
<td>116</td>
</tr>
<tr>
<td>Sum</td>
<td>182</td>
<td>325</td>
<td><strong>454</strong></td>
<td>403</td>
<td>252</td>
<td>111</td>
<td>104</td>
<td>180</td>
<td><strong>227</strong></td>
<td>207</td>
</tr>
</tbody>
</table>

The accordance between the results for the previous and the subsequent days 1, 2, and 3 in Table IV. is quite extraordinarily close. In other words, the primary pulse obtained by taking the excess of "character" figures for selected disturbed and adjacent days over the corresponding figures for selected quiet and adjacent days is almost perfectly symmetrical as between time previous and time subsequent. We cannot hope to see equal symmetry in the associated pulses, whose form is necessarily more dependent on accident, but there is at least no marked α-symmetry in the second and third associated pulses. If curves were drawn to represent these, they would not be markedly steeper on one side of the maximum than the other. This suggests adding the two sets of results, as has been done in the last line of Table IV., and applying the sums to the evaluation of the period. The most orthodox way probably would be to fit an algebraic curve to each of the successive sets of figures, and calculate the abscissa of its maximum ordinate. But as there is nothing to guide one as to what the theoretical shape of such a curve should be, rougher methods may not
unlikely be quite as satisfactory. As an example of the methods actually used, take the data for days 52 to 57 in Table IV. The maximum obviously comes between days 54 and 55, say at 54 + x. Assume the slopes from the maximum down to days 54 and 55 to be the same, and to be the arithmetic means of the slopes from days 53 and 54 (129 per diem), and from days 55 to 56 (151 per diem).

Then we have

\[ 454 + 140x = 403 + 140 (1-x) \]

or

\[ x = 0.318. \]

Thus twice the period is 54.318 days, i.e., the period is 27.16 days.

If we take the same days, but assume the slope on the two sides of the maximum to be the mean of those from days 52 to 54 and from days 55 to 57, the only difference is that we replace 140 in the above calculation by 141, and again find for the single period 27.16 days.

Treating the data for days 79 to 84 in the same way, taking first the arithmetic mean of the slopes from days 80 to 81 and 82 to 83, and then the arithmetic mean of the slopes from days 79 to 81 and 82 to 84, we get as estimates for the triple period 81.29 and 81.31 days, both giving 27.10 days for the single period.

§12. An inspection of fig. 2 suffices to show that the ratio borne by the maximum ordinate of the first associated pulse—whether for disturbed or quiet days—to the maximum ordinate of the primary pulse is notably less than the ratio borne by the maximum ordinate of the second associated pulse to that of the first. These ratios and those between the maximum ordinates of the several associated pulses are fairly alike, whether we take subsequent or previous days, and whether we take disturbed or quiet days. Thus the most accurate information on the subject is probably that derivable from the data in the last line of Table IV. The ratios between the successive maximum ordinates deduced from the data in question are as follows:

<table>
<thead>
<tr>
<th>Primary</th>
<th>First associated</th>
<th>Second associated</th>
<th>Third associated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.284</td>
<td>0.191</td>
<td>0.096</td>
</tr>
</tbody>
</table>

The maximum ordinates of the first, second, and third associated pulses stand to one another almost exactly in the ratio 3:2:1. It is easily seen in fact in fig. 2 that the summits of corresponding first, second, and third associated pulses lie nearly on straight lines, which, if produced, would cut the zero line at points answering roughly to days ±110. This linearity in the summits cannot well represent the true phenomenon exactly, because it would imply that no finite associated pulse existed except those shown in fig. 2, whereas there can be but little doubt that if data existed for a really long series of years, pulses could be recognised considerably beyond the
range of the figure. At first sight, one might have expected to find the maximum ordinates in successive pulses decreasing after an exponential law. But two things have to be remembered. First, the breadth of successive pulses increases as the height diminishes, representing a distribution of energy over a greater and greater number of days; and secondly, as has been already remarked, the true maxima do not seemingly fall on exact days, so that the true maxima are not available. We should, for instance, accepting the figures in Table IV., put the true maximum for the second associated pulse between days 54 and 55, and the numerical value corresponding would thus naturally be in excess of 454, the value found for day 54. A similar remark applies to the other associated pulses, so that the ratios given above are at best only approximations to the truth.

§ 13. Evidence that the results of §§ 8 to 12 are not confined to the period 1906 to 1911, nor due to any peculiarity in international "character" data, was derived from a study of data for 1890 to 1900. The results of this investigation are summarised in Table V. They were derived from days associated with disturbed days. Only the

<table>
<thead>
<tr>
<th>Table V.—Primary Disturbance Pulse and Associated Pulses, Years 1890 to 1900. (Unit = 0‘001 “Character” Unit.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day . . .</td>
</tr>
<tr>
<td>Day . . .</td>
</tr>
<tr>
<td>Day . . .</td>
</tr>
<tr>
<td>Day . . .</td>
</tr>
</tbody>
</table>

first previous pulse was considered, but the investigation extended to the fourth associated subsequent pulse. The entries in the table are the excesses of the mean "character" figures for the days stated over the normal figure 0‘697 derived from all days of the 11 years. To avoid decimals the unit employed is 0‘001 of the "character" unit, as in Table IV. The associated disturbed day had a "character" figure in excess of the normal, except in the one case in which a negative sign appears in the table. The representative disturbed day is described as day 0, as in Table IV. The maximum for each pulse is in heavy type.

Uncertainties arising from variations in the normal "character" figure appropriate
at times corresponding to the several groups of subsequent days, naturally become less the longer the period of years dealt with. The fact that the commencing months of both 1890 and 1901 were all very quiet is also to the advantage of the 11-year group, as compared with the 6-year group. Still, I should have preferred, but for considerations of time, to have included quiet as well as disturbed day data for the 11 years, employing the Astronomer Royal’s quiet days for the former.

The data for the previous associated pulse, and the first, second, and third subsequent associated pulses in Table V. are very fairly smooth; but those for the fourth associated subsequent pulse seem unduly affected by “accidental” phenomena, which depress the entry for day 108 and raise that for day 110. The eleven years were dealt with in four groups—

(A) Sunspot minimum years, 1890, 1899, and 1900;
(B) Sunspot maximum years, 1892, 1893, and 1894;
(C) Highly disturbed years, 1891, 1895, and 1896;
(D) Other years, 1897 and 1898.

The largest “character” figure for the five days 107 to 111 occurred on day 111 in group (A) and day 110 in group (B), but on day 107 in groups (C) and (D); while the lowest figure occurred on day 108 in group (A), and on day 111 in groups (C) and (D). Considering this variability, much weight cannot be attached to details in the results for the fourth associated subsequent pulse. The fact, however, that the figures for all five days 107 to 111 are so decidedly in excess of the normal seems clear evidence that this pulse is by no means negligible.

The primary pulse in Table V. shows the two characteristics noted in the discussion of fig. 2. The third day prior to the representative disturbed day is decidedly quieter than the average day. The rise to the maximum in the primary pulse is considerably more rapid than the subsequent fall. This $a$-symmetry is also clearly shown by the first and second associated subsequent pulses.

The ratio borne by the excess of the maximum “character” figure for the primary pulse over the normal to the corresponding excesses for the associated pulses are as follows:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Previs.</td>
<td>Subsequent.</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>0.323</td>
<td>0.298</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.310</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

These ratios are fairly similar to those derived in §12 from the combined disturbed and quiet day data of the 6-year period. In the present case, however, we have
very nearly for the ratios of the amplitudes of the three associated subsequent pulses:—

First : Second : Third :: 1 : 0.62 : (0.62)^2.

Thus the amplitudes of the successive associated pulses do, in this instance, decrease nearly in geometrical progression. At this rate we should have had the amplitude of the fourth associated subsequent pulse in Table V. about 60.

The remarks made on the sources of uncertainty affecting corresponding data in § 12 apply here equally.

§ 14. It seemed desirable to make sure that no period shorter than 27 days was indicated by days previous to the selected disturbed days. Mean "character" figures were accordingly calculated for all days up to the 35th prior to the selected disturbed days of the 11 years 1890 and 1900. The "character" figures thus deduced appear in the first line of Table VI. The second line supplies for comparison

Table VI.—"Character" Figures on Previous and Subsequent Days associated with the Selected Disturbed Days of the 11 years 1890 to 1900.

<table>
<thead>
<tr>
<th></th>
<th>1.</th>
<th>2.</th>
<th>3.</th>
<th>4.</th>
<th>5.</th>
<th>6.</th>
<th>7.</th>
<th>8.</th>
<th>9.</th>
<th>10.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preceding days . . . . . . .</td>
<td>0.064</td>
<td>0.69</td>
<td>0.63</td>
<td>0.67</td>
<td>0.72</td>
<td>0.79</td>
<td>0.80</td>
<td>0.83</td>
<td>0.90</td>
<td></td>
</tr>
<tr>
<td>Following days . . . . . . .</td>
<td>0.64</td>
<td>0.64</td>
<td>0.63</td>
<td>0.65</td>
<td>0.71</td>
<td>0.59</td>
<td>0.70</td>
<td>0.64</td>
<td>0.61</td>
<td></td>
</tr>
</tbody>
</table>

The corresponding figures for the 35 days subsequent to the selected disturbed days, as given in S.M. The "character" of the representative disturbed day was 1.51. Figures in excess of the normal value 0.70 are in heavy type.
There is a faint suggestion of a period of about $13\frac{1}{2}$ days, but if it exists its amplitude is very small.

The first subsequent pulse is not clearly shown in Table VI. before day 25, while the first previous pulse clearly persists until day 24 if not day 22. Also the previous pulse is not clearly shown until day 30, while the subsequent pulse obviously extends until day 32.

The differences arise undoubtedly in the main from the fact already noticed in connection with the 6-year period, that the first previous and subsequent pulses both follow the primary in having the rise to the maximum more rapid than the subsequent fall. The primary pulse itself in Table VI. is not clearly manifest until the second day before the selected disturbed day, while it clearly persists until the fourth day thereafter. But, in addition to this, there is at least a suggestion that the interval between the crests of the primary and the first previous pulse is shorter than that between the crests of the primary and the first subsequent pulse. This result is also suggested by the 6-year data in Table III.

Even if we accept the figures as mathematically exact, a real difference in period does not necessarily follow. The phenomenon may be a consequence of the diurnal variation which undoubtedly exists in disturbance. Analysing the list of Greenwich magnetic storms between 1848 and 1903 given by Mr. MAUNDER,* I found that accepting the times of commencement assigned, 60 per cent. of the storms began between noon and 8 p.m., leaving only 40 per cent. for the remaining 16 hours. Again at Potsdam, where individual hours have their disturbance "character" classified, $55\frac{1}{2}$ per cent. of the hours counted as disturbed from 1892 to 1901 fell between 4 p.m. and midnight. The natural inference is that the disturbances which give the "character" to the day at Kew occur in the majority of instances in the afternoon. Thus, supposing the period to be somewhat over 27 days, the occasions when the associated subsequent disturbance falls on the 28th day following would naturally be more numerous than the occasions when the associated previous disturbance fell on the 28th day previous. This marked diurnal variation of disturbance is a difficulty, whatever plan is adopted. It might seem at first sight that the international "character" data would be unaffected. This might be so if the stations were uniformly distributed in longitude, but in reality there are but few stations in the hemisphere whose central meridian is 180° from Greenwich, and European stations largely predominate.

§ 15. The same mean "character" figure may be arrived at in many ways. For example, in the case of the 11 years, when 660 selected days were dealt with, a mean "character" 1'00 might arise from a 1 on each day, or from a 2 on 330 days and a 0 on the remaining 330 days; or, more generally, from $p$ cases of 0, $p$ cases of 2, and $660-2p$ cases of 1, where $p$ may be any positive integer not exceeding 330. It thus appeared desirable to ascertain whether there was an essential difference between the

ways in which subsequent and previous associated pulses were made up. The enquiry was confined to the first of the previous and subsequent pulses associated with the disturbed days of the 11 years. That period was preferred because a greater definiteness attached to the individual “character” figures. When international data are taken, the figure assigned to any individual day may be built up in a large variety of ways.

Table VII shows the results of the enquiry; only the days containing the main part of the pulses were considered. The data for the subsequent days were derived from S.M. The representative disturbed day is counted as day 0.

Table VII.—Analysis of “Character” Figures during the First Previous and the First Subsequent Pulses associated with Selected Disturbed Days of 1890 to 1900.

<table>
<thead>
<tr>
<th>Days</th>
<th>Previous pulse</th>
<th>Normal</th>
<th>Subsequent pulse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of “2’s”</td>
<td>99 105 139 155 126 80</td>
<td>79</td>
<td>96 120 148 132 101 90</td>
</tr>
<tr>
<td>” ” “1’s”</td>
<td>284 319 312 323 341 359</td>
<td>302</td>
<td>275 305 324 343 354 343</td>
</tr>
<tr>
<td>Disturbed days</td>
<td>383 424 451 478 467 439</td>
<td>381</td>
<td>371 425 472 475 455 433</td>
</tr>
<tr>
<td>Quiet ” ”</td>
<td>277 236 209 182 193 221</td>
<td>279</td>
<td>289 233 188 185 206 227</td>
</tr>
</tbody>
</table>

Disturbed days in Table VII. include all of “character” 2 or 1, those of “character” 0 being called quiet; so that the sum of the disturbed and quiet together necessarily amounts to 660. The distribution one would have had in 660 average days appears under “normal.” As regards the number of 2’s, days +27 and -27 decidedly overtop their neighbours. The incidence of 2’s in the pulses is more alike if we invert the order of days in the previous pulse, i.e., regard days -25 and +25, &c., as corresponding. But in both pulses the marked tendency is for days of moderate disturbance to follow the crest. No significance probably attaches to the fact that 2’s are slightly more numerous in the previous than in the subsequent pulse; because, while the highest “character” figure in the first previous pulse exceeds that in the first subsequent pulse in the case of the 11-year period, it does not do so in the 6-year period.

§ 16. Table VIII., p. 265, represents the results of an enquiry into the possible variation of the 27-day period throughout the year. The 11-year and 6-year periods were treated separately. The 660 selected days of the former period gave 55 January days and so on. These 55 January days and the subsequent days associated with them are treated as a separate group in Table VIII. The first two columns give the mean character figures for the selected disturbed days of the 12 months, for the two periods. Columns 3 to 8 give the mean character figures for days 25 to 30 subsequent
to the selected disturbed days of the 11 years; columns 9 to 14 do the same for the 6 years. The largest "character" figure found in days \( n+25 \) to \( n+30 \) is in heavy type, and the ratio borne by this maximum to the character figure on day \( n \) (i.e., the ratio of the maximum for the first subsequent pulse to that of the primary pulse) is given for the two periods separately in columns 15 and 16. Column 17 gives the arithmetic mean of the ratios in the two previous columns.

Investigations by Mr. W. Ellis and Mr. E. W. Maunder, covering a very long series of years, showed that whether one considers magnetic storms—averaging about 13 a year—or days of large and moderate disturbance—averaging about 77 a year—the frequency of occurrence of disturbance at Greenwich is above the average in the 4 equinoctial months, and below it in the 4 summer months, May to August; the numbers in the equinoctial months standing to those in the summer months roughly in the ratio of 8 to 5.

A preponderance of disturbances in the equinoctial months has been noticed at many other stations, but there is reason to doubt whether it is universal. Dr. W. van Bemmel's lists of disturbances at Batavia, averaging about 60 a year, showed but a very slight excess in the equinoctial months, and the records of Captain Scott's expedition in the Antarctic during 1902 to 1904 indicated a marked maximum of disturbance at midsummer. Still the equinoctial months are undoubtedly the most disturbed at Kew, or at the average station on which the international figures depend.

In both periods of years the order in which the months come as regards disturbance is not quite the same when one takes the mean character figure of the selected disturbed days, given in Table VIII., as when one takes the mean character figure of all days of the month, or when one takes the number of days of character "2."

In the 6-year period the months of March, September, February, and October appear to have been distinctly the most disturbed. In the 11-year period, March and February were clearly the most disturbed, and judging by the number of days of "character" "2," October came next. Thus both periods manifested the usual tendency to an increase of disturbance towards the equinoxes, but that season was less prominent than in Ellis and Maunder's lists. Also the want of smoothness in the sequence of the figures in the first two columns of Table VIII. suggests that a considerably longer series of years would be required for the elimination of "accidental" features.

All months in Table VIII. show the first subsequent pulse clearly, the crest generally falling on the 27th day itself. The maximum in the subsequent pulse is considerably larger in some months than others, but the months in which it is largest, or smallest, are not the same for the two periods. In both, the maximum figure is above its average in January, February, March, August, and September; but these months represent Winter, Summer, and Equinox.

Judging by the differences between the two periods, and between successive
TABLE VIII.—"Character" Figures on Representative Disturbed Day and during First subsequent Pulse for the 12 Months of the Year.

<table>
<thead>
<tr>
<th></th>
<th>Representative disturbed day, (n)</th>
<th>Subsequent days.</th>
<th>Ratio of maximum &quot;character&quot; figure during the first subsequent pulse to that in day (n).</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>1.56 1.29</td>
<td>0.85 0.96 1.05 0.98 0.96 0.89</td>
<td>0.89 1.04 1.02 1.04 1.00 0.91</td>
</tr>
<tr>
<td>February</td>
<td>1.76 1.43</td>
<td>0.93 1.09 1.11 1.09 0.98 0.91</td>
<td>0.75 0.75 0.90 0.85 0.80 0.81</td>
</tr>
<tr>
<td>March</td>
<td>1.73 1.48</td>
<td>0.82 0.87 1.00 0.98 0.93 0.84</td>
<td>0.70 0.81 0.87 0.83 0.80 0.75</td>
</tr>
<tr>
<td>April</td>
<td>1.47 1.22</td>
<td>0.67 0.75 0.83 0.87 0.76 0.76</td>
<td>0.57 0.62 0.71 0.79 0.77 0.71</td>
</tr>
<tr>
<td>May</td>
<td>1.55 1.31</td>
<td>0.51 0.60 0.71 0.76 0.76 0.67</td>
<td>0.60 0.72 0.70 0.65 0.55 0.53</td>
</tr>
<tr>
<td>June</td>
<td>1.40 1.15</td>
<td>0.65 0.76 0.91 0.95 0.82 0.73</td>
<td>0.49 0.52 0.65 0.61 0.57 0.62</td>
</tr>
<tr>
<td>July</td>
<td>1.47 1.21</td>
<td>0.55 0.75 0.82 0.75 0.69 0.73</td>
<td>0.71 0.86 0.97 0.86 0.78 0.68</td>
</tr>
<tr>
<td>August</td>
<td>1.29 1.33</td>
<td>0.67 0.84 0.89 0.98 0.67 0.80</td>
<td>0.69 0.85 1.03 0.99 0.83 0.68</td>
</tr>
<tr>
<td>September</td>
<td>1.49 1.48</td>
<td>0.78 0.98 1.15 0.98 0.98 0.80</td>
<td>0.67 0.70 0.91 0.88 0.79 0.75</td>
</tr>
<tr>
<td>October</td>
<td>1.53 1.40</td>
<td>0.71 0.76 0.98 0.91 0.82 0.84</td>
<td>0.61 0.78 0.79 0.76 0.70 0.62</td>
</tr>
<tr>
<td>November</td>
<td>1.40 1.26</td>
<td>0.69 0.76 0.85 0.89 0.78 0.73</td>
<td>0.61 0.67 0.70 0.69 0.57 0.59</td>
</tr>
<tr>
<td>December</td>
<td>1.45 1.26</td>
<td>0.65 0.78 0.87 0.89 0.95 0.82</td>
<td>0.49 0.64 0.82 0.70 0.80 0.82</td>
</tr>
<tr>
<td>Means</td>
<td>1.51 1.32</td>
<td>- - - - - -</td>
<td>- - - - - -</td>
</tr>
</tbody>
</table>
months, a high value in the maximum for the subsequent pulse is in considerable measure accidental, but even if accepted as a physical fact, it might have more than one interpretation.

When we took from amongst the selected disturbed days those whose "character" figure exceeded 1.5, the amplitude of the associated pulse was decidedly larger than that associated with the full choice of 5 days a month. Consequently, the amplitude of the subsequent pulse increases with that of the primary pulse. Thus a large maximum in columns 3 to 8, or 9 to 14, of Table VIII. is naturally regarded as due at least in part to a large corresponding value in columns 1 or 2. But it might also arise from a greater potency of the 27-day period at one season of the year than another, or simply from a large average amount of disturbance during the month in which the subsequent pulse falls. If the principal cause of a large amplitude in the subsequent pulse is large amplitude in the primary, then, apart from accident, one would expect only minor variations in the ratios of these two quantities given in the three last columns of Table VIII. If, on the other hand, the 27-day period is markedly more potent at one season than another, one would expect the values of the ratio to show a marked annual variation, and this to be at least approximately the same in columns 15 and 16.

In column 15 the highest value exceeds the lowest by 0.28, or 44 per cent. of the mean value 0.64. In column 16 the corresponding excess is 40 per cent. of the mean value. Thus the fluctuations are considerable. But the variations, especially in column 16, do not suggest any regular law, and they do not follow a parallel course in the two columns.

It will be found that there is a distinct tendency for the figure in column 16 to be high or low, according as the corresponding figure in column 2 is less or greater than the figure for the immediately subsequent month. In January and July, for instance, the ratio given in column 16 is very high, while the January and July figures in column 2 are considerably less than those for February and August. The same phenomenon may be traced in columns 15 and 1.

To see the extent to which this phenomenon prevails, the values were calculated of the ratio borne by the maximum figure in any month in columns 3 to 8 to the figure assigned to the next subsequent month in column 1, and the same calculation was repeated for the 6-year period. The twelve monthly ratios thus obtained for the 11-year period had the same mean value 0.64 as the ratios in column 15, but they ranged only from 0.75 in September to 0.55 in May. Their average departure, irrespective of sign, from their arithmetic mean was only 0.040, as compared with 0.053 for the ratios in column 15. In the case of the 6-year period, the corresponding figures were respectively 0.054 and 0.076.

The days which are from 25 to 30 days subsequent to a given selected disturbed day fall, in the majority of instances, in the subsequent month. Thus the natural inference from the previous figures is that the amplitude of the first subsequent pulse
depends more on the character of the month in which that pulse falls than on the amplitude of the primary disturbance with which it is associated.

On the whole, Table VIII suggests no special development of the 27-day period at any particular season. If, for example, we take the three months clustering round each equinox (i.e., February to April, and August to October), the mean of the ratios in column 17 is 0'653 as compared with 0'635 from the other six months. A very similar conclusion follows if we take the ratios in which the second member is the character of the representative disturbed day in the month subsequent to the primary pulse.

When a sufficiently long series of years is available, it will be possible to replace the ratios in columns 15 to 17 by others sufficiently smooth to show the real nature of the annual variation, if such exists. The investigation might then be extended to the second and third subsequent pulses, and to the previous pulses. When this is done, in the case both of selected disturbed and selected quiet days, results of interest may be expected.

§ 17. The primary object of S.M. was to investigate the nature of the connection, if any, between sunspots and the daily range of H (horizontal force). Use was made of the Greenwich projected sunspot areas. The 5 days of largest spot area in each month of 1890 to 1900 formed the selected days, and the mean H ranges at Kew were found for days previous and subsequent to the selected days. Denoting by n the representative selected day of large sunspot area, the H range showed a marked pulse with its crest at day n + 4. Moreover, when curves were drawn having time for abscissae, the ordinate being in the one case sunspot area and in the other H range, the rise of the latter curve to a maximum and its subsequent decline closely resembled the course of the former curve, but with a lag of about 4 days.

If we take the H trace for an individual highly disturbed day, it may be difficult even for an expert to recognise the influence of the regular diurnal variation. But if a number of such days are combined, a regular diurnal inequality emerges, which in the case of H differs little from that characteristic of quiet days, except in being of larger amplitude. Even on days of character "2," the H range owes an appreciable amount to the regular diurnal inequality, and on the average day—especially in a quiet year—the regular diurnal inequality is the principal contributor. Thus there were strong \textit{a priori} reasons for regarding the relation described above as involving the regular diurnal inequality rather than magnetic disturbance. This view was supported by an examination of the Kew "character" figures for days previous and subsequent to the selected days of the 11 years. The mean "character" figure of each column was derived from 5x12x11, or 660 days. Of the 660 days occurring in column n + 4, where the crest of the pulse in the H ranges appeared, 86 were of "character" "2." Out of 660 average days of the 11 years, 82 had a "character" "2"; thus the excess of days of "character" "2" in column n + 4 was only 4, and of the 31 columns from n − 15 to n + 15, 10 showed an excess larger than this, the excess

2 m 2
being in one case 14. The pulse in the H range curve owed its crest at day \( n+4 \) almost entirely to the frequency of days of “character”’ “1.” The columns containing most “2’s” were \( n-12 \) with 94, \( n-11 \) with 96, and \( n-10 \) with 94. The concentration of 2’s in these columns proved to be the chief, if not the sole, cause of a subsidiary pulse in the H range curve, to which there was no corresponding feature in the sunspot curve.

H range data were not available for 1906 to 1910, so no further comparison of them with sunspots was possible. But a comparison was made between sunspots and international “character” figures, taking the same fundamental days as in the previous part of this paper. In the present case, then, the basis of selection was the “character” figure, whereas in S.M. it was the sunspot area. The research was limited to the 5 years 1906 to 1910, as Greenwich spot areas for 1911 were not published at the time. There were thus \( 5 \times 12 \times 5 \), i.e., 300, representative days \( n \). Spot areas were entered in 32 columns, \( n-20 \) to \( n+11 \), and the columns were summed. The

Table IX.—Projected Sunspot Areas on Days of Largest International “Character” and on Previous and Subsequent Days, as Percentages of the Mean Area for the Five Years 1906 to 1910.

<table>
<thead>
<tr>
<th>Day</th>
<th>( n-20 )</th>
<th>( n-19 )</th>
<th>( n-18 )</th>
<th>( n-17 )</th>
<th>( n-16 )</th>
<th>( n-15 )</th>
<th>( n-14 )</th>
<th>( n-13 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage</td>
<td>91·7</td>
<td>90·1</td>
<td>90·6</td>
<td>89·8</td>
<td>89·0</td>
<td>88·7</td>
<td>89·3</td>
<td>90·9</td>
</tr>
<tr>
<td>Day</td>
<td>( n-12 )</td>
<td>( n-11 )</td>
<td>( n-10 )</td>
<td>( n-9 )</td>
<td>( n-8 )</td>
<td>( n-7 )</td>
<td>( n-6 )</td>
<td>( n-5 )</td>
</tr>
<tr>
<td>Percentage</td>
<td>91·3</td>
<td>90·2</td>
<td>91·9</td>
<td>94·3</td>
<td>97·7</td>
<td>99·7</td>
<td>102·6</td>
<td>104·8</td>
</tr>
<tr>
<td>Day</td>
<td>( n-4 )</td>
<td>( n-3 )</td>
<td>( n-2 )</td>
<td>( n-1 )</td>
<td>( n )</td>
<td>( n+1 )</td>
<td>( n+2 )</td>
<td>( n+3 )</td>
</tr>
<tr>
<td>Percentage</td>
<td>108·4</td>
<td>108·5</td>
<td>110·3</td>
<td>112·8</td>
<td>114·1</td>
<td>112·2</td>
<td>110·5</td>
<td>109·1</td>
</tr>
<tr>
<td>Day</td>
<td>( n+4 )</td>
<td>( n+5 )</td>
<td>( n+6 )</td>
<td>( n+7 )</td>
<td>( n+8 )</td>
<td>( n+9 )</td>
<td>( n+10 )</td>
<td>( n+11 )</td>
</tr>
<tr>
<td>Percentage</td>
<td>105·0</td>
<td>103·2</td>
<td>102·1</td>
<td>101·0</td>
<td>100·0</td>
<td>99·5</td>
<td>97·8</td>
<td>94·5</td>
</tr>
</tbody>
</table>

mean projected areas for the years 1906 to 1910 were in order 1047, 1453, 952, 941 and 357, the unit being the one-millionth of the sun’s apparent disc. Thus the total area for 300 average days, 60 from each year, comes to 283,000. The figures appearing in Table IX. represent percentages of this number. The three last selected days were December 22, 28 and 29, 1910; thus two days in each of the columns
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from $n+4$ to $n+9$, and three days in columns $n+10$ and $n+11$, fell in 1911. As no sunspot data for 1911 were available, while sunspot areas in December, 1910, were very small, it was decided to treat the few days specified as spotless. The percentage figures in columns $n+4$ to $n+11$ may thus be slightly too small, but the error* is unlikely to exceed $0.2$.

The highest and lowest percentages in the table are in heavy type.

Table IX. appears at first sight to demonstrate a very definite relationship between contemporaneous sunspot area and magnetic disturbance. It shows a regular pulse in sunspot area whose crest absolutely synchronises with that in magnetic "character." The form however of the two pulses is widely different. This is readily seen on consulting fig. 3, which represents graphically the sunspot figures in Table IX., and

* July 28, 1913.—The correction required is $+0.1$ from day $n+7$ to day $n+11$. 

![Figure 3](image-url)
throughout. Thus there is nothing in the observed sunspot variation to account for the rapidity of the variation in magnetic "character."

Taking the individual years, the largest sunspot area occurred in 1906 in column \(n-7\), in 1907 in column \(n+3\), in 1908 in column \(n+10\), in 1909 in column \(n-2\), and only in 1910—a year of small sunspot area—did it occur in column \(n\). Thus the occurrence in column \(n\) of the highest percentage met with in Table IX. is a fact of somewhat doubtful significance. A considerably longer series of years would be required to give a result whose representative character could be relied on.

§ 18. In S.M., in the comparison made between sunspot area and magnetic "character," the representative days \(n\) were the days of largest spot area. On the average of the 11 years 1890 to 1900, magnetic "character" was below its mean from days \(n-7\) to \(n\) inclusive, and above its mean from days \(n+1\) to \(n+11\). The highest "character" figures appeared in columns \(n+4\) to \(n+6\), that in column \(n+4\) being slightly the highest. In this case the sunspot area (primary) pulse was much more concentrated than the "character" (secondary) pulse, and there was a marked "character" crest in columns \(n-12\) to \(n-10\), but little inferior to that in columns \(n+4\) to \(n+6\) to which nothing in sunspot areas corresponded. Thus the apparent connection between magnetic "character" and sunspot area was much more ambiguous than that between \(H\) daily ranges and sunspots. Still the 11-year mean "character" figure in column \(n+s\) was very decidedly in excess of that in column \(n-s\), for all values of \(s\) from 1 to 7, and the natural inference was that in the average year there is a distinct tendency for maxima in magnetic disturbance to follow maxima in sunspot area. Thus one would have expected to find in Table IX., not an array of figures symmetrical about column \(n\) but a decided excess of the figure in column \(n-s\) over that in column \(n+s\) for small values of \(s\), the largest value occurring prior to day \(n\).

It was obviously desirable to ascertain whether the departure from the result anticipated represented a real difference between the two periods dealt with, or arose from the difference in the procedure followed. Accordingly a second investigation was made, adopting the same procedure for 1906 to 1910 as had been followed in the case of 1890 to 1900, the selected days \(n\) being now the 5 days of largest projected spot area in the month.

The calculations were made for the "character" figures assigned at Kew alone, as well as for the international choice at de Bilt, in view of the possibility that the results for 1890 to 1900 in S.M. might have been influenced by some peculiarity in the choice of Kew "character" figures. This contingency could be provided for only in part, because the date at which "character" figures were assigned to the years 1890 to 1900 was subsequent to 1910, and undoubtedly 2's were more freely given than in dealing with the years 1906 to 1910. During the latter 5 years 2's were given only 48 times at Kew, as compared with 37 times at Greenwich; whereas in 1911 the number of 2's was 38 at Kew, as against 6 at Greenwich. The Kew
standard was intentionally changed in 1911; whereas the Greenwich standard has, I believe, remained nearly uniform, a "2," these being roughly equivalent to the "magnetic storm" of Ellis and Maunder. The number of magnetic storms in Mr. Maunder's list averaged about 18 a year, while the number of 2's awarded to the years 1890 to 1900 at Kew averaged about 44 per annum.

The investigation referred to above was confined to days \( n - 2 \) to \( n + 4 \), except that day \( n - 11 \) was added for the Kew data. The results appear in Table X. The absolute values are given of the mean "character" figure for the stated days of the individual years. Values above the normal—or mean value from all days—are in heavy type. The percentage figures for 1906 to 1910 express the arithmetic means of the "character" figures in column \( n - 2 \), &c., as percentages of the corresponding mean of the normal day values. The two last lines give comparative percentage results for the 11 years 1890 to 1900, and the last five years of that period respectively.

Table X, confirms the physical reality of the difference between the two periods 1890 to 1900 and 1906 to 1910, but the percentage figures obtained for the later period in columns \( n - 2 \) to \( n + 4 \) bear a remarkable resemblance to those applying to the five years 1896 to 1900.

In the 11-year period, 1890 to 1900, it was the contribution of the sunspot maximum years, 1892 to 1894, which mainly determined the excess of the "character" figures in columns \( n + s \) over those in columns \( n - s \). Since 1900 sunspot development has been somewhat poor and irregular, and the results derived from the shorter period, 1906 to 1910, would naturally be less representative than those derived from 1890 to 1900. Still, it would be desirable to have results from several 11-year periods before dogmatising on this point.

In the case of the Kew "character" figures for 1906 to 1910 there were thirteen occurrences of "2" in days \( n + 3 \) and \( n + 4 \), as compared with eleven occurrences on days \( n - 2 \), \( n - 1 \), and \( n + 2 \), and nine occurrences on days \( n \). But the number of disturbed days (i.e., days of "2" and "1" combined) was most numerous on day \( n \), being greater by one for that day than for day \( n + 3 \).

Day \( n - 11 \), in 1906 to 1910, had only five occurrences of "2," or nearly three below normal, and occurrences of "0" were five above normal; whereas in 1896 to 1900, as in 1890 to 1900, day \( n - 11 \) had fewer occurrences of "0" than normal. Taking the whole 11 years, 1890 to 1900, day \( n - 11 \), it will be remembered, had more 2's than any other. This was the reason for including it in Table X.

§ 19. The Greenwich volumes of heliographic results give "corrected" as well as "projected" areas of sunspots. The corrected areas allow for foreshortening, and take as unit the one-millionth of the visible hemisphere. Projected and corrected areas are also given for faculae. It was decided to replace the projected spot areas of the investigation in § 17 by corrected spot areas, projected faculae, and Wolfer's sunspot frequencies in turn. The fundamental days \( n \), as in § 2, were the 300 selected
Table X.—Magnetic "Character" on Selected Days of Largest Sunspot Area and Associated Days.

<table>
<thead>
<tr>
<th>Year</th>
<th>Normal day.</th>
<th>International &quot;character&quot; figures.</th>
<th>Kew &quot;character&quot; figures.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>n - 2. n - 1. n. n + 1. n + 2. n + 3 n + 4.</td>
<td>n - 11. n - 2. n - 1. n. n + 1. n + 2. n + 3 n + 4.</td>
</tr>
<tr>
<td>1906</td>
<td>0·65</td>
<td>0·64 0·62 0·60 0·59 0·61 0·69 0·74</td>
<td>0·56 0·52 0·58 0·57 0·55 0·50 0·57 0·63 0·68</td>
</tr>
<tr>
<td>1907</td>
<td>0·66</td>
<td>0·67 0·72 0·70 0·65 0·66 0·68 0·65</td>
<td>0·71 0·67 0·65 0·73 0·73 0·67 0·68 0·73 0·65</td>
</tr>
<tr>
<td>1908</td>
<td>0·68</td>
<td>0·76 0·74 0·75 0·75 0·76 0·79 0·76</td>
<td>0·65 0·65 0·77 0·68 0·70 0·70 0·72 0·75 0·73</td>
</tr>
<tr>
<td>1909</td>
<td>0·62</td>
<td>0·70 0·69 0·71 0·70 0·67 0·68 0·74</td>
<td>0·58 0·60 0·67 0·65 0·65 0·65 0·60 0·62 0·70</td>
</tr>
<tr>
<td>1910</td>
<td>0·72</td>
<td>0·73 0·75 0·83 0·86 0·84 0·77 0·69</td>
<td>0·64 0·58 0·72 0·72 0·77 0·77 0·73 0·72 0·63</td>
</tr>
<tr>
<td>&quot; 1890 ,, 1900</td>
<td>— — — — — — — —</td>
<td>— 105 91 95 98 101 102 106 109</td>
<td></td>
</tr>
<tr>
<td>&quot; 1896 ,, 1900</td>
<td>— — — — — — — —</td>
<td>— 106 107 110 105 107 110 110</td>
<td></td>
</tr>
</tbody>
</table>
highly disturbed days of the five years 1906 to 1910. The investigation was restricted to the seven days $n-3$ to $n+3$. The results for the several years appear in Table XI, the figures being expressed as percentages of the normal value of the quantity concerned for the year in question.

Table XI.—Relation of Sunspot Areas and Frequencies and of Faculae to Magnetic Disturbance ($n$ being Representative Day of Large Disturbance).

<table>
<thead>
<tr>
<th>Year</th>
<th>Projected.</th>
<th>Corrected.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n-3$, $n-2$, $n-1$, $n$, $n+1$, $n+2$, $n+3$.</td>
<td>$n-3$, $n-2$, $n-1$, $n$, $n+1$, $n+2$, $n+3$.</td>
</tr>
<tr>
<td>1906</td>
<td>91 83 84 85 87 88 88</td>
<td>100 94 91 88 86 88 87</td>
</tr>
<tr>
<td>1907</td>
<td>108 114 120 124 128 133 134</td>
<td>107 112 113 118 119 124 128</td>
</tr>
<tr>
<td>1908</td>
<td>113 112 114 114 106 97 93</td>
<td>109 114 110 114 106 99 98</td>
</tr>
<tr>
<td>1909</td>
<td>121 124 123 121 112 104 100</td>
<td>118 118 111 114 111 109 103</td>
</tr>
<tr>
<td>1910</td>
<td>120 134 139 141 137 137 137</td>
<td>117 127 128 134 129 132 130</td>
</tr>
<tr>
<td></td>
<td>First mean</td>
<td>111 113 116 117 114 112 110</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Year</th>
<th>Wolfer's frequencies.</th>
<th>Greenwich faculae projected areas.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n-3$, $n-2$, $n-1$, $n$, $n+1$, $n+2$, $n+3$.</td>
<td>$n-3$, $n-2$, $n-1$, $n$, $n+1$, $n+2$, $n+3$.</td>
</tr>
<tr>
<td>1906</td>
<td>90 92 92 91 89 92 93</td>
<td>98 97 101 96 95 90 94</td>
</tr>
<tr>
<td>1907</td>
<td>109 113 108 114 113 112 113</td>
<td>101 101 93 92 97 98 103</td>
</tr>
<tr>
<td>1908</td>
<td>108 111 106 101 96 95 93</td>
<td>105 112 111 103 95 93 96</td>
</tr>
<tr>
<td>1909</td>
<td>110 111 107 105 104 104 101</td>
<td>112 99 101 103 108 105 102</td>
</tr>
<tr>
<td>1910</td>
<td>114 125 129 119 119 115 119</td>
<td>116 114 104 110 101 102 101</td>
</tr>
<tr>
<td></td>
<td>First mean</td>
<td>106 110 108 106 104 104 104</td>
</tr>
<tr>
<td></td>
<td>Second mean</td>
<td>105 108 105 104 103 102 102</td>
</tr>
</tbody>
</table>

Of the two sets of mean values given in the last two lines of Table XI., the first are arithmetic means of the percentages for the individual years; the second were obtained by summing the area or frequency figures for the 300 days in each column, and expressing the mean as a percentage of the corresponding mean derived from all days of five years.

Table XI. shows that, at least for the years considered, it does not much matter whether projected or corrected spot areas are taken for comparison with magnetic disturbances. If anything, the projected percentages are a trifle the larger. We
should have expected a marked difference if the proximity of the spot—supposed to create disturbance—to the central meridian had been an important element. Wolfer's frequencies give results of the same general character as spot areas, but the percentages are decidedly smaller. Also the percentages in the last two lines derived from the Wolfer frequencies are less symmetrical with respect to column $n$, being distinctly larger for the previous than for the succeeding days. This $a$-symmetry is still more developed in the percentages based on faculae.

On the average of the five years, the maximum magnetic disturbance was preceded by two days by the Wolfer frequency maximum, and by at least four days by the maximum faculae area.

On the average highly disturbed day, the faculae area was almost exactly normal.

Whether we take spot areas or frequencies, 1906 shows a markedly diminished solar activity for days $n-2$ to $n+3$; and 1910—a year of small solar activity and very quiet magnetically—is the year which most strongly suggests a parallel variation between magnetic disturbances and solar activity.

§20. It appeared desirable to ascertain the extent to which a 27–28 day period of the type here considered manifests itself in sunspots themselves. The selected days of the investigation were the five days of largest projected spot area in each month of the five years 1906 to 1910. Projected spot areas were entered in the columns for days $n-30$, $n-28$, $n-27$, $n-25$, $n$, $n+25$, $n+27$, $n+28$, and $n+30$. That seemed likely to be a sufficient choice of days to show the nature and amplitude of the anticipated phenomenon. The results obtained are given in Table XII. In the first five lines the projected spot areas are expressed in terms of the Greenwich unit. The five subsequent days associated with the five selected days of December 1910 had to be omitted, so the entries for columns $n+25$ to $n+30$ in that year were based on 55 days only.

The results for the 300 (or 295) days included in each column were summed, and each sum was expressed as a percentage of that for the normal day.

The last line in Table XII. gives for comparison corresponding results calculated for the first previous and first subsequent pulses in "character" figures, the selected days $n$ in this case being those of maximum "character" for the sixty months of the five years.

If we take a mean from the previous and subsequent pulses in Table XII., the largest excess above the normal in the first subsidiary pulse bears to that in the primary pulse the ratio 27:122, or 0'221:1, for the spot areas, and 21'5:98, or 0'219:1, for magnetic "character." This is a very striking resemblance. It did not, however, extend to individual years. Thus the previous and subsequent sunspot area curves were better developed in 1907 than in the other years, but the development of the previous "character" pulse was best in 1908, and that of the subsequent "character" pulse was better in 1908, 1909, and 1910 than in 1907.

A noteworthy difference is that the crests of the subsidiary sunspot area pulses in
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Table XII appear on days \( n-28 \) and \( n+28 \), and not on days \( n-27 \) and \( n+27 \) as in the case of magnetic "character." It is also curious that the spot area on day \( n-28 \) should so largely exceed that on day \( n+28 \). As this phenomenon, however, is not shown in 1906 it may be "accidental." The sunspot area pulses, both primary and secondary, appear considerably rounder than those in magnetic "character," and this is probably responsible for the greater variability in the position of the crest in the subsidiary pulses of sunspot area than in those of magnetic "character." Thus in 1909 and 1910 the largest spot area in the subsidiary pulses appear on day \( n-30 \); while the spot areas on day \( n-25 \) in 1909, and on days \( n+27, n+28 \) and \( n+30 \) in 1910 are actually below the normal.

§ 21. The results obtained in S.M. and in the present paper put it beyond a doubt that there is in terrestrial magnetism a period of about 27 days, in the sense that if day \( n \) is either decidedly more or decidedly less disturbed than the normal day, then days \( n \pm 27 \) show a distinct tendency to differ from the normal day in the same direction as day \( n \). The characteristic is just as clearly shown by quiet days as by disturbed days. The phenomenon appears in disturbed years and in quiet years, in years of many and in years of few sunspots. It was particularly prominent in 1911 when sunspots were few, and it was also well developed in 1910, a year in which only one day was awarded character " 2 " at Greenwich.

Prof. Schuster, as is well known, has adduced arguments which appear fatal to the view that a magnetic storm on the earth can be due to any limited jet of electrified particles emanating from the sun. It may thus seem a waste of time to consider other difficulties, in the way of jet theories, suggested by the present enquiry. There are, however, physicists, with whom I to some extent sympathise, who have a feeling that demonstrations of the impossibility of some physical hypothesis may prove in the long run less conclusive than was at first supposed. Fresh physical discoveries may remove what seemed at one time insuperable barriers. Thus it may not be
wasted effort to direct attention to the difficulty which seems to be raised by the conspicuous nature of the 27-day period in quiet days. The rapidity of the decline in disturbance and the rapidity of its resuscitation after the representative quiet day are prominent facts. It will hardly, I think, be suggested that there are limited solar areas—similar to sunspots in dimensions—whose direct presentation to the earth exerts a soothing or damping influence on magnetic disturbance on the earth, removing or diminishing disturbances which otherwise would have made their presence felt.

§ 22. A serious difficulty in the way of an exact determination of the period is that magnetic storms, and magnetically quiet times, are events usually covering a large number of hours. A magnetic storm is seldom confined to a single day. Successive magnetic storms do not as a rule present closely similar features, nor are they usually of closely similar length. There is thus as a rule no such thing as a definite interval between them. In the majority of cases opinions would differ—often by hours—as to when a magnetic storm begins, and still more so as to when it ends. The uncertainty is least about the time of commencement, and that is presumably the reason why Mr. MAUNDER calculated his intervals from the times of commencement. If, however, one could assign exact intervals for the beginning and ending, the natural interval would seem to be, not the time from beginning to beginning, but the time from centre to centre. If we accept a jet theory, then if the second of two magnetic storms is shorter than the first, the jet and so presumably the corresponding solar area has contracted. In the absence of definite knowledge to the contrary, the most natural hypothesis would seem to be that the jet has contracted uniformly about its centre.

If successive magnetic storms were of roughly equal duration, and if in a number of instances they both had what are termed “sudden commencements,” much less uncertainty would attach to the interval. As I pointed out, however, in a review of Mr. MAUNDER’s first paper, Nature but seldom presents this simple case. Of the 276 magnetic storms which Mr. MAUNDER’s list gave for Greenwich between 1882 and 1903, only 77 had “sudden commencements.” Of the 91 storms which he regarded as showing a 27–28 day period during these 22 years, only 15 had “sudden commencements,” and there were only four cases in which two successive storms of his sequence groups had both “sudden commencements.”

The definition of a magnetic storm is purely arbitrary. A striking example of this is afforded by the Kew and Greenwich lists of “character” figures supplied in 1906, 1907 and 1908 to de Bilt. In both lists the days of character “2”—i.e., magnetic storms according to Greenwich standard—numbered 29, but only 19 of these days were common to the two lists. Both lists gave eleven 2’s in 1907; but the Greenwich list gave nine 2’s in each of the years 1906 and 1908, while the Kew list gave five in the former year and thirteen in the latter. Thus if attention is confined to “magnetic storms,” where one man gets a sequence of approximately the right period, another gets no sequence at all. If, on the other hand, one takes disturbances moderate as well as large, the number is so great that it does not require
any great skill to find between pairs of them intervals of 27 days, or of any other number of days which the individual desires.

The difficulties in the way of treating disturbances individually exist in at least equal measure in the case of quiet days. On some occasions a quiet time ends with great precipitancy, but to say exactly when it commences would usually prove an impossible task.

I have referred to this aspect of the problem because it is rather a fashion amongst experimentalists to regard statistical enquiries such as the present with suspicion. They are unable wholly to purge themselves of the popular superstition that statistics can prove anything which the statistician desires. In the present case, however, the popular view is the exact opposite of the truth. The statistics employed are in large part international data, published before the enquiry commenced, and based on estimates of magnetic “character” made independently, at observatories scattered over the world, by individuals none of whom had any suspicion of the purpose to which they would be put. The observational data, on the other hand, are usually of so complex a nature, and so influenced by the latitude and longitude of the station, that the observer does not know what to regard as essential and what to consider secondary. Moreover, the record is in nearly all cases photographic. Except in a few of the better staffed observatories, the fact that a magnetic storm has occurred is not known until a day or two afterwards, when the photographic sheets have been developed. If a continuous succession of solar pictures and contemporary magnetic changes could appear side by side during the actual progress of a magnetic storm, an observer would have a better chance of framing the right guess as to the nature of the solar link, provided corresponding events on the sun and earth are nearly simultaneous, or are separated by a constant small interval of time. In the case, however, of sunspots and the amplitude of the daily H range at Kew, during the eleven years 1890 to 1900, the results reached in S.M. indicated a clear lag of about four days in the magnetic range, and they were at least consistent with a similar lag in magnetic “character.” The results of the present paper do not suggest a lag in magnetic “character,” but the rate of change of sunspot area near the time of maximum “character,” as shown in Table 1X. and fig. 3, is slow, so that the question of lag in “character” is still an open one. If there is a lag, and especially if the lag is of variable amount—as might well be the case if cathode rays or electrified particles are concerned—the difficulties in the way of direct observation will be materially increased.

We have seen that magnetic “character” and sunspots have both periods of from 27 to 28 days. In some years the phenomena are, so to speak, in phase, in other years not in phase. The period seems better developed in some years than in others, and the years in which it is best developed do not seem to be necessarily the same for the two sets of phenomena.

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(Communicated by the Astronomer Royal.)

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Introduction.

§ 1. While the observational study of terrestrial magnetism is receiving ever more and more attention, and being rewarded with success by the acquisition of new and important data, the theoretical side of the subject shows a much less rapid advance. The search for a physical theory of the earth's magnetism and its changes is fascinating but elusive. Perhaps in one case only—that of Schuster's important theory of the diurnal variations of the magnetic state of the earth—has there been put forward a clearly outlined theory which promises to explain the real mechanism of any magnetic phenomenon.

On this theory, the solar diurnal variations are attributed to the action of electromotive forces produced in masses of conducting air in the upper atmosphere, by their motion across the permanent magnetic field of the earth. The magnetic field of the resulting electric currents is identified with that which produces the observed diurnal changes. Schuster has shown that if the motion of the air is taken to be substantially that which is indicated by the barometric variations, the atmosphere being supposed to oscillate as a whole, the conductivity required by the theory is not unreasonable, considering the ionization of the tenuous upper atmosphere by ultra-

* 'Phil. Trans.' A, vol. 208, p. 163.
violet radiation from the sun.* The fundamental assumptions are in accordance with Schuster's demonstration† that the magnetic variations are principally due to a system of currents above the earth's surface. In order to explain the relative magnitudes of the diurnal and semi-diurnal terms in the magnetic potential, it is necessary to suppose that the conductivity of the atmosphere varies with the solar hour angle, which is certainly à priori probable: the great excess of the summer variation over the winter variation is unexplained, however, as the usual rapid rate of recombination of ions makes it difficult to believe that the solar ionization is slowly cumulative.

There is at present much uncertainty as to the numerical constants of the potential of the magnetic field responsible for the solar diurnal variations, as the only two calculations yet made‡ show serious disagreement. A new determination of this potential is now in progress at the Royal Observatory, Greenwich. Whatever be the result of this calculation, however, there will remain several important features of the phenomenon which require explanation—in particular, the seasonal changes. By the elucidation of these difficulties, terrestrial magnetism may throw light on the ionization of the upper atmosphere. The variables at disposal in the theory are, unluckily, too numerous to get very definite knowledge of any one of them from a single source, and therefore it is peculiarly fortunate that there is a kindred but independent set of phenomena, produced by the moon jointly with the sun, which promises veryvaluably to supplement the knowledge furnished by the solar diurnal variations. It should be specially instructive to compare the seasonal changes of the two sets of phenomena.

§ 2. The general outlines of this paper may be briefly indicated here. The principal known facts regarding the lunar magnetic variation are first summarized, and it is shown that, so far as they go, they seem most easily explicable in the manner proposed by Schuster for the solar diurnal variations. Nothing in the nature of a proof is yet possible however. Some new facts, deduced by harmonic analysis of existing material for the lunar variation at the separate phases of the moon, are then described, and it is pointed out how they confirm the hypothesis of the variable conductivity of the atmosphere in a very direct way, and provide a powerful means of quantitatively investigating the changes of the conductivity. The details of the calculation of these new harmonic terms in the lunar variation, and the actual tables of results, are collected in Part III. of the paper. In order to discuss the bearing of these observational results on the theory, it is necessary to extend Schuster's calculation of the effect of an atmospheric oscillation, under the influence of the earth's radial magnetic forces and the variable conductivity of the air, in producing

* That there is a highly conducting layer in the upper atmosphere is also indicated by the bending of electric waves round the earth.
† 'Phil. Trans.,' A, vol. 180, p. 467.
‡ Schuster, 'Phil. Trans.,' A, vol. 180, p. 467; and Fritsche, St. Petersburg, 1902.
magnetic diurnal variations. The calculations are given in Part II., in a very general form; the work is in some respects simpler and more direct than in Schuster's investigation, owing to the adoption of the resistivity, instead of the conductivity, as the variable. The formal results (which as yet, however, are at a somewhat incomplete stage) are reduced to numerical form and compared with the observed data. The whole of the discussion is collected in Part I., and it is shown that the fourth harmonic component of the lunar variation favours the assumption that the atmospheric conductivity may fall to a very small value during the night hours. The question of the seasonal variations, as affecting both the solar and lunar effects, is barely touched on, since though it arises naturally from the calculations in Part II., better observational material is necessary to realize the proper use of the theoretical work. A fuller discussion is reserved therefore till the new determination of the potential of the solar variation, already mentioned, is completed.

**PART I.—General Discussion.**

§ 3. The magnetic elements show regular periodic changes depending on the lunar hour angle; just as on the solar hour angle: the latter variations are considerably the greater of the two, and almost entirely mask the lunar variations. Kreil,* of Prague, in 1841, first established the existence of these changes, and since then a very limited number of investigators† have confirmed and extended Kreil's discovery. Owing to the nearly equal length of the solar and lunar days, the separation of the two effects involves considerable rearrangement of the observed data as usually tabulated, and the smallness of the lunar variation renders it necessary to deal with a large quantity of material in order to eliminate accidental errors. The determination of the lunar diurnal variation for the three magnetic elements at a single station is therefore a laborious undertaking, and hardly any observatory, as yet, includes such an examination of its observations in its scheme of work. If the potential of the magnetic field producing these variations is to be found, however, they must be computed not merely for one, but for several stations, well distributed on the earth's surface. This formidable task would be much

* Bohemian Society of Sciences, 1841.

Also the published observations at St. Helena, Toronto, Hobarton, and Cape of Good Hope (edited by Sabine), and at Melbourne, Dublin, and Philadelphia. Also Airy, 'Greenwich Observations,' 1859 and 1867.

Also Moos, 'Bombay Magnetic Observations,' 1846-1905, vol. II. (1910); and van Bemmelen, 'Met. Zeitschr.,' May, 1912.
expedited if various observatories would undertake the reduction of their own data on a uniform plan, and it is partly in the hope that some may be induced to co-operate in this work that the present preliminary paper has been written.

§ 4. When determined from the mean of a number of whole lunations, the lunar diurnal variation is found to be always of the same character, for every element and at every station: it consists solely of a very regular semi-diurnal oscillation. Other harmonic components of relatively small amplitude may be present, but their lack of regularity and consistency proves them to be accidental inequalities which are no real part of the phenomenon. This simplicity makes it probable that the lunar diurnal variation will be easier to explain than the solar diurnal variation.

Schuster's theory of the latter naturally suggests that the former is due to the lunar tidal oscillations of the atmosphere. These oscillations have very little effect upon the barometer, the ordinary diurnal barometric variation being a thermal and not a tidal effect; but a lunar barometric tide does exist, and has been evaluated with a considerable degree of accuracy at some tropical stations (St. Helena, Singapore, and Batavia).* The explanation gains weight from the fact that at perigee the lunar magnetic variations are of distinctly greater amplitude than at apogee,† and there is some evidence that the ratio of the amplitudes at the two seasons is that which would be predicted by the tidal theory (1.23), though the observational results do not suffice, as yet, to establish this definitely.

§ 5. Dr. van Bemmelen, at Batavia, has recently collected all the existing determinations of the lunar magnetic variation for different stations, and has examined this material, together with newly computed data for other stations, to see whether the magnetic field which produces these effects has a potential, and whether the latter has its source above or below the earth's surface.‡ He finds that most of the field, at any rate, has a potential, and that this arises partly above and partly below the earth's surface, but that the internal field is too great to be merely a secondary induction effect. This result should be accepted with some reserve, at present, not only on account of the imperfections of the data, but also because the seasonal change of the variations was disregarded; in certain elements at some stations the summer and winter variations are of opposite sign, and this renders it unsafe to take the mean variation for the whole year. At many stations, unfortunately, the data so far computed apply only to the whole year, so that if this material was to be used, no course was possible save to adopt the mean of the year for all. One important result of van Bemmelen's work was to show that the principal term in the potential of the lunar variation field was of the form \( Q_3 \) (in the usual language of harmonic analysis, a tesseral harmonic of the second kind and third

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* SABINE, 'Phil. Trans.' 1847; 'Batavian Observations,' 28 (1905).
† See 'Trevandrum Observations' (BROWN), vol. I., p. 137, and SABINE's and FIGEE's discussions already cited.
‡ 'Met. Zeitschr.,' May, 1912.
degree). This is in accordance with the theory that the lunar atmospheric tide is the main cause of the phenomenon, although, of course, it does not prove this to be the case.

§ 6. So far reference has been made entirely to the lunar variation as determined from a number of whole lunations, as has been generally done (the exceptions are Trevandrum, Bombay, and Batavia). It will be remembered that Schuster's theory of the solar diurnal variation involved the hypothesis of a variable conductivity depending on the sun's hour angle. This should, of course, also affect the electric currents which arise from the lunar atmospheric tide, and so make the lunar magnetic variations depend on the sun as well as on the moon. In the course of a lunation, however, the angle between the sun and moon, viewed from the earth, changes from 0 to $2\pi$, and the mean lunar variation for such a period cannot be expected to show any special dependence on solar time. At any particular lunar phase, however, the solar day hours, during which (over a given part of the earth) the atmospheric conductivity is greatest, occur at a definite part of the lunar day, this part changing with the lunar phase; and it has, in fact, been found* that the lunar variation determined from the mean of a number of days all at the same lunar phase is not of the semi-diurnal form. The variation curve goes through a regular cycle of change with lunar phase, in such a manner as to leave the mean variation over a whole lunation of the simple form already described. The magnetic needle is most mobile during the day hours: at certain seasons of the year, Broun found that the amplitude of the lunar diurnal variation of magnetic declination at Trevandrum was five times as great during the solar day hours as during the night hours.† These facts clearly show that the conductivity of the medium in which the electric currents flow to produce the lunar magnetic variation depends on the position of the sun; and since it is unreasonable to suppose that the mechanisms concerned in producing the lunar and solar diurnal magnetic variations are materially different, the assumption of variable conductivity in Schuster's theory is confirmed in a very definite and independent way; in Schuster's discussion two barometric oscillations, diurnal and semi-diurnal, were concerned, and it was necessary to explain why the resulting magnetic variations, deduced on the assumption of uniform conductivity, did not bear the proper ratio to one another. This might be because the conductivity was not uniform, or because the ratio of the two oscillations was different in the upper regions of the atmosphere from that indicated by the barometer. This latter uncertainty is absent in the case of the lunar variations, where there is only a single barometric oscillation, from which arise magnetic variations of other periods, depending on the solar hour angle.

§ 7. In order to examine the effect of this variable conductivity, it is natural to determine the harmonic components of the lunar diurnal variation for different lunar

* By Broun, Chambers, Figee, and Moos in the investigations already cited.
† 'Trevandrum Observations,' vol. I, p. 121.
phases, but (rather strangely) this has only once been done hitherto, and then without result.* Chambers† obtained an analytical expression for the variation and its dependence on phase, which satisfactorily represents the observations, but it is not of a simple character. His formula was

\[ f_{\text{e,2}}(h) \cos 2 \left( \frac{2\pi}{P} t \right) + f_{\text{s,2}}(h) \sin 2 \left( \frac{2\pi}{P} t \right), \]

where \( h \) is the hour of the solar day, \( P \) is the mean period of a lunation in solar days, and \( t \) is the age of the moon in solar days; \( f_{\text{e,2}}(h) \) and \( f_{\text{s,2}}(h) \) are the observed variations at new moon and one-eighth phase respectively. This formula, it will be noticed, expresses the lunar variation as, in reality, a solar diurnal variation (\( h \), the solar time, being the variable) which merely runs through a cycle of change depending on the age of the moon. This, in fact, was Chambers' view—he termed the variation "luni-solar." It will be seen later, however, that there is a true lunar semi-diurnal variation which remains unchanged throughout the course of \( \pi \) lunation, as well as luni-solar components governed by the position of both bodies. As to Chambers' expression for the variation, while it is numerically correct, it does not aid in interpreting the phenomenon, because it depends on two complex curves \( f_{\text{e,2}}(h) \) and \( f_{\text{s,2}}(h) \), for which no analytical expression was obtained; these two curves are not independent, as will appear later.

§ 8. Figee determined the harmonic coefficients of the diurnal and semi-diurnal components of the variation at each lunar phase, and came to the conclusion that "a regular variation of the movement of the magnetic needle with the moon's phases is not indicated by the observations at Batavia."‡ It will be shown, on the contrary, that the Batavian observations agree with those made at other places in manifesting considerable regularity of change with lunar phase.

§ 9. Moos§ has made the valuable suggestion that the luni-solar variation may be regarded as a simple lunar variation the amplitude of "part of which goes through a series of wave-like changes in the course of a lunation." He multiplies each hourly value of the mean lunar variation determined from a whole month by \( 1 + \cos (t + \nu) \), where \( t \) is the lunar time reckoned from upper culmination (one hour equalling 15°), and \( \nu \) is the angular measure of the moon's age, reckoned as 0° at new moon, and changing through 360° in the course of a month. Curves showing the results of this calculation are exhibited for comparison with the observed curves, for the eight lunar phases, for the element of declination. The general similarity of the two sets of curves is sufficiently striking to show that the suggestion is in the right direction. It will be seen that this idea is, formally, much akin to Schuster's idea of variable

* 'Batavian Observations,' XXVI., Appendix, p. 195, § 44.
† Chambers, 'Phil. Trans.,' A, vol. 178 (1887).
‡ 'Batavian Observations,' XXVI., Appendix, § 44.
conductivity, and is most naturally interpreted in that way. Moos, however, seems not to have thought of the matter in this simple light, but speaks of changes in the radio-activity of the earth's crust, due to a tidal action, as possibly responsible for the luni-solar changes, perhaps by ionizing the atmosphere indirectly; and also of the reflection by the moon of ionizing radiation from the sun.*

§ 10. Since the mean variation of any element over a whole lunation is almost exactly a semi-diurnal wave, Moos's expression is equivalent to

\[ 2 \cos (2t + t_0) [1 + \cos (t + \nu)] = \cos (t + t_0 - \nu) + 2 \cos (2t + t_0) + \cos (3t + t_0 + \nu), \quad (A) \]

though he did not himself write it out formally thus. The examination of the data by harmonic analysis, which is effected in the third part of this paper, is the best means of numerically testing Moos's suggestion, being preferable to a mere comparison of two sets of curves by eye. The desire to apply this test partly occasioned the present re-examination of the existing data, which also has in view the comparison of the results of these past determinations of the lunar magnetic variation (on which enormous labour has been spent) to see how far they confirm one another, and gauge the possibility of obtaining accurate information from them.

Moos's suggestion implies the presence, in the lunar diurnal variation at a particular lunar phase, of first and third harmonic components of amplitude equal to half that of the semi-diurnal component, and with phase angles which respectively decrease and increase by 45° with each change of lunar phase, the epoch of the second component remaining constant. No other relations or components would satisfy the above equation.

§ 11. The calculations from the observational data show that while first and third harmonic components possessing the above phase relations are present, the amplitudes are not generally in accordance with Moos's equation. Moreover, a fourth harmonic component, which was calculated in the first instance merely because to do so involved scarcely any trouble after the other components had been computed, was also found to be present, of quite appreciable amount, and obeying an unexpected phase law; its phase angle increases during each lunation by \(4\pi\), twice the amount of change in the phases of the first and third components.

There is considerable accidental error in the determinations of the phase angles and amplitudes at each lunar phase, as, of course, the material is much subdivided. While, however, the phase angles go through an easily recognizable monthly cycle, the amplitudes show no regular variation with lunar phase (the mean of a number of lunations is dealt with, of course, so that perigee and apogee occur at different phases during the period). The mean of the amplitudes at the separate phases gives, therefore, the best determination of the amplitudes of the first, third, and fourth

* 'Bombay Magnetical Observations,' 1846-1905, vol. II., § 527. It may be mentioned that earlier investigators had regarded the lunar variations as possibly due to the direct or indirect action of induced magnetism in the moon, arising from solar or terrestrial magnetism, or both.
components, as well as of the second; and similarly, by correcting the separate phase angles by the amount indicated by the regular phase law, and taking their mean, the accidental error of the determined phase angle at any particular lunar phase can be much reduced. In this way, as described more fully in § 27, the expression of the lunar variation at every period of the lunation, complete as far as the fourth harmonic term, is obtained. It is found that the amplitudes of the first and third harmonics are often unequal; sometimes their amplitude exceeds that of the second component, but generally they are less, down to about half this amount. The determined values of C and \( t_0 \) in the formula

\[
C_1 \cos (t + t_0' - \nu) + C_2 \cos (2t + t_0'') + C_3 \cos (3t + t_0''' + \nu) + C_4 \cos (4t + t_0'''' + 2\nu), \tag{B}
\]

which has been found to fit the observations, are given in Tables XI., XII., and XIII. for all the stations and elements for which data were available. Moos's representation, it is seen, though it pointed in the right direction, is of too simple a character to represent the phenomenon; the solar excitation which it indicates is a matter which concerns the whole earth, and this action cannot be represented by a simple harmonic factor at each individual station.

§ 12. Schuster* has calculated the effect of an atmospheric oscillation with a velocity potential \( Q_s^2 \) (which is also the main component of a lunar diurnal tide)† in producing, under the influence of a variable conductivity of amount

\[
\rho = \rho_0 (1 + \gamma \cos \omega), \quad \gamma \leq 1
\]

(where \( \omega \) is the zenith distance of the sun from each particular point on the earth's surface), magnetic variations of one, two, three and more periods in the solar day. Adopting the rather more general expression

\[
\rho = \rho_0 [1 + \gamma' \cos \theta + \gamma \sin \theta \cos (\lambda + t)], \tag{C}
\]

where \( \theta \) is the colatitude, \( \lambda \) is the longitude, and \( \lambda + t \) is the local time, he finds that the resulting magnetic potential (apart from a constant factor) is of the form

\[
\sum_{n=0}^{\infty} \frac{n+1}{2n+1} p_n^* Q_n^* \sin \{\sigma (\lambda + t) - \alpha\} + \sum_{n=1}^{\infty} \frac{n+1}{2n+1} q_n^* Q_n^* \sin \{\sigma (\lambda + t) + \alpha\}, \tag{D}
\]

where \( Q_n^* \) are numerical constants which depend on \( \nu \) and \( \nu' \); their values are tabulated in the paper referred to.

It is shown in Part II. of the present paper that the above equation (D) holds good, whatever be the functional relation between \( \rho \) and \( \omega \), and this calculation is

† \( Q_n^* \) represents the tesseral function \( \sin^* \theta d\sigma P_n^* d\sigma \), where \( P_n^* \) is the zonal harmonic of degree \( n \).
adapted, in §23, to cover the case of the luni-solar magnetic variations. It is there shown that the equivalent expression to (D) is in this case (apart from a constant factor)

\[
\sum_{\sigma=0}^{\infty} \frac{n+1}{2n+1} p^* Q^* \sin \{\sigma(\lambda+\ell') - \alpha + (\sigma-2) \nu\} + \sum_{\sigma=0}^{\infty} \frac{n+1}{2n+1} q^* Q^* \sin \{\sigma(\lambda+\ell') + \alpha + (\sigma+2) \nu\} \quad \ldots \ldots \quad (E)
\]

This expression, it should be noticed, consists of series of harmonic components of one, two, three, and more periods in the lunar day, with phase angles which depend on the age of the moon. In the second series the phase angles increase by \(2(\sigma+2)\pi\) per lunation; this phase change is very rapid, even for the diurnal term, and with the lunation divided up into not more than eight parts, hardly comes within the range of observation, even if the coefficients \(q^*\) were of the same order of magnitude as the \(p^*\) coefficients. The theoretical values of \(q^*\) are, however, much less than those of the important members of the \(p^*\) set of coefficients, and therefore this part of the magnetic potential can be neglected. The other part consists of terms of period \(2\pi\nu\), whose phase angles increase by \(2(\sigma-2)\pi\) per lunation; thus the phase of the first harmonic decreases by \(2\pi\) each lunation, that of the second component remains constant, while the third, fourth, and higher components increase by amounts \(2\pi, 4\pi, 6\pi,\) and so on. This, however, is exactly the law of phase change which is indicated by the formula (B), which was determined empirically from the observations.

At new moon, when \(\nu = 0\), the formula indicates that all the harmonic components should have the same phase angle, or differ by 180 degrees exactly (since the coefficients may be of different sign). The data obtained in this paper show a very satisfactory agreement with this conclusion, when the extreme smallness of the whole phenomenon is considered.

§13. The amplitudes must next be considered. The actual calculations necessary for the comparison of theory and observation are given in §25, and only the results obtained will be cited here. It appears that as regards the relative magnitudes of the first three components in the lunar variation, there is tolerably good agreement with the results derived either from Schuster's simple theory \(p/p_1 = 1+\cos \omega\), or from the more general theory of Part II of this paper. The numerical constants \((p/p_0 = 1+3 \cos \omega + \frac{3}{4} \cos^2 \omega)\) might be altered to fit the observations better, but it seems hardly worth while to do this till better observational material is available. The given constants were chosen to represent a function which should have a large maximum at midday, and should be small and nearly constant during the night hours.

§14. The deciding factor between the two expressions for \(p/p_0\) is found to be the amplitude of the fourth harmonic component. Three tables are given in §25 to illustrate this. They give the ratio of the amplitudes of the four harmonic components to that of the second component, for the three elements \(X, Y, Z\). The first
table is that calculated on the hypothesis \( \rho/\rho_0 = 1 + \cos \omega \), the second that calculated from \( \rho/\rho_0 = 1 + 3 \cos \omega + \frac{3}{2} \cos^2 \omega \), and the third gives the observed values. The simple form of \( \rho \) gives altogether too small a value for \( C_1/C_2 \), while the second expression for \( \rho \) gives values of the right order, at any rate. Perhaps the detailed calculations in Part II. have not been carried to a sufficient degree of approximation, as the expressions for \( p_\theta \) do not converge very rapidly. When better data are available, this point must receive consideration. Enough evidence, however, has been brought forward to show that the fourth harmonic component of the lunar variation favours the hypothesis that the conductivity during the night hours is small compared with its value during the daytime, and that the rate of recombination of ions in the upper atmosphere (assuming this to be the seat of the effect) is rapid, as would naturally be expected.

The proper discussion of the observations, whether of the lunar or solar magnetic variations, can only be made on the basis of a reliable determination of the numerical coefficients of the various tesseral harmonics in the potential, derived from a number of observatories properly distributed over the globe. The significance of the lower harmonics in the lunar variation makes it desirable to obtain the terms in the potential down to those of the fourth type \( (Q_4) \)—not only for the lunar variation, but also for the solar variation; its fourth harmonic shows a sufficient degree of constancy, at most observatories, to entitle it to respect as having definite physical significance.

**PART II.—Mathematical Theory.**

§ 15. The problem in hand is to determine the current function of the electric currents induced in a spherical shell of fluid by its quasi-tidal motion across a radial magnetic field of force, the electric conductivity of the fluid at any point being a known function of the angular distance between that point and another (that with the sun at its zenith) which uniformly rotates round the axis of the sphere. The velocity potential \( \psi \) of the motion will be expressed as the sum of a number of terms such as

\[
Q_m \sin (\tau \cdot \lambda + t - \alpha),
\]

where \( Q_m \) is a surface harmonic of degree \( m \) and type \( \tau \), and \( \lambda \) is the longitude measured towards the east from some standard meridian, at which the local time is \( t \). The colatitude and zenith distance of the sun will be denoted by \( \theta \) and \( \omega \) respectively;

* It is not asserted that any observational evidence has been brought forward in favour of the particular numerical constants here chosen for \( \rho \), but only that the observations indicate the presence of an appreciable term in \( \rho \) depending on \( \cos 2\omega \), and that this term, if present, may be expected, on general physical grounds, to be of such a sign as to diminish the value of \( \rho \) at night as compared with the value by day.—June 11, 1913.
if $\delta$ is the declination of the sun, evidently we have

$$\cos \omega = \sin \delta \cos \theta + \cos \delta \sin \theta \cos (\lambda + t)$$

$$\equiv x + 2y\mu,$$

where

$$x = \sin \delta \cos \theta, \quad 2y = \cos \delta \sin \theta, \quad \mu = \cos (\lambda + t).$$

The conductivity and resistivity at the point $(\theta, \lambda)$ will be denoted by $\rho$ and $\kappa'$ respectively, $\rho \kappa'$ being, of course, equal to unity. For the present we shall suppose that $\rho$ and $\kappa'$ are finite and continuous functions of $\omega$, so that they can be expressed as FOURIER's series in $\cos n\omega$ over the range $0, \pi$; $\rho$ will certainly satisfy this condition, and the case of $\rho = 0, \kappa' = \infty$ will be considered later. Further, it will be assumed possible to express $\kappa'$ as a TAYLOR's series in $\cos \omega$, and it is in this form that we shall suppose the resistivity to be given, as one of the data of the problem. Theoretically this is a limitation of the problem, as there are some functions which cannot be expressed in the form stated; for instance, if the conductivity were proportional to $\cos \omega$ in that hemisphere on which the sun is shining, and zero or constant over the other hemisphere, $\kappa'$ could not be so expressed. But in reality nothing of value is lost, as any continuous function can be approximately expressed in the form of a TAYLOR's series to any desired degree of accuracy.

[Some further explanation of this use of series may be desirable. The series used in the analysis are all written as infinite ones, for the sake of formal simplicity and theoretical completeness. In the detailed execution of the work, however, only a finite number of these terms can be utilized, as workable general expressions for the coefficients in the current function $R$ cannot be obtained. The actual procedure, therefore, must be to take a finite number of terms of the FOURIER's series for $\rho$, transform this into a polynomial in $\cos \omega$ (this also, of course, will have only a finite number of terms), and work out the coefficients of $R$ in terms of the coefficients of this polynomial to as great a degree of accuracy as is practicable and desirable. This is the course of the work in §§18–20, where the terms $(a + b \cos \omega + c \cos 2\omega)$ of the FOURIER's series for $\rho$ are taken, and the expression for $R$ is worked out as far as concerns the terms in $a, b, b^2,$ and $c$. The resistivity $1/\rho$ is introduced into the calculations for purely mathematical reasons, on account of certain analytical advantages which it seems to offer. The results obtained in this way, in terms of the coefficients of $\rho$, might be got otherwise by an extension of the method used by SCHUSTER. This identity of results is clear from the fact that if the FOURIER coefficients of $\rho$ are small enough the TAYLOR's series for $1/\rho$ is absolutely convergent, and the legitimacy of the use of $1/\rho$ is in this case immediately evident; the formal results, however, do not depend on any property of convergence, so that the results obtained by using $1/\rho$ remain equally valid with those obtained in any other way, even though the series for $1/\rho$ should become non-convergent. This is one of many instances in which it is possible and advantageous to use expressions which may

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become non-convergent to obtain results which can be got less simply in other ways. Whether the final result is convergent depends in this case only on \( \rho \), and not on the processes of analysis used to deduce \( R \) from \( \rho \).—Added June 11, 1913.]

We shall write, therefore,

\[
\kappa = Cae \sum_0^\infty d_p \cos^p \omega, \quad \equiv Cae \kappa,
\]

where \( C, e, \) and \( \alpha \) are constants (introduced for convenience) whose meaning will be explained later, and the coefficients \( d_p \) are given numbers. Expanding \( \cos^p \omega \) in terms of \( \mu \), we have

\[
\kappa = \sum_0^\infty d_p (x + 2y\mu)^p
\]

\[
= \sum_0^\infty e_p \cdot (2y)^p \cdot \mu^p
\]

\[
= f_0 + 2 \sum_1^\infty f_p \cos \rho (\lambda + t),
\]

where

\[
e_p = \sum_{l=0}^\infty l+l C_p \cdot d_{l+p} \cdot x^l,
\]

and (since

\[
2^{m-1} \mu^m = 2^{m-1} \cos^m (\lambda + t) = \cos^m (\lambda + t) + m C_1 \cos (m-2)(\lambda + t) + m C_2 \cos (m-4)(\lambda + t) + \ldots
\]

\( f_p \) is given (for \( p \geq 0 \)) by

\[
f_p = \sum_{q=0}^\infty p+2q C_q \cdot e_{p+2q} \cdot y^{p+2q},
\]

\[
\equiv y^p \sum_{q=0}^\infty \alpha_{p,q} y^{2q}.
\]

Here we have written

\[
\alpha_{p,q} \equiv p+2q C_q e_{p+2q}.
\]

In virtue of the definitions of \( e_p \) and \( \alpha_{p,q} \), we have

\[
\frac{de_p}{dx} = (p+1) e_{p+1}, \quad \frac{d\alpha_{p,q}}{dx} = (q+1) \alpha_{p-1,q+1}.
\]

Next we consider the differential coefficients of \( \kappa \). We have

\[
\frac{d\kappa}{d\lambda} = -2 \sum_1^\infty f_p \sin \rho (\lambda + t),
\]

\[
\frac{d\kappa}{d\theta} = \frac{df_p}{d\theta} + 2 \sum_1^\infty \frac{df_p}{d\theta} \cos \rho (\lambda + t).
\]

We shall write \( f'_p \) for \( \frac{df_p}{d\theta} \); evidently we have

\[
f'_p = \frac{1}{2} y^{p-1} \cos \delta \cos \theta \sum_0^\infty (p+2q) \alpha_{p,q} y^{2q} \sin \delta \sin \theta (q+1) \alpha_{p-1,q+1} y^{2q}.
\]
§ 16. Following Schuster's notation and treatment, the earth will be regarded as a uniformly magnetized sphere of radius \( a \), whose magnetic potential may be resolved into the zonal harmonic of the first degree and the tesseral harmonic of the first degree and type. The former harmonic is much the larger of the two, as the inclination \( \phi \) of the magnetic to the geographical axis is small. The radial force can be expressed as

\[
V = C \cos \theta + C \tan \phi \sin \theta \cos \lambda,
\]

where \( C \) is a constant not differing much from \(-\frac{1}{3}\) (the force being measured positive outwards), and \( \lambda \) is now the longitude measured from the meridian (68° 31' west of Greenwich) containing the magnetic axis.

The components of electric force, \( X \) and \( Y \), measured towards the south and east respectively, are

\[
X \alpha = \frac{V}{\sin \theta} \frac{d\psi}{d\lambda}, \quad Y \alpha = -\frac{V}{\sin \theta} \frac{d\psi}{d\theta},
\]

\( \psi \) being the velocity potential.

If we express \( X \) and \( Y \) in the form

\[
X = \frac{dS}{d\theta} + \kappa' \frac{dR}{e \sin \theta \, d\lambda}, \quad Y = \frac{dS}{\sin \theta \, d\lambda} - \kappa' \frac{dR}{e \, d\theta},
\]

where \( \kappa' \) is the known resistivity and \( e \) the thickness of the conducting atmospheric shell, the function \( R \) will be the current function of the electric currents produced by \( X \) and \( Y \) (neglecting electric inertia). The function \( S \) is the potential of a system of electric forces which in the steady state are balanced by a static distribution of electricity revolving round the earth, and causing a variation in the electrostatic potential which is found to be too weak to affect our instruments.

To determine \( R \) we shall eliminate \( S \), thus obtaining the equation

\[
\frac{dX}{d\lambda} - \frac{d}{d\theta} \left( Y \sin \theta \right) = \frac{1}{\sin \theta} \frac{d}{d\lambda} \left( Ca \kappa \frac{dR}{d\lambda} \right) + \frac{d}{d\theta} \left( Ca \kappa \sin \theta \frac{dR}{d\theta} \right).
\]

Instead of using the resistivity \( \kappa' \), Schuster worked with the conductivity \( \rho \) (using the special form \( 1 + k \cos \phi \)), in order to avoid the difficulties introduced by "the high and possibly infinite values which \( \kappa' \) would take when the conductivity sinks low or vanishes."* These difficulties, however, are found not to be serious, and the work is greatly simplified by the use of \( \kappa' \), which enables \( R \) to be determined directly, without first evaluating \( S \), as is necessary when \( \rho \) is kept as the variable quantity. The investigation can also be made much more general, without formal complexity, when \( \kappa' \) is used.

Recalling the expressions for $X$, $Y$ in terms of the velocity potential $\psi$, the left-hand side of the last equation, after division by $Ca\sin\theta$, may be written

$$2m+1 \left[ m (m+2) (m-\tau+1) Q_{m+1} + (m-1) (m+1) (m+\tau) Q_{m-1} \right] \sin (\tau, \lambda + t - \alpha)$$

$$+ \frac{\tan \phi}{2(2m+1)} \left[ \{ (m-1) (m+1) Q_{m+1} - m (m+2) Q_{m-1} \} \sin (\tau, \lambda + t + \lambda - \alpha) \right.$$

$$\left. - (m-1) (m+1) (m+\tau-1) Q_{m+1} - m (m+2) (m-\tau+1) (m-\tau+2) Q_{m-1} \} \sin (\tau, \lambda + t - \lambda - \alpha) \right].$$

The right-hand side of our equation for $R$, after division by $Ca\sin\theta$, becomes equal to

$$\frac{\kappa}{\sin^2 \theta} \left( \frac{d^2}{d\lambda^2} + \sin \theta \frac{d}{d\theta} \sin \theta \frac{d}{d\theta} \right) R + \left( \frac{dR}{\sin \theta d\lambda} \cdot \frac{d\kappa}{\sin \theta d\lambda} + \frac{dR}{d\theta} \cdot \frac{d\kappa}{d\theta} \right).$$

We suppose $R$ to be expressed as the sum of a number of tesseral harmonics $p^n Q^\sigma \sin (\sigma \lambda' - \alpha')$, where $p^n$ is a numerical coefficient, $\lambda'$ has been written for $\lambda + t$, and $\sigma$ ranges (possibly) from $-\infty$ to $+\infty$. The contribution of each such term to the total value of the last expression is easily seen to be the product of $p^n Q^\sigma$ into

$$-n (n+1) Q^\sigma \sin (\sigma \lambda' - \alpha') \left\{ f_0 + 2 \sum_{1}^{\infty} f_p \cos p\lambda' \right\},$$

$$-\frac{2\sigma Q^\sigma}{\sin^2 \theta} \cos (\sigma \lambda' - \alpha') \sum_{1}^{\infty} p f_p \sin p\lambda',$$

$$+ 2 \frac{dQ^\sigma}{d\theta} \sin (\sigma \lambda' - \alpha') \left\{ f_0' + 2 \sum_{1}^{\infty} f_p' \cos p\lambda' \right\},$$

where we have inserted the values of $\kappa$ and its differential coefficients, and have transformed the first line by means of LAPLACE’S equation

$$\sin \theta \frac{d}{d\theta} \sin \theta \frac{dQ^\sigma}{d\theta} + \frac{d^2 Q^\sigma}{d\lambda^2} + n (n+1) \sin^2 \theta \cdot Q^\sigma = 0.$$
This enables us to write expression (3) in the form

\[-n(n+1)Q_n^\sigma \sum_{-\infty}^{\infty} f_p \sin (\sigma + p \lambda' - \alpha'),\]

\[-\frac{\sigma Q_n^\sigma}{\sin^2 \theta} \sum_{-\infty}^{\infty} f_p \sin (\sigma + p \lambda' - \alpha'),\]

\[+ \frac{dQ_n^\sigma}{d\theta} \sum_{-\infty}^{\infty} f_p \sin (\sigma + p \lambda' - \alpha'),\]

\[= \sum_{p=\infty}^{n} R_n^\sigma (p) \sin (\sigma + p \lambda' - \alpha'),\]

where

\[R_n^\sigma (p) = \left( -n(n+1)Q_n^\sigma - \frac{p \sigma Q_n^\sigma}{\sin^2 \theta} \right) f_p + f_p \frac{dQ_n^\sigma}{d\theta}.\]

When \( p \) is positive, substituting our expressions for \( f_p \) and \( f'_p \) we find

\[R_n^\sigma (p) = \sum_{q=0}^{\infty} \alpha_{p,q} y^{pq} \left[ -n(n+1)Q_n^\sigma - \frac{p \sigma Q_n^\sigma}{\sin^2 \theta} \right] y^p,\]

\[+ \sum_{q=0}^{\infty} (p+2q) \alpha_{p,q} y^{pq} \left( \frac{1}{2} \cos \theta \cos \theta \frac{dQ_n^\sigma}{d\theta} \right) y^{p-1},\]

\[- \sum_{q=0}^{\infty} (q+1) \alpha_{p-1,q+1} y^{pq} \sin \theta \left( \frac{dQ_n^\sigma}{d\theta} \right) y^p,\]

\[= \frac{1}{2} \cos \delta \sum_{q=0}^{\infty} \alpha_{p,q} y^{pq} \left[ \left( p+q \right) \left( \cos \theta \frac{dQ_n^\sigma}{d\theta} - \frac{\sigma Q_n^\sigma}{\sin \theta} \right) + q \left( \cos \theta \frac{dQ_n^\sigma}{d\theta} + \frac{\sigma Q_n^\sigma}{\sin \theta} \right) \right.\]

\[-n(n+1)Q_n^\sigma \sin \theta \left] y^{p-1},\right.\]

\[-\sin \delta \sum_{q=0}^{\infty} (q+1) \alpha_{p-1,q+1} y^{pq} \sin \theta \frac{dQ_n^\sigma}{d\theta} y^p.\]

Since, in the original equation for \( R_n^\sigma (p) \), a change in the sign of \( p \) only affects the term \( \sigma Q_n^\sigma / \sin \theta \) in the first term, from our last expression we may at once write down the value of \( R_n^\sigma (-p) \), \( p \) being positive, by changing the sign of \( \sigma Q_n^\sigma / \sin \theta \). Thus,

\[R_n^\sigma (-p) = \frac{1}{2} \cos \delta \sum_{q=0}^{\infty} \alpha_{p,q} y^{pq} \left[ \left( p+q \right) \left( \cos \theta \frac{dQ_n^\sigma}{d\theta} + \frac{\sigma Q_n^\sigma}{\sin \theta} \right) + q \left( \cos \theta \frac{dQ_n^\sigma}{d\theta} - \frac{\sigma Q_n^\sigma}{\sin \theta} \right) \right.\]

\[-n(n+1)Q_n^\sigma \sin \theta \left] y^{p-1},\right.\]

\[-\sin \delta \sum_{q=0}^{\infty} (q+1) \alpha_{p-1,q+1} y^{pq} \sin \theta \frac{dQ_n^\sigma}{d\theta} y^p.\]

§ 17. For convenience and clearness, some well-known formulae of transformation will now be set down. These have been much used by Schuster in his papers, and
the equations, and some formulœ derived from them, will be denoted by the same Roman letters which he uses.\* 

\[
(2n+1) \cos \theta Q_n^\sigma = (n-\sigma+1) Q_{n+1}^{\sigma+1} + (n+\sigma) Q_{n-1}^{\sigma-1}, \quad \ldots \quad (A)
\]

\[
(2n+1) \sin \theta Q_n^\sigma = Q_{n+1}^{\sigma+1} - Q_{n-1}^{\sigma-1}, \quad \ldots \quad (B)
\]

\[
= (n+\sigma)(n+\sigma-1)Q_{n-1}^{\sigma-1} - (n-\sigma+2)(n-\sigma+1)Q_{n+1}^{\sigma-1}, \quad \ldots \quad (C)
\]

\[
\frac{2nQ_n^\sigma}{\sin \theta} = (n+\sigma)(n+\sigma-1)Q_{n-1}^{\sigma-1} + Q_{n-1}^{\sigma+1}, \quad \ldots \quad (D)
\]

\[
= Q_{n+1}^{\sigma+1} + (n-\sigma+2)(n-\sigma+1)Q_{n+1}^{\sigma-1}, \quad \ldots \quad (E)
\]

\[
\frac{2dQ_n^\sigma}{d\theta} = (n+\sigma)(n-\sigma+1)Q_{n-1}^{\sigma-1} - Q_{n+1}^{\sigma+1}, \quad \ldots \quad (F)
\]

\[
(2n+1) \left( \cos \theta \frac{dQ_n^\sigma}{d\theta} - \frac{\sigma Q_n^\sigma}{\sin \theta} \right) = -\left\{ nQ_{n+1}^{\sigma+1} - (n+1)Q_{n-1}^{\sigma+1} \right\}, \quad (G)
\]

\[
(2n+1) \left( \cos \theta \frac{dQ_n^\sigma}{d\theta} + \frac{\sigma Q_n^\sigma}{\sin \theta} \right) = \left\{ n(n-\sigma+2)(n-\sigma+1)Q_{n+1}^{\sigma-1} + (n+1)(n+\sigma)(n+\sigma-1)Q_{n-1}^{\sigma-1} \right\}, \quad (H)
\]

\[
(2n+1) \sin \theta \frac{dQ_n^\sigma}{d\theta} = n(n-\sigma+1)Q_{n+1}^{\sigma+1} - (n+1)(n+\sigma)Q_{n-1}^{\sigma-1}. \quad (I)
\]

Making use of these equations, we obtain the following expressions:

\[
R_n^\sigma (p) = \frac{\cos \delta}{2(2n+1)} \sum_{q=0}^{\infty} \alpha_p, q y^q \left[ -\left\{ n(n+p+q+1)Q_{n+1}^{\sigma+1} + (n+1)(n-p-q)Q_{n-1}^{\sigma+1} \right\} \\
+ q \left\{ n(n-\sigma+2)(n-\sigma+1)Q_{n+1}^{\sigma-1} + (n+1)(n+\sigma)(n+\sigma-1)Q_{n-1}^{\sigma-1} \right\} \right] y^{p-1} \\
- \frac{\sin \delta}{2n+1} \sum_{q=0}^{\infty} (q+1) \alpha_{p-1, q+1} y^q \left[ n(n-\sigma+1)Q_{n+1}^{\sigma-1} - (n+1)(n+\sigma)Q_{n-1}^{\sigma-1} \right] y^{p-1}.
\]

In $R_n^\sigma (-p)$, the second term remains the same, while the expression in square brackets in the first term becomes

\[
\left\{ n(n+p+q+1)(n-\sigma+2)(n-\sigma+1)Q_{n+1}^{\sigma-1} - (n-p-q)(n+1)(n+\sigma)(n+\sigma-1)Q_{n-1}^{\sigma-1} \right\} \\
- q \left\{ nQ_{n+1}^{\sigma+1} + (n+1)Q_{n-1}^{\sigma+1} \right\}.
\]

These expressions for $R_n^\sigma (\pm p)$ are of the type

\[
\alpha y^q Q_v^{\sigma+1} y^{-p-1}, \quad \alpha y^q Q_v^{\sigma-1} y^{-p-1}, \quad \alpha y^q Q_v^{\sigma-1} y^{-p-1}.
\]

Now by equations (B) and (C), it is evident that $Q_v^{\sigma+1} y^{-p-1}$, $Q_v^{\sigma-1} y^{-p-1}$, and $Q_v y^p$ can be expressed as the sum of a number of tesseral harmonics all of type $\sigma+p$ or all of type $\sigma-p$ (at will), and of degrees ranging, by steps of 2, from $\nu \pm (p-1)$, $\nu \pm (p+1)$ and $\nu \pm p$ respectively. Further multiplication by $y^p$ can be so arranged as

to leave the type unchanged, while extending the range of the degrees by 4\(q\). Also by equation (A), the coefficient \(\alpha\), which, it will be remembered, is a power series in \(\cos \theta\), leaves the type unchanged while it increases the range of the degrees of the resulting tesseral harmonics. In every case, therefore (\(p\) positive or negative), \(R_n^p\) can be expressed as the sum of a number of terms such as \(Q_{n^*p}\). Therefore if we write \(\Psi\) for the sum of all the expressions (2) resulting from each term \(Q_n^\alpha \sin (\tau \lambda' - \alpha)\) in the velocity potential \(\varphi\), the fundamental equation (1) for \(R\) takes the form

\[
\Psi = \sum_{p=-\infty}^{\infty} \{ \sum_{n,s} k_{n,s}^p Q_{n^*p} \sin (\sigma + p' \lambda' - \alpha') \},
\]

where \(k_{n,s}^p\) is a coefficient whose value can be determined in terms of \(p_n^p\) and the coefficients \(d_n^p\) in the TAYLOR’s series for \(x\). By equating the coefficients of harmonics of the same degree and type, on the two sides of the equation, we obtain equations to determine the coefficients \(p_n^p\) in terms of the \(d_n^p\)'s and the known constants of the velocity potential. In practice this must be done by a process of successive approximation. Knowing, from the form of the above equation, which is linear in \(p_n^p\) and \(d_n^p\), that every coefficient \(p_n^p\) can be expressed as a TAYLOR’s series in \(d_0^p, d_1^p, d_2^p, \ldots\), and so on, we can determine this series by successively assuming that all save one particular variable \(d_0^p\) are zero, and considering this variable alone, it may easily be seen that the phase angle of every term in \(R\) arising from a particular term in \(\Psi\) is the same as that of the latter.

§ 18. SCHUSTER has worked out the values of \(p_n^p\) for the special form of conductivity already mentioned, and for the two terms \(Q_1^1 \sin (\lambda' - \alpha)\) and \(Q_2^2 \sin (2\lambda' - \alpha)\) in the velocity potential, to the fourth order of approximation, and he finds that the numerical coefficients of the terms are such that only the first order term (depending, in our notation, on \(d_1/d_0\)) are large enough to be detectable by observation. The present calculation will not be carried so far, therefore, and will not include terms of higher order than \(d_1/d_0\) or \((d_1/d_0)^2\). Also, since in the expression for \(\Psi\) the term depending on the inclination of the magnetic to the geographical axis is multiplied by the small factor tan \(\phi\), the part of \(R\) depending on this term will only be calculated as far as the first order \(d_1/d_0\). Further, since the actual atmospheric oscillations seem to be mainly performed in the simplest mode possible, so that \(m = \tau\) for the principal terms, the second order terms will be neglected for the smaller harmonics in the velocity potential \(\varphi\), for which \(m \neq \tau\).

We therefore consider the terms in \(R\) which depend upon a term \(A_m^\tau Q_{m-1^\tau}\) in \(\Psi\), where \(m'\) and \(\tau'\) are quite general, except that in the terms depending on \(d_1/d_0\) we shall suppose \(m' = \tau' + 1\) (since \(Q_{m'} = 0\) when \(\tau' > m'\), the term in (2) depending on \(Q_{m-1}^\tau\) vanishes when \(m = \tau\)).

It will first be necessary to write out the developed expressions for \(R_n^\tau(0), R_n^\tau(\pm 1), R_n^\tau(\pm 2)\) as far as the terms in \(d_2\). No other values of \(p\) in \(R_n^\tau(p)\) give
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terms containing \(d_n, d_1, d_2\). It is convenient to write down first of all the values of

\[ f_0 = e_0 + 2d_2 y^2, \quad e_0 = d_0 + d_1 x + d_2 x^2, \]

\[ f_1 = e_1 y, \quad e_1 = d_1 + 2d_2 x, \]

\[ f_2 = e_2 y^2, \quad e_2 = d_2, \]

\[ f'_0 = 2e_2 y \cos \delta \cos \theta - e_1 \sin \delta \sin \theta, \]

\[ f'_1 = \frac{1}{2} e_1 \cos \delta \cos \theta - 2e_2 y \sin \delta \sin \theta, \]

\[ f'_2 = e_2 y \cos \delta \cos \theta. \]

\[ R_2^* (0) = -n (n + 1) d_2 Q_n^s \]

\[-\frac{d_2}{2n+1} \sin \delta \left[ 2(n+2) Q_{n+1}^s + (n-1)(n+1)(2n-1) Q_{n-1}^s \right] \]

\[-\frac{d_2}{(2n+1)(2n+3)} \sin \delta \left[ 6(n+3) Q_{n+2}^s + (6n^3 + 13n^2 - 7n - 6) Q_n^s \right] \]

\[ + \frac{d_2}{(2n+1)(2n+3)} \cos \delta \left[ 3n(n+3) Q_{n+2}^s + (4n^2 - n^2(n+1) + 6) \right] \]

\[-(n+1)(2n+3) Q_n^s \].

As already stated, in the last two lines we have substituted \(\sigma = n - 1\) in working out the numerical coefficients, as also in all other terms with \(d_2\) as factor.

\[ R_2^* (1) = \frac{d_1}{2} \cos \delta \left[ -n (n+2) Q_{n+1}^{\sigma+1} + (n^2 - 1) Q_{n-1}^{\sigma+1} \right] \]

\[ + \frac{d_2}{(2n+1)(2n+3)} \cos \delta \left[ -2(n+3) Q_{n+2}^{\sigma+1} - (2n^3 + 3n^2 - 5n - 3) Q_n^{\sigma+1} \right] \]

\[ R_2^* (-1) = \frac{d_1}{2} \cos \delta \left[ n(n+2)(n-\sigma+2)(n-\sigma+1) Q_{n+1}^{\sigma-1} \right] \]

\[ - (n-1)(n+1)(n+\sigma)(n+\sigma-1) Q_{n-1}^{\sigma-1} \]

\[ + \frac{d_2}{(2n+1)(2n+3)} \cos \delta \sin \delta \left[ 24n(n+3) Q_{n+2}^{\sigma-1} + 2n(2n-1)(n+6) \right] \]

\[-2(n+1)(2n+3)(2n^2 - 4n + 3) Q_n^{\sigma-1} \].

\[ R_2^* (2) = -\frac{n(n+3)}{4(2n+1)(2n+3)} d_2 \cos \delta \cdot Q_{n+2}^{\sigma+2}. \]

\[ R_2^* (-2) = \frac{d_2}{(2n+1)(2n+3)} \left[ -30n(n+3) Q_{n+2}^{\sigma-2} \right] \]

\[ + \left\{ 3n(n+3)(2n-1)(n-1) + 3(n-1)(n-2)(n+1)(2n+1) \right\} Q_n^{\sigma-2} \]

\[-(n-2)^2(n^2 - 1)(2n-3)(2n+1) Q_{n-2}^{\sigma-2}. \]

If in the above expressions the term \(Q_n^{\sigma-1}\) occurs, it may be replaced by

\(-Q_n^{\sigma}/n(n+1)\).
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§ 19. Consider now the terms arising from a term $A_m r Q_m r' \sin (\tau' \lambda' - \alpha)$ in $\Psi$. The only term of corresponding type involving $d_0$, on the right-hand side of equation (4), which does not vanish, is

$$- p_m r' m' (m' + 1) d_0 Q_m r' \sin (\tau' \lambda' - \alpha').$$

Consequently, to order $1/d_0$,

$$p_m r' = - \frac{A_m r'}{m' (m' + 1) d_0}, \quad \text{and} \quad \alpha' = \alpha,$

and no other term is of order $1/d_0$. Next, taking terms of order $d_1/d_0$, it is evident that these can only arise from $R_m r' (0)$, $R_m r' (\pm 1)$, which involve $d_1 Q_m r' \pm 1$, $d_1 Q_m r' \pm 1$.

Equating the sum of the coefficients of the terms containing these harmonics, with factors $d_0$ and $d_1$, to zero, we get the following general expressions for $p_m r' \pm 1$, $p_m r' \pm 2$, to order $d_1/d_0$:

$$
\begin{align*}
 p_{m' + 1} r' &= - d_1 \sin \delta \frac{m' (m' - r + 1)}{d_0 (m' + 1) (2m' + 1)} p_{m'} r', \\
 p_{m' - 1} r' &= - d_1 \sin \delta \frac{(m' + 1) (m' + r)}{m' (2m' + 1)} p_{m'} r', \\
 p_{m' + 1} r' + 1 &= - \frac{1}{2} d_1 \cos \delta \frac{m'}{d_0 (m' + 1) (2m' + 1)} p_{m'} r', \\
 p_{m' - 1} r' + 1 &= + \frac{1}{2} d_1 \cos \delta \frac{m' + 1}{m' (2m' + 1)} p_{m'} r', \\
 p_{m' + 1} r' - 1 &= + \frac{1}{2} d_1 \cos \delta \frac{m' (m' - r + 2) (m' - r' + 1)}{(m' + 1) (2m' + 1)} p_{m'} r', \\
 p_{m' - 1} r' - 1 &= - \frac{1}{2} d_1 \cos \delta \frac{(m' + 1) (m' + r') (m' + r' - 1)}{m' (2m' + 1)} p_{m'} r'.
\end{align*}
$$

(5)

If the type of any of these coefficients exceeds the degree, it must be set equal to zero. No other terms are of order $d_1/d_0$.

So far no restriction has been placed on $m'$ and $r'$. In making a further approximation, we shall not write out general expressions, but shall consider the effect of the second order terms ($d_2/d_0$ and $d_1^2/d_0^2$) on two specific terms in the velocity potential of the atmosphere, viz., $Q \sin (\lambda' - \alpha)$ and $Q \sin (2\lambda' - \alpha)$, which give rise in $\Psi$ to terms

$$- Q_{r'} \sin (\lambda' - \alpha) \quad \text{and} \quad - \frac{1}{2} Q_{r'} \sin (2\lambda' - \alpha).$$

The terms of the proper order on the right-hand side of (4) are (a) those involving $d_2$ from $R_{m' r'} (\pm 2)$, and (b) those involving $d_1^2/d_0^2$ from $R_{m' \pm 1} (p)$ and $R_{m' \pm 1} (p)$ where $p = 0, \pm 1.$
Considering the diurnal variation first, the terms (a) are found, from the formulæ at the foot of p. 296, to be

$$\frac{1}{n}d_2 p_z \left[ \left( 6Q_4' - 13Q_4'' \right) \cos^2 \delta \sin \left( \lambda' - \alpha_1 \right) - \left( 12Q_4' + 16Q_4'' \right) \sin^2 \delta \sin \left( \lambda' - \alpha_1 \right) \right. \left. - \left( 4Q_4' + 3Q_4'' \right) \sin \left( 2\lambda' - \alpha_1 \right) - \left( 48Q_4'' - 6Q_4'' \right) \sin \left( -\alpha_1 \right) \right] \sin \delta \cos \delta$$

$$- \frac{1}{2} Q_4'' \cos^2 \delta \sin \left( 3\lambda' - \alpha_1 \right) + 3 \left( Q_4' - Q_4'' \right) \cos^2 \delta \sin \left( -\lambda' - \alpha_1 \right),$$

and the terms (b) are

$$p_0 d_1 \left[ \left( -\frac{1}{3} Q_4' - \frac{4}{3} Q_4'' \right) \sin \left( 2\lambda' - \alpha_1 \right) + \left( \frac{1}{3} Q_4' + \frac{3}{3} Q_4'' \right) \sin \left( \lambda' - \alpha_1 \right) \right. \left. \right] \cos \delta$$

$$- \left( \frac{1}{3} Q_4' + \frac{3}{3} Q_4'' \right) \sin \delta \sin \left( \lambda' - \alpha_1 \right),$$

$$+ p_1 d_1 \left[ - \frac{1}{3} Q_4' \sin \left( 2\lambda' - \alpha_1 \right) + Q_4'' \sin \left( -\alpha_1 \right) \right] \cos \delta - Q_4'' \sin \delta \sin \left( \lambda' - \alpha_1 \right),$$

$$+ p_0 d_1 \left[ - \frac{1}{3} Q_4' \sin \left( 3\lambda' - \alpha_1 \right) + \left( \frac{1}{3} Q_4' - \frac{5}{3} Q_4'' \right) \sin \left( \lambda' - \alpha_1 \right) \right] \cos \delta$$

$$- \left( \frac{1}{3} Q_4' + \frac{5}{3} Q_4'' \right) \sin \delta \sin \left( 2\lambda' - \alpha_1 \right),$$

$$+ p_0 d_1 \left[ \left( -\frac{1}{3} Q_4' + \frac{4}{3} Q_4'' \right) \sin \left( \lambda' - \alpha_1 \right) + \left( -\frac{5}{3} Q_4' + \frac{4}{3} Q_4'' \right) \sin \left( -\lambda' - \alpha_1 \right) \right] \cos \delta$$

$$- \left( \frac{5}{3} Q_4' + \frac{4}{3} Q_4'' \right) \sin \delta \sin \left( -\alpha_1 \right) \right],$$

$$+ p_1 d_1 \left[ - \frac{1}{3} Q_4' \sin \left( \lambda' - \alpha_1 \right) - \frac{1}{2} Q_4'' \sin \left( -\lambda' - \alpha_1 \right) \right] \cos \delta - 2 Q_4'' \sin \delta \sin \left( -\alpha_1 \right).$$

The sum of the coefficients of any particular term $Q_n^\sigma \sin \left( \sigma \lambda' - \alpha_1 \right)$ in the above, must be equated with $p_\sigma n \left( n + 1 \right) Q_n^\sigma \sin \left( \sigma \lambda' - \alpha_1 \right)$. The values of $p_\sigma^\sigma$ thus calculated are given in Table I. It should be remarked that $Q_n^{-1}$ has been replaced by $-Q_n'/n(n + 1)$, and the coefficient of $Q_n^1 \sin \left( \lambda' + \alpha_1 \right)$ will be denoted by $q_n^1$.

§ 20. The second order terms arising from the semi-diurnal atmospheric oscillation are similarly written down as follows:

\[(a) \quad p_0^2 d_2 \left[ \left( \frac{4}{3} Q_4^2 - 4 Q_4'' \right) \cos^2 \delta - \left( \frac{1}{3} Q_4^2 + 4 Q_4'' \right) \sin^2 \delta \right] \sin \left( 2\lambda' - \alpha_3 \right)\]

\[+ \left( \frac{1}{3} Q_4^2 - Q_4'' \right) \sin \left( 3\lambda' - \alpha_2 \right) + \left( \frac{4}{3} Q_4^2 - 6 Q_4'' \right) \sin \left( \lambda' - \alpha_2 \right) \right] \sin \delta \cos \delta\]

\[- \frac{1}{3} Q_4^4 \cos^2 \delta \sin \left( 4\lambda' - \alpha_2 \right) + 12 Q_4'^2 - \frac{1}{2} Q_4'^2 - \frac{2}{3} Q_4'' \right] \cos^2 \delta \sin \left( -\alpha_2 \right),\]

\[(b) \quad p_1^2 d_1 \left[ - \frac{1}{3} Q_4^2 \sin \left( 2\lambda' - \alpha_2 \right) + \left( \frac{2}{3} Q_4^2 - \frac{3}{3} Q_4' \right) \sin \left( -\alpha_2 \right) \right] \cos \delta\]

\[- \left( \frac{1}{3} Q_4^2 + \frac{3}{3} Q_4' \right) \sin \delta \sin \left( \lambda' - \alpha_2 \right),\]

\[+ p_1^2 d_1 \left[ \left( \frac{4}{3} Q_4^2 + \frac{8}{3} Q_4'' \right) \sin \left( 2\lambda' - \alpha_2 \right) + \left( \frac{4}{3} Q_4^2 - \frac{8}{3} Q_4'' \right) \sin \left( -\alpha_2 \right) \right] \cos \delta\]

\[- \left( \frac{3}{3} Q_4^2 + \frac{8}{3} Q_4' \right) \sin \delta \sin \left( \lambda' - \alpha_2 \right),\]

\[+ p_2^2 d_1 \left[ \left( -\frac{3}{3} Q_4^2 + \frac{8}{3} Q_4'' \right) \sin \left( 3\lambda' - \alpha_2 \right) + \left( \frac{3}{3} Q_4^2 - \frac{1}{3} Q_4' \right) \sin \left( \lambda' - \alpha_2 \right) \right] \cos \delta\]

\[- \frac{3}{3} Q_4^2 \sin \delta \sin \left( 2\lambda' - \alpha_2 \right),\]

\[+ p_3^2 d_1 \left[ \left( -\frac{3}{3} Q_4^2 + \frac{8}{3} Q_4'' \right) \sin \left( 3\lambda' - \alpha_2 \right) + 16 Q_4'^2 - 25 Q_4'' \right] \sin \left( \lambda' - \alpha_2 \right) \right] \cos \delta\]

\[- \left( 8 Q_4^2 + 10 Q_4'' \right) \sin \delta \sin \left( 2\lambda' - \alpha_2 \right),\]

\[+ p_4^2 d_1 \left[ \left( -\frac{3}{3} Q_4^2 + \frac{8}{3} Q_4'' \right) \sin \left( 4\lambda' - \alpha_2 \right) + 8 Q_4'^2 - 35 Q_4'' \right] \sin \left( 2\lambda' - \alpha_2 \right) \right] \cos \delta\]

\[- \left( \frac{1}{3} Q_4^2 + \frac{3}{3} Q_4' \right) \sin \delta \right].\]

The values of $p_\sigma$ calculated from the above expressions are given in Table II.
TABLE I.—Velocity Potential \( Q_1 \sin (\lambda + t - \alpha) \).

\[
p_2^1 = \frac{1}{6d_o} - \frac{5}{432} \frac{d_z^2 \cos^2 \delta}{d_o^3} - \frac{13}{252} \frac{\cos^2 \delta}{d_o^2} \left( \frac{d_z}{d_o} - \frac{d_z}{d_o} \right),
\]

\[
p_1^0 = -\frac{3}{20} \frac{d_z \cos \delta}{d_o^2},
\]

\[
p_3^0 = \frac{1}{15} \frac{d_z \cos \delta}{d_o^3},
\]

\[
p_2^1 = -\frac{3}{20} \frac{d_z \sin \delta}{d_o^2},
\]

\[
p_3^1 = -\frac{2}{45} \frac{d_z \sin \delta}{d_o^3},
\]

\[
p_2^2 = -\frac{1}{90} \frac{d_z \cos \delta}{d_o^3},
\]

\[
p_2^0 = \frac{1}{72} \frac{d_z \sin \delta \cos \delta}{d_o^3} - \frac{1}{42} \frac{\sin \delta \cos \delta}{d_o^2} \left( \frac{d_z}{d_o} - \frac{d_z}{d_o} \right),
\]

\[
p_2^0 = \frac{2}{35} \frac{\sin \delta \cos \delta}{d_o^3} \left( \frac{d_z}{d_o} - \frac{d_z}{d_o} \right),
\]

\[
p_2^1 = -\frac{1}{140d_o^2} \left( 3 \cos^2 \delta - 1 \right) \left( \frac{d_z}{d_o} - \frac{d_z}{d_o} \right),
\]

\[
p_z^0 = \frac{\sin \delta \cos \delta}{144} \frac{d_z^2}{d_o^3} - \frac{\sin \delta \cos \delta}{84d_o^2} \left( \frac{d_z}{d_o} - \frac{d_z}{d_o} \right),
\]

\[
p_2^2 = -\frac{\sin \delta \cos \delta}{210d_o^2} \left( \frac{d_z}{d_o} - \frac{d_z}{d_o} \right),
\]

\[
p_4^3 = -\frac{\cos^2 \delta}{1680d_o^2} \left( \frac{d_z}{d_o} - \frac{d_z}{d_o} \right),
\]

\[
q_z^1 = -\frac{\cos^2 \delta}{144} \frac{d_z^2}{d_o^3} + \frac{\cos^2 \delta}{84d_o^2} \left( \frac{d_z}{d_o} - \frac{d_z}{d_o} \right),
\]

\[
q_1^1 = -\frac{\cos^2 \delta}{280d_o^3} \left( \frac{d_z}{d_o} - \frac{d_z}{d_o} \right) + 2q_2
\]
Table II.—Velocity Potential $Q_x^2 \sin (2\lambda + \ell - \alpha)$. 

\[
p_3 = \frac{2}{15d_o} - \frac{d_1^2}{270d_o^3} \sin (2\lambda + \ell - \alpha)
\]

\[
p_4 = \frac{16}{63d_0^2} \cos \delta
\]

\[
p_4 = \frac{3}{70d_0^2} \cos \delta
\]

\[
p_5 = \frac{8}{63d_0^2} \sin \delta
\]

\[
p_6 = \frac{1}{35d_0^2} \sin \delta
\]

\[
p_7 = \frac{1}{140d_0^2} \cos \delta
\]

\[
p_1 = \frac{8}{35d_0^2} \cos \delta \left(d_2 - \frac{d_1^2}{d_0}\right)
\]

\[
p_2 = \frac{8}{36d_0^2} \cos^2 \delta \left(d_2 - \frac{d_1^2}{d_0}\right) + \frac{2}{15d_0^2} \cos \delta \left(d_2 - \frac{d_1^2}{d_0}\right)
\]

\[
p_3 = \frac{4}{105d_0^2} \cos \delta \left(d_2 - \frac{d_1^2}{d_0}\right)
\]

\[
p_4 = \frac{8}{35d_0^2} \sin \delta \cos \delta \left(d_2 - \frac{d_1^2}{d_0}\right)
\]

\[
p_5 = \frac{d_1^2}{72d_0^2} \sin \delta \cos \delta + \frac{\sin \delta \cos \delta}{15d_0^2} \left(d_2 - \frac{d_1^2}{d_0}\right)
\]

\[
p_6 = \frac{16}{525d_0^2} \sin \delta \cos \delta \left(d_2 - \frac{d_1^2}{d_0}\right)
\]

\[
p_7 = \frac{2}{525} \left(3 \cos^2 \delta - 1\right) \left(d_2 - \frac{d_1^2}{d_0}\right)
\]

\[
p_8 = \frac{d_1^2}{432d_0^2} \sin \delta \cos \delta - \frac{\sin \delta \cos \delta}{90d_0^2} \left(d_2 - \frac{d_1^2}{d_0}\right)
\]

\[
p_9 = \frac{4}{1575d_0^2} \sin \delta \cos \delta \left(d_2 - \frac{d_1^2}{d_0}\right)
\]

\[
p_{10} = \frac{\cos^2 \delta}{3150d_0^2} \left(d_2 - \frac{d_1^2}{d_0}\right)
\]
As this paper is primarily concerned with the lunar diurnal variation of the earth's magnetism, the numerical values of the coefficients $p_n$ arising from expression (2) owing to the inclination of the magnetic to the geographical axis will not be written down here. This can at once be done, when necessary, from the equations (5), as also the terms in the current potential arising from diurnal and semi-diurnal atmospheric oscillations of degree higher than the type.

§ 21. The main general result of our investigation is the same in form as that of SCHUSTER's more special calculations, viz., that the current function $R$ of electric flow induced under the action of the vertical force $C \cos \theta$ in a shell of air oscillating with a velocity potential $A_n Q_n \sin (\tau \lambda + t - \alpha)$, under the influence of a variable resistivity depending on the zenith distance ($\omega$) of the sun, is

$$
A_n \left[ \sum_{n=0}^{\infty} p_n Q_n \sin \{ \sigma (\lambda + t) \} + \sum_{n=1}^{\infty} q_n Q_n \sin \{ \sigma (\lambda + t) \} \right].
$$

In order to obtain the magnetic potential of the variation caused by the flow of air, a factor $-4 \pi (n+1)/(2n+1)$ must be inserted before each term $Q_n$. We have considered only those terms in the resistivity which depend on $\cos \omega$ and $\cos^2 \omega$, though the general theory has been given for any number of terms. If then

$$
\kappa = Cae (d_0 + d_1 \cos \omega + d_2 \cos^2 \omega),
$$
we have for the conductivity $\rho$, to the same degree of approximation,

$$
\rho = \frac{1}{Cae d_0} \left[ 1 - \frac{d_1}{d_0} \cos \omega - \frac{1}{d_0} \left( d_2 - \frac{d_1^2}{d_0} \right) \cos^2 \omega \right].
$$

If we put

$$
\frac{1}{Cae d_0} = \rho_0, \quad -\frac{d_1}{d_0} = \rho_1, \quad -\frac{1}{d_0} \left( d_2 - \frac{d_1^2}{d_0} \right) = \rho_2,
$$
this becomes

$$
\rho = \rho_0 + \rho_1 \cos \omega + \rho_2 \cos^2 \omega.
$$

In SCHUSTER's calculation, the last term was omitted, so that $\rho_2$ was taken equal to zero, while $\rho_1 \cos \delta$ and $\rho_1 \sin \delta$ were written $\rho_{0}'$ and $\rho_{0}''$ respectively. If we make these substitutions in Tables I. and II., it is readily verified that the present results, as far as they go, reduce to those obtained by SCHUSTER. The extra terms depending on $d_2 - \frac{d_1^2}{d_0}$ give the effect of the term $\cos^2 \omega$ in $\rho$.

§ 22. Finally, a word must be said with regard to the legitimacy of our analysis, considering the fact that if $\rho$ falls to zero, $\kappa$, the resistivity, must become infinite. Regarding the matter physically, it is evident that an infinite resistivity is not likely to introduce spurious terms into the current potential, and an examination of the equation (1) for $R$ will show that an actual infinity in $\kappa$ would only lead to a zero term in $P$. But such an infinite term should not occur in the analysis, and it
is clear that by altering the constant term in \( \rho \), so that \( \rho \) never falls to zero, the above calculations become formally and really legitimate; when we wish to return to the actual case we must appeal to the "law of continuity," and the fact that our mathematics is applied to an ordinary physical problem, to allow us to pass to the limiting value of \( d_0 \) in the final result. The latter is expressed as a power series in \( 1/d_0 \) and if \( d_0 \) is sufficiently diminished, this series might become non-convergent. But the actual results do not indicate any such behaviour, and are, as we have seen, identical with those obtained by SCHUSTER's method (in which the conductivity only was considered), so far as the scope of the two calculations is the same.

\[ 23. \] So far the calculations have been kept quite general, in that no relation between the causes of the variable conductivity and of the atmospheric oscillation has been assumed. Thus they may both be caused by the sun, in which case the mathematics is that applicable to the theory of the solar diurnal variations of the earth's magnetism. Without much modification, however, they may equally well be adapted to the case of the lunar diurnal variations. We shall consider it sufficient, for our purpose, to regard the solar and lunar periods as equal at any one time, allowing for the slow cumulative effect of their inequality by introducing a variable phase angle \( \nu \) into the expression for \( \cos \omega \), the quantity on which \( \rho \) and \( \alpha' \) depend. Thus

\[
\cos \omega = \sin \delta \cos \theta + \cos \delta \sin \theta \cos (\lambda + t' + \nu),
\]

where \( t' \) is now the local lunar time of the standard meridian (measured from upper culmination), and \( \nu \) measures the lunar phase, increasing from 0 to \( 2\pi \) from one new moon to the next. The velocity potential will be \( Q_2^2 \sin (2\lambda' + t' - \alpha) \). The calculations will be formally the same if we now change the meaning of \( \lambda' \) to \( \lambda + t' + \nu \), so that the velocity potential becomes

\[
Q_2^2 \sin (2\lambda' - \alpha - 2\nu).
\]

Thus by equation (6) the current function obtained is

\[
\sum_{\sigma=0}^{\infty} p_\sigma Q\sigma \sin (\sigma\lambda' - \alpha - 2\nu) + \sum_{\sigma=1}^{\infty} q_\sigma Q\sigma \sin (\sigma\lambda' + \alpha + 2\nu)
\]

\[
= \sum_{\sigma=0}^{\infty} p_\sigma Q\sigma \sin (\sigma\lambda + t' - \alpha + (\sigma - 2)\nu) + \sum_{\sigma=1}^{\infty} q_\sigma Q\sigma \sin (\sigma\lambda + t' + \alpha + (\sigma + 2)\nu).
\]

The terms on the left of the last line change in phase through an angle \( 2(\sigma - 2)\pi \) each month, viz., \(-2\pi\) for the diurnal term, zero for the semi-diurnal term, \(+2\pi\) for the third component, and \(+4\pi\) for the fourth component, as the observations indicated. The terms on the left change phase by \( 2(\sigma + 2)\pi \) each month, a change so rapid that it would be difficult to detect in the observations, affected as these are by accidental error. The coefficients \( q_\sigma \), moreover, are very small, so that altogether these terms are negligible.

One interesting result of the analysis may be noticed here, viz., that the main
lunar term in the magnetic variation, $Q_a^2$, has a coefficient $p_a^*$ which does not (to the order of accuracy of our calculations) show any dependence on solar declination. Thus any seasonal change in this term of the magnetic potential cannot be referred to the effect of the varying declination of the sun. This is not quite the case with regard to the main diurnal term in the solar diurnal magnetic variation.

§ 24. We will now consider what are likely values of $\rho_a/\rho_0$ and $\rho_d/\rho_0$ to substitute in our formulae, in order to get a comparison with the observed data. The conductivity should rise to a maximum during the daytime and fall to a minimum about midnight. It cannot actually be less than zero, though it is not so clear that it is better to have the least value of $\rho$ zero than to have it slightly less, in order to make the mean nightly conductivity small in amount. However, we will keep to this condition, and make $\rho_{\text{min.}} = 0$; it is found that the following is a very satisfactory expression for the representation of a function of $\theta$ which is large for values of $\theta$ up to $\frac{\pi}{2}$, and much smaller, while never negative, from $\theta = \frac{\pi}{2}$ to $\pi$:

$$\rho = \rho_0 (1 + 3a + \frac{3}{2}b^2).$$

The following table and figure gives the value of $4\rho/\rho_0$ for every 10°. It is seen that the mean of the nine day values is 24·1 times that of the nine night values. The function has a physically false maximum at midnight, but this is of very small amount, and some such feature cannot be avoided with so simple an expression for $\rho$:

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>0°</th>
<th>10°</th>
<th>20°</th>
<th>30°</th>
<th>40°</th>
<th>50°</th>
<th>60°</th>
<th>70°</th>
<th>80°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4\rho/\rho_0$</td>
<td>25·0</td>
<td>24·5</td>
<td>22·2</td>
<td>21·1</td>
<td>18·5</td>
<td>15·4</td>
<td>12·2</td>
<td>9·2</td>
<td>6·4</td>
<td>4·0</td>
</tr>
<tr>
<td>$\omega$</td>
<td>100°</td>
<td>110°</td>
<td>120°</td>
<td>130°</td>
<td>140°</td>
<td>150°</td>
<td>160°</td>
<td>170°</td>
<td>180°</td>
<td></td>
</tr>
<tr>
<td>$4\rho/\rho_0$</td>
<td>2·2</td>
<td>0·9</td>
<td>0·2</td>
<td>0·0</td>
<td>0·1</td>
<td>0·4</td>
<td>0·6</td>
<td>1·0</td>
<td>1·0</td>
<td></td>
</tr>
</tbody>
</table>

* I might remark here that in working out Part II. of this paper I had not contemplated the possibility of the coefficients of $\rho/\rho_0$ being greater than unity, as seems to be necessary if the atmospheric conductivity is small and nearly constant at night. The size of these coefficients makes it necessary to carry the calculations some steps further than I have already done, before a sufficient degree of approximation is arrived at. The present work suffices, however, to establish the point with which I am most immediately concerned, viz., that the size of the fourth harmonic in the lunar variation is inexplicable with the form $a + b \cos \omega$ for $\rho$, while the addition of a term $c \cos^2 \omega$ introduces a fourth harmonic in the theoretical result, which agrees, as to order of magnitude, with the observed quantity. Better observed data are now being obtained, and concurrently I shall proceed to carry the theoretical calculations further, in order to test the exact numerical agreement between theory and observation.—June 11, 1913.
Diagram illustrating the assumed form for the atmospheric conductivity

\[ \rho = \rho_0 (1 + 3 \cos \omega + \frac{3}{8} \cos^2 \omega). \]

§ 25. Substituting the values

\[ \frac{d_1}{d_0} = +3, \quad \left( \frac{d_2 - \frac{d_3}{d_0}}{d_0} \right) = +\frac{3}{4} \]

in the expressions for \( \rho_n^* \) in Table II. (the table which relates to the lunar variation), we get the following values for \( \rho_0 \text{Caep}_n^* \). The terms for which \( \sigma = 0 \) are omitted, as they merely produce a monthly change in the mean magnetic elements.

\[ \rho_0 \text{Caep}_n^*. \]

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>1. ( \frac{1}{3} \sin \delta \cos \delta )</th>
<th>2. ( \frac{4}{21} \cos \delta )</th>
<th>3. ( \frac{1}{16} \sin \delta \cos \delta )</th>
<th>4. ( \frac{-3}{10} \cos \delta )</th>
<th>5. ( \frac{3}{10} \sin \delta \cos \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{8}{21} \sin \delta )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{4} \sin \delta \cos \delta )</td>
<td>( \frac{1}{10} \cos \delta )</td>
<td>( \frac{1}{100} \cos^2 \delta )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{8}{21} \sin \delta )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{4} \sin \delta \cos \delta )</td>
<td>( \frac{1}{10} \cos \delta )</td>
<td>( \frac{1}{100} \cos^2 \delta )</td>
</tr>
</tbody>
</table>
The following are the values of the corresponding tesseral harmonics:

\[
\begin{align*}
Q_1 &= \sin \theta, & Q_3 &= 3 \sin \theta \cos \theta, & Q_5 &= \frac{3}{2} \sin \theta (5 \cos^2 \theta - 1), \\
Q_2 &= \frac{1}{2} \sin \theta (70 \cos^2 \theta - 15 \cos \theta), & Q_4 &= \frac{3}{2} \sin \theta (63 \cos^4 \theta - 84 \cos^2 \theta + 3), \\
Q_3 &= 3 \sin^2 \theta, & Q_5 &= 15 \sin^2 \theta \cos \theta, & Q_6 &= \frac{1}{2} \sin^2 \theta (14 \cos^2 \theta - 1), \\
Q_4 &= 1 \sin^3 \theta, & Q_6 &= 105 \sin^3 \theta \cos \theta, & Q_7 &= \frac{1}{2} \sin^3 \theta (9 \cos^2 \theta - 2), \\
Q_5 &= 105 \sin^4 \theta, & Q_7 &= 945 \sin^4 \theta \cos \theta.
\end{align*}
\]

Since all the stations for which we have observational data, in Part III., are tropical, we shall consider the values of X, Y, and Z for such stations only. Hence, in our expression for V, the magnetic potential (which we must now use instead of the current function), all terms containing \( \cos^2 \theta \) may be neglected, and will be omitted. Thus we get

\[
V = \left( \frac{3}{2} \cos \delta \sin \theta \cos \theta \frac{\sin \delta \cos \delta \sin \theta}{r^2} + \frac{3}{2} \sin \delta \cos \delta \frac{\sin \theta}{r^2} \right) \cos (\lambda' + \nu - \alpha),
\]

\[
+ \left( \frac{3}{2} \sin \delta \cos \delta \frac{\sin \theta \cos \theta}{r^2} + \frac{3}{2} \sin \delta \cos \delta \frac{\sin \theta}{r^2} \right) \cos (2\lambda' - \alpha),
\]

\[
+ \left( \frac{3}{2} \cos \delta \cos \delta \frac{\sin \theta \cos \theta}{r^2} + \frac{3}{2} \cos \delta \cos \delta \frac{\sin \theta}{r^2} \right) \cos (3\lambda' - \nu - \alpha),
\]

\[
+ \frac{3}{2} \cos \delta \cos \delta \frac{\sin \theta \cos \theta}{r^2} \cos (4\lambda' - 2\nu - \alpha).
\]

In the above expression, the terms depending on \( \sin \delta \) represent the main seasonal effect. Since

\[
\alpha X = \frac{dV}{d\theta}, \quad \alpha Y = \frac{dV}{\sin \theta d\lambda}, \quad Z = -\frac{dV}{dr},
\]

it is evident that when \( \cos \theta \) is put equal to zero after the differentiation, only the terms in \( V \) which do not contain \( \cos \theta \) will contribute any result to \( Y \) and \( Z \). But the above equation shows also that these terms always contain \( \sin \delta \), so that at equatorial stations \( Y \) and \( Z \) change sign in passing from summer to winter. Tables XI. and XIII. corroborate this sufficiently well, especially when it is remembered that the stations are not quite equatorial, and that the obliquity of the magnetic axis also produces a disturbing effect. A further interesting consequence of the above equations is to indicate that at the equator the terms in \( X \) which depend on \( \sin \delta \), i.e., the seasonal terms in the horizontal force variation, vanish. This agrees with the known fact that at tropical stations the \( X \) variation hardly changes throughout the year. Table XII. illustrates this, especially for the most nearly equatorial observatory, Batavia (6° S.).

For comparison with observation we shall write down the values of the ratios of the amplitudes of the first, third, and fourth harmonic components to that of the
second; for X we take the mean value of $\cos \delta$ in our equations, and neglect the seasonal changes; for Y and Z the terms in $\cos \theta$ and $\sin \delta$ are taken separately. The values of the amplitudes of the second component in the several cases are also given. It should be remarked that our calculations have not been carried sufficiently far to give the seasonal variation of the fourth component, but it is less important than the term in $\cos \theta$, for such stations as Bombay. We thus obtain the following table:

<table>
<thead>
<tr>
<th></th>
<th>$C_1/C_2$</th>
<th>$C_2/C_2$</th>
<th>$C_3/C_2$</th>
<th>$C_4/C_2$</th>
<th>$C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0.61</td>
<td>0.47</td>
<td>0.13</td>
<td>2.61</td>
<td></td>
</tr>
<tr>
<td>Y ($\cos \theta$)</td>
<td>0.31</td>
<td>0.70</td>
<td>0.27</td>
<td>5.22$\cos \theta$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td>0.55</td>
<td></td>
<td>1.01$\sin \delta$</td>
<td></td>
</tr>
<tr>
<td>Z ($\cos \theta$)</td>
<td>0.41</td>
<td>0.62</td>
<td>0.22</td>
<td>7.83$\cos \theta$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.49</td>
<td></td>
<td>1.52$\sin \delta$</td>
<td></td>
</tr>
</tbody>
</table>

From Schuster's calculations, taking $\rho/\rho_0 = 1 + \cos \omega$, the following table of values of $C/C_2$, in which the seasonal changes are disregarded, is obtained:

<table>
<thead>
<tr>
<th></th>
<th>$C_1/C_2$</th>
<th>$C_2/C_2$</th>
<th>$C_4/C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0.67</td>
<td>0.38</td>
<td>0.002</td>
</tr>
<tr>
<td>Y ($\cos \theta$)</td>
<td>0.33</td>
<td>0.58</td>
<td>0.003</td>
</tr>
<tr>
<td>Z ($\cos \theta$)</td>
<td>0.62</td>
<td>0.46</td>
<td>0.0025</td>
</tr>
</tbody>
</table>

Our observational data only allow us to make the roughest possible comparison with these calculations, and the following table is enough to give an idea of what agreement is present. It is got by taking the mean amplitudes at Bombay, Batavia, and Trevandrum (as many as afford data in each case) for the whole year, combining the columns April to September and October to March together by simply averaging the amplitudes regardless of phase.

<table>
<thead>
<tr>
<th></th>
<th>$C_1/C_2$</th>
<th>$C_2/C_2$</th>
<th>$C_4/C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0.94</td>
<td>0.42</td>
<td>0.23</td>
</tr>
<tr>
<td>Y</td>
<td>0.50</td>
<td>0.64</td>
<td>0.23</td>
</tr>
<tr>
<td>Z</td>
<td>0.85</td>
<td>1.06</td>
<td>0.47</td>
</tr>
</tbody>
</table>

The size of the fourth harmonic shows that the term $\cos^2 \omega$ in $\rho/\rho_0$ has distinct importance, for without the presence of such a term, as the second of the above tables show, there should be no appreciable fourth harmonic at all. As regards the other harmonics, there is little to chose between the two expressions for $\rho/\rho_0$ though
the more complex one might be made to fit better than the above figures indicate, if the constants of the formula were altered a little. This, however, is not worth while doing till better observational material is to hand.

**PART III.—The Observational Material.**

§ 26. The following are the data which were available for examining the dependence of the lunar magnetic variation upon lunar phase:

<table>
<thead>
<tr>
<th>Station and period</th>
<th>Sub-division of month</th>
<th>Seasonal division</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trevandrum (1854–64)</td>
<td>Four quarters of month</td>
<td>Separate months of year</td>
</tr>
<tr>
<td>&quot; (Moos) (1872–89)</td>
<td>&quot; &quot;</td>
<td>Nov.–Jan.</td>
</tr>
<tr>
<td>Batavia (1883–99)</td>
<td>&quot; &quot;</td>
<td>April–Sept., Oct.–March.</td>
</tr>
<tr>
<td>Bombay (CHAMBERS) (1846–73)</td>
<td>Eight phases</td>
<td>As for declination.</td>
</tr>
<tr>
<td>&quot; (Moos) (1872–89)</td>
<td>&quot; &quot;</td>
<td>Nov.–Jan.</td>
</tr>
<tr>
<td>&quot; &quot; (1873–79, 1881–1883–85)</td>
<td>&quot; &quot;</td>
<td>May–July.</td>
</tr>
<tr>
<td>Batavia (1883–99)</td>
<td>&quot; &quot;</td>
<td>April–Sept., Oct.–March.</td>
</tr>
<tr>
<td>Bombay (Moos)</td>
<td>Eight phases</td>
<td>As for declination.</td>
</tr>
<tr>
<td>Batavia (1883–99)</td>
<td>&quot; &quot;</td>
<td>&quot; &quot;</td>
</tr>
</tbody>
</table>

For purposes of comparison, the Trevandrum results for the separate months of the year have been combined into the four quarters and the two half years (as for Bombay); also the 25 hourly values have been reduced to 24.

The separate tables of the 24 hourly values will not be repeated here, nor the $a$ and $b$, and $C$ and $\theta$ coefficients of the first four harmonic components which have been calculated from those tables. The harmonic formula used has been

$$a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + a_3 \cos 3t + b_3 \cos 3t + a_4 \cos 4t + b_4 \sin 4t,$$

$$C_1 \sin (t + \theta_1) + C_2 \sin (2t + \theta_2) + \ldots .$$

In the case of all the coefficients $a$, $b$, $C$, the adopted unit is $10^{-7}$ C.G.S. units of force (the declination results were also reduced in terms of force), and this was reckoned positive towards the North, West, and upwards ($H$, $D$, $V$).

§ 27. The tables of harmonic coefficients showed that they were subject to an accidental error of amount small in itself, but quite a considerable fraction of the
whole effect. This is not surprising when the minuteness of the lunar variation is considered. The tables showed some outstanding features, however, in particular the constancy (within reasonable limits) of $C_2$ and $\theta_2$; $C_1$, $C_3$, $C_4$ are generally smaller and rather more irregular in amount for the eight phases. The phase angles $\theta_n$ showed a fairly regular increase through $2\pi$ with the moon's age, while $\theta_1$ showed a less regular decrease of the same amount. No regular change in $\theta_1$ was noticed, partly because $C_4$ is small and $\theta_4$ therefore not well determined, till the Batavian summer declination results were considered; in this case the fourth harmonic happened to be exceptionally large, and the phase therefore better determined. This clue having once been obtained, the same feature, viz., a monthly increase of $4\pi$ in the phase angle $\theta_n$, was verified to be present in most other cases, where $C_4$ was not too small. The examination of the phase laws followed by the harmonic components was first undertaken by means of vector diagrams, and independently of the theoretical considerations which suggested themselves later, and which are embodied in §§ 12, 23.

The real test of the phase laws suggested by the vector diagrams was made, of course, by correcting the phases by the amount through which the law indicated they had changed from the period of new moon. The corrected values, $\theta'$ (where

$$\theta'_1 = \theta_1 + \nu,$$

$$\theta'_2 = \theta_2,$$

$$\theta'_3 = \theta_3 - \nu,$$

$$\theta'_4 = \theta_4 - 2\nu,$$

$\nu$ being the moon's age, in angular measure, at the particular lunar phase considered) should then all be the same (for the same value of the suffix and different values of $\nu$), apart from accidental error. The Tables III. to X. show that this is the case, generally, as far as we have any right to expect, though, in some instances, the agreement is not very apparent. Even in these cases, however, the mean value of $\theta'$ frequently agrees so closely with the mean value of $\theta'_2$ as to show that the phase law is acting, though its manifestation is obscured by large accidental error. This agreement between $\theta'_1$, $\theta'_2$, $\theta'_3$, and $\theta'_4$ is a noticeable feature, of which, as well as of the monthly changes of phase, the theory of the lunar variation gives a satisfactory account (§ 23). On general grounds, too, it is to be expected that if any simple relation exists at all between the phase angles of the four harmonic components, this relation should assume the simplest form (which proves to be equality) at new moon, when the sun and moon are on the same meridian. The equality of the phase angles at new moon points to a single exciting cause (the lunar atmospheric tide being suggested) of the four components.

The regular monthly change in the values of $\theta_1$, $\theta_3$, and $\theta_4$ results in the
disappearance of the corresponding harmonic components from the lunar variation, as determined from the mean of a whole number of months. It is found, indeed, that any such component still remaining is of purely accidental character.

As to the amplitude of the various components, this appears to be independent of the lunar phase, the irregularities being accidental. The mean of the amplitudes at the separate phases has therefore been taken as the best value of the true amplitude, except that a correction has been applied to allow for the fact that the instantaneous amplitude is greater than that deduced from the mean of a few days, during which the phase is varying. Thus, if we tabulate a function \( c \cos (n\theta + kv) \), where \( \theta \) is the lunar hour angle (one hour = 15°) and \( v \) the age of the moon in angular measure, in lunar hours for successive days over an interval of the month \( v_1 \) to \( v_2 \), the mean result may be taken as

\[
c(\cos n\theta \cos kv - \sin n\theta \sin kv)
\]

where \( \cos kv, \sin kv \) are the mean values of these functions over the range \( v_1 \) to \( v_2 \). This equals

\[
\frac{\sin \frac{k}{2}(v_1 - v_2)}{C \frac{k}{2} \cos \left(n\theta + \frac{k}{2}v_1 + v_2\right)}
\]

showing that the phase of the mean wave is equal to the true phase at the mean time, but that the amplitude is reduced in the ratio

\[
\frac{\sin \frac{k}{2}(v_1 - v_2)}{\frac{k}{2}(v_1 - v_2)}
\]

The corresponding factors to counterbalance this are for Trevandrum, where \( v_2 - v_1 \) is one-quarter of a month, or \( \frac{\pi}{2} \),

\[
1.11 (C_1, C_3) \quad \text{and} \quad 1.57 (C_4),
\]

and at other stations, where \( v_2 - v_1 = \frac{\pi}{4} \),

\[
1.02 (C_1, C_3) \quad \text{and} \quad 1.11 (C_4).
\]

The values of the mean amplitudes, thus corrected, and of the phases of the four components, reduced to new moon, for all the stations, are summarized in Tables XI.–XIII.

The resolved parts of the amplitudes in the direction of the mean phase (where the separate values of \( \theta' \) depart much from the mean) might have been taken, but this would not have altered the mean amplitude greatly, and seemed hardly worth
while in view of the large accidental variations of the determined amplitudes. In Tables XI.–XIII. the values of the mean phases $\theta'$ have been characterized by affixes 1, 2, 3, 4, 5, representing (in descending order of merit) the reliability of the mean as judged from the accordance of the separate values of $\theta'$. Only the numbers marked 1 to 3 can be considered at all satisfactory.

As regards the accordance of the results from the same or different stations, the best feature is the extremely good agreement between Chambers' and Moos's values for Bombay, for different periods of time and for different instruments.*

* Van Bemmelen, in his paper in the 'Met. Zeitschr.,' May, 1912, already referred to, remarks that the two computations do not agree at all, but this must evidently be due to a mistaken reduction of Chambers' results, which he quotes at three times their proper value.

**Table III.**—Trevandrum. Declination West.

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<tr>
<th>Lunar phase</th>
<th>$C_1$</th>
<th>$\theta'_1$</th>
<th>$C_2$</th>
<th>$\theta'_2$</th>
<th>$C_3$</th>
<th>$\theta'_3$</th>
<th>$C_4$</th>
<th>$\theta'_4$</th>
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<td>220</td>
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<td>263</td>
<td>66</td>
<td>286</td>
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The unit of force in the tables of amplitude is $10^{-7}$ C.G.S. unit.
### Table III. (continued)

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<th>$\theta_{1}$</th>
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<th>$\theta_{2}$</th>
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<th>$\theta_{3}$</th>
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The unit of force in the tables of amplitude is $10^{-7}$ C.G.S. unit.
DR. S. CHAPMAN ON THE DIURNAL VARIATIONS OF THE

Table IV.—Bombay (Chambers). Declination West.

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<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
<th>C₅</th>
<th>C₆</th>
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Mean . . 44 103 83 129 51 115 18 117

April–September.

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<th>C₃</th>
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<th>C₅</th>
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Mean . . 45 96 66 109 52 106 14 106

October–March.

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<th>C₃</th>
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Mean . . 30 246 63 241 14 251 12 232

The unit of force in the tables of amplitude is $10^{-7}$ C.G.S. unit.
### Table IV. (continued)

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The unit of force in the tables of amplitude is $10^{-7}$ C.G.S. unit.
DR. S. CHAPMAN ON THE DIURNAL VARIATIONS OF THE

TABLE V.—Bombay (Moos). Declination West.

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TABLE VI.—Batavia. Declination West.

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<th>C₂</th>
<th>θ₂</th>
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|          |     |     |     |     |     |     |     |     |
| October–March. |    |    |    |    |    |    |    |    |
| 1 | 85 | 280 | 238 | 278 | 153 | 291 | 16 | 299 |
| 2 | 44 | 177 | 172 | 266 | 126 | 290 | 61 | 300 |
| 3 | 34 | 346 | 175 | 268 | 131 | 288 | 62 | 237 |
| 4 | 83 | 171 | 130 | 257 | 104 | 289 | 51 | 395 |
| 5 | 84 | 276 | 237 | 288 | 148 | 296 | 68 | 320 |
| 6 | 68 | 178 | 154 | 264 | 144 | 283 | 68 | 301 |
| 7 | 12 | 347 | 181 | 264 | 131 | 282 | 32 | 304 |
| 8 | 53 | 173 | 148 | 260 | 97 | 284 | 23 | 323 |
| Mean | 58 | 243 | 179 | 268 | 129 | 287 | 48 | 310 |

The unit of force in the tables of amplitude is $10^{-7}$ C.G.S. unit.
### Table VII.—Bombay (Chambers). Horizontal Force.

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| February–April. |
|-----------------|-------|-------------|-------|-------------|-------|-------------|-------|-------------|
| 1               | 68    | 185         | 98    | 164         | 18    | 190         | 15    | 34          |
| 2               | 96    | 191         | 90    | 165         | 47    | 202         | 52    | 190         |
| 3               | 78    | 184         | 110   | 156         | 42    | 172         | 8     | (135)       |
| 4               | 97    | 148         | 93    | 182         | 27    | 285         | 64    | 222         |
| 5               | 77    | 145         | 48    | 179         | 11    | (297)       | 24    | 178         |
| 6               | 143   | 181         | 45    | 155         | 21    | 199         | 12    | 124         |
| 7               | 24    | (25)        | 30    | (250)       | 30    | 344         | 29    | 135         |
| 8               | 189   | 181         | 70    | 176         | 54    | 225         | 52    | 95          |
| Mean ...        | 97    | 174         | 73    | 168         | 31    | 231         | 32    | 140         |

| May–July. |
|-----------|-------|-------------|-------|-------------|-------|-------------|-------|-------------|
| 1         | 62    | 225         | 50    | 275         | 20    | 158         | 18    | 46          |
| 2         | 71    | 157         | 88    | 163         | 9     | 103         | 9     | 60          |
| 3         | 52    | 214         | 83    | 174         | 47    | 135         | 12    | 12         |
| 4         | 33    | 151         | 50    | 101         | 20    | 82          | 18    | 8           |
| 5         | 35    | 126         | 16    | 184         | 17    | 164         | 16    | 82          |
| 6         | 50    | 283         | 60    | 243         | 9     | 25          | 11    | 62          |
| 7         | 46    | 237         | 16    | 297         | 13    | 66          | 34    | 66          |
| 8         | 152   | 152         | 60    | 172         | 39    | 269         | 38    | 24          |
| Mean ... | 64    | 193         | 53    | 202         | 22    | 125         | 20    | 34          |

The unit of force in the tables of amplitude is $10^{-7}$ C.G.S. unit.

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Table VII. (continued).

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| **April–September.** |       |           |       |           |       |           |       |           |
| 1           | 82    | 211       | 43    | 239       | 9     | 237       | 20    | 54        |
| 2           | 74    | 195       | 91    | 163       | 8     | 123       | 18    | 141       |
| 3           | 77    | 234       | 76    | 181       | 8     | 162       | 20    | – 15      |
| 4           | 72    | 160       | 53    | 158       | 24    | 219       | 9     | 80        |
| 5           | 35    | 162       | 33    | 197       | 15    | 151       | 17    | 110       |
| 6           | 33    | 212       | 35    | 197       | 12    | 127       | 7     | 26        |
| 7           | 23    | 224       | 18    | 278       | 3     | –         | 22    | 68        |
| 8           | 136   | 118       | 61    | 171       | 16    | 264       | 11    | – 31      |
| **Mean**    | 66    | 190       | 51    | 198       | 12    | 183       | 15    | 54        |

| **October–March.** |       |           |       |           |       |           |       |           |
| 1           | 78    | 173       | 115   | 177       | 34    | 176       | 15    | 171       |
| 2           | 58    | 201       | 92    | 181       | 51    | 206       | 27    | 139       |
| 3           | 74    | 140       | 93    | 149       | 50    | 187       | 10    | 138       |
| 4           | 101   | 153       | 115   | 179       | 8     | 180       | 29    | 205       |
| 5           | 71    | 147       | 93    | 171       | 37    | 231       | 32    | 205       |
| 6           | 149   | 185       | 83    | 172       | 39    | 253       | 14    | 287       |
| 7           | 52    | 186       | 85    | 187       | 30    | 201       | 30    | 183       |
| 8           | 170   | 193       | 113   | 169       | 52    | 201       | 13    | 166       |
| **Mean**    | 94    | 172       | 100   | 173       | 38    | 204       | 21    | 187       |

The unit of force in the tables of amplitude is $10^{-7}$ C.G.S. unit.
TABLE VIII.—Bombay (Moos). Horizontal Force.

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The unit of force in the tables of amplitude is $10^{-7}$ C.G.S. unit.
DR. S. CHAPMAN ON THE DIURNAL VARIATIONS OF THE

Table IX.—Batavia. Horizontal Force.

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Table X.—Bombay (Moos). Vertical Force (upwards).

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The unit of force in the tables of amplitude is \( 10^{-7} \) C.G.S. unit.
Table X. (continued.—Batavia. Vertical Force.

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October—March.

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The unit of force in the tables of amplitude is \(10^{-7}\) C.G.S. unit.
Table XI.—Declination West.

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Trevandrum.

Bombay (Chambers).

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Bombay (Moos).

Batavia.

The unit of force in the tables of amplitude is $10^{-7}$ C.G.S. unit.
**EARTH'S MAGNETISM PRODUCED BY THE MOON AND SUN.**

### TABLE XII.—Horizontal Force.

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**Batavia.**

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**Bombay (Moos).**

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**Batavia.**

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The unit of force in the tables of amplitude is 10\(^{-7}\) C.G.S. unit.

**VOL. CCXIII.—A.**
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**Note**: The table continues with more data entries that are not fully visible in the image provided.

By W. M. Hicks, F.R.S.

Received June 7,—Read June 26, 1913.

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ABBREVIATIONS.

[I.] and [II.] denote the two previous parts of this discussion published respectively in the 'Phil. Trans.,' A, vol. 210 (1910), and vol. 212 (1912).

The formula for a line is \( n = \frac{N}{D_m^2} - \frac{N}{D_n^2} \).

\( \frac{N}{D_m^2} \) is the limit or value when \( m = \infty \).

\( \xi \) denotes the correction to be added to any limit adopted to give the true value.

\( \frac{N}{D_n^2} \) is referred to as the V part (variable).

\( D_m \) is referred to as the denominator of the line.

"Separation" of two lines is the difference of their wave numbers.

"Difference" of two lines is the difference of their denominators.

"Mantissa" is the decimal part of the denominator.

\( v \) denotes the separation of two lines of a doublet.

\( \Delta \) is used for the denominator difference of the two lines which produces \( v \).

\( v_1, v_2, \Delta_1, \Delta_2 \) are similar quantities for triplets.

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W denotes the atomic weight, \( w = W/100 \).
\( \delta_i \) denotes the "own" \( = 90 \cdot 47\text{\textsuperscript{a}}\).
\( \delta_i = n\delta_i \), but \( \delta \) is used for \( \delta_i = 361 \cdot 89\text{\textsuperscript{a}} \).

O - C is used for the difference in wave-length between an observed line and its calculated value.
O denotes the maximum possible error of observation.

In general, figures in brackets before lines denote intensities, and in brackets after, possible errors of observation.

The doublet and triplet separations in the spectra of elements are, as has long been known, roughly proportional to the squares of their atomic weights, at least when elements of the same group of the periodic table are compared. In the formula which give the series lines these separations arise by certain terms being deducted from the denominator of the typical sequences. For instance, in the alkalies if the \( p \)-sequence be written \( N/N_{m}^2 \), where \( D_{m} = m + \mu + \alpha/m \), the \( p \)-sequence for the second principal series has denominator \( D_ - \Delta \), and we get converging doublets; whereas the constant separations for the \( S \) and \( D \) series are formed by taking \( S_1(\infty) = D_1(\infty) = N/D_1^2 \) and \( S_2(\infty) = D_2(\infty) = N/(D_1 - \Delta)^2 \). It is clear that the values of \( \Delta \) for the various elements will also be roughly proportional to the squares of the atomic weights. For this reason it is convenient to refer to them as the atomic weight terms. We shall denote them by \( \Delta \) in the case of doublets and \( \Delta_1 \) and \( \Delta_2 \) in the case of triplets, using \( \nu \) as before to denote the separations. Two questions naturally arise. On the one hand what is the real relation between them and the atomic weights, and on the other what relation have they to the constitution of the spectra themselves? The present communication is an attempt to throw some light on both these problems.

The Dependence of the Atomic Weight Term on the Atomic Weight.

The values of the \( \Delta \) can be obtained with very considerable accuracy, especially in the case of elements of large separations, \( i.e., \) of large atomic weight. If, therefore, the definite relation between these quantities can be obtained, not only may it be expected to give some insight into the constitution of the vibrating systems which give the lines, but it may afford another avenue whereby the actual atomic weights of elements may be obtained, and the solution of the problem is therefore of importance to the chemist as well as to the physicist.

It may be interesting to note the steps which first led the author to the solution which follows, and incidentally may add some weight to the formal evidence in its favour. It has long been known that in the case of triplets the ratio of \( \Delta_1 : \Delta_2 \) is always slightly larger than 2. It was natural, therefore, in an attempt to discover their relation to the atomic weight to consider the values of \( \Delta_1 - 2\Delta_2 \). These were calculated for several cases, \( \Delta_1 \) and \( \Delta_2 \) being expressed in terms of the squares of the atomic weights. It was at once noticed that in several cases these differences were multiples of the same number, in the neighbourhood of 360, \( e.g., \) Ca 1, Sr 3, Ba 8,
Hg 19, and, further, that in many cases $\Delta_1$ and $\Delta_2$ were also themselves multiples of the same number. As, however, Mg with a difference of 450 and Zn of 543 could not possibly be brought into line with the others, this line of attack was given up. But later the case of Zn, which at first had seemed to stand in the way of an explanation on these lines, gave cause for encouragement. The series for Zn are well defined, the measures good, and the formulæ reproduce the lines with great accuracy.* Great confidence can thus be put in the values for $\Delta_1$ and $\Delta_2$, and it was noticed that they were both extremely exact multiples of the difference $\Delta_1-2\Delta_2$. In fact, the values are $\Delta_1 = 31 \times 543'446 w^2$ and $\Delta_2 = 15 \times 543'476 w^2$. This relation could hardly be due to mere chance, especially when it was also noticed that 543'44 is very close to 3/2 the former 360, and, further, the 450 of Mg is about 5/4 the same. In other words, with the rough values used 360 = 4 x 90, 450 = 5 x 90, and 540 = 6 x 90. This looked so promising that a systematic discussion of all the data at disposal with limits of possible variation was undertaken. The theory to be tested then is that the $\Delta$ of any element which give its doublet or triplet separations are multiples of a quantity proportional to the square of the atomic weight. We will denote this by $\delta = qw^2$. It will be convenient, in general, to deal with the 360 quantity, and $\delta$ will be used to denote this. If other multiples are dealt with as units a subscript unit will be used giving the multiple of the 90. Thus $\delta_1$ denotes the smallest, $\delta_1 = 542'7 w^2$, and so on. The results are given in Table I. below.

The value of $\Delta$ is obtainable as the difference of two decimals with six significant figures. It is convenient therefore to tabulate the values of $10^6\Delta$. The exactness of the calculated value depends on (1) the correctness of the adopted value of $S(\infty)$, (2) the exactness with which $\nu$ is measured, and (3), when expressed in terms of $w^2$, the exactness of the value of $W$ or the atomic weight. In the case of the latter the value $W/100 = w$ is used and the values of $10^6\Delta/w^2$ are tabulated. The method adopted may best be seen by taking an actual example, say that of calcium. The values of $v_1, v_2$ as found by least squares from the S-series are 105'89, 52'09. The value of $S(\infty)$ as given in Table I. of [II.] is 33983'45, and the correct value is supposed to be $\xi$ larger. The numbers 33983'45, 34089'34, 34141'43 are then thrown into the form N/D, and the denominators are 1796470, 1793679, 1792310, giving for differences $\Delta_1 = '002791$, $\Delta_2 = '001369$, which are tabulated as 2791, 1369. The corrections for the error $\xi$ are found to be $-2\xi$ and $-1\xi$. Moreover, the last digits of $10^6D$ may be 5 wrong and the value of the $\Delta$ be $\pm 1$ out. In cases where the $\nu$ are known to three decimal places, the calculations are carried out with 9-figure logarithms, and the values of $\Delta$ determined without this ambiguity. The values of $\nu$ may be wrong by $dv$, i.e., 105'89 + $dv$, &c. This will produce a variation in $\Delta_1$ of $26'3dv$—in general $dv$ is a fraction <1. Thus

$$\Delta_1 = 2791 \pm 1 - 2\xi + 26'3dv.$$  

* See Table I. of Part II.
The atomic weights are supposed to be those given by Brauner,* +x, where x is a number to be added to the fourth significant figure in Brauner’s value. Brauner gives for Ca 40.124. \( \Delta_i \) is then divided by \((40.124)^2\).

The result is

\[
\Delta_i = (17336.1 \pm 6.22 - 1.24\xi + 163d_v - 8.64x) \, u^2.
\]

This is

\[
= 48 (361.169 \pm 1.3 - 0.025\xi + 3.4d_v - 1.80x) \, u^2.
\]

Table I. gives the values for those elements in which the series have been established. The second column contains the atomic weight as given by Brauner, except for the few belonging to volumes of Abegg’s ‘Handbuch’ not yet published, with estimated possible error beneath. In the third column the top number gives \( v \) and the second \( 10^4 \Delta \). For triplets there are therefore two sets. The fourth column gives \( 10^5\Delta / u^2 \), and the multiples of which it is composed. In general the 360 ratio is taken, but in several cases it is necessary to take \( 2 \times 90 \) or 180 and \( 1 \times 90 \). The second line of columns 5, 6, 7, 8 gives the coefficient of the error corrections to be applied to this number 360, or 180, &c., as the case may be, and the upper line of figures gives the maximum errors estimated, which have, in general, been less than those permissible by the observations. The last column gives the difference between 361.8 and the factor given in the fourth column, except that when it is not the \( 4 \times 90 \) term it is brought up to it by multiplying by 2 if it is 180 and 4 if 90. The maximum errors are also attached.

In many cases it will be seen that the number of significant figures in the calculated numbers is larger than in the data from which they are derived. In these cases the number of significant figures in the data must be supposed to be made up to the proper number by the addition of zeros. This enables new calculated values to be determined when the data are improved without the trouble of recalculation.

**Table I.—Evaluation of \( \delta \) and of \( m \).**

*Notation.—\( W = \) atomic weight; \( qw^2 = \delta \) with \( w = (\) atomic weight\)/100; \( \xi, \) error in \( w \); \( dv, \) error in \( v; \) \( x, \) error in \( w \) on the fourth significant figure.*

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<th>( dv )</th>
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*Abegg, ‘Handbuch der Anorganischen Chemie.’
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<td>15 x 543 x 383</td>
<td>0</td>
<td>0.015</td>
<td>135</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta_1 + \Delta_2 =$</td>
<td>46 x 543 x 366</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cd</td>
<td>112.3</td>
<td>1170.848</td>
<td>18321.33</td>
<td>0</td>
<td>0.007</td>
<td>0.15</td>
<td>0.321</td>
<td>408 646</td>
</tr>
<tr>
<td></td>
<td></td>
<td>23105.56</td>
<td>1014 x 180 x 504</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>541.892</td>
<td>8221.64</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>10368.54</td>
<td>91 x 90 x 348</td>
<td>0</td>
<td>0.003</td>
<td>0.16</td>
<td>0.160</td>
<td>644</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta_1 + \Delta_2 =$</td>
<td>49 x 541 x 50</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eu</td>
<td>151.93</td>
<td>2630.5</td>
<td>22191.06</td>
<td>0</td>
<td>0.007</td>
<td>0.068</td>
<td>0.236</td>
<td>94 68</td>
</tr>
<tr>
<td></td>
<td></td>
<td>51223</td>
<td>123 x 180 x 41</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1004</td>
<td>7940.57</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>18329</td>
<td>22 x 360 x 93</td>
<td>0</td>
<td>0.013</td>
<td>0.136</td>
<td>0.472</td>
<td>94 68</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta_1 + \Delta_2 =$</td>
<td>33 x 90 x 485</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hg</td>
<td>200.3</td>
<td>4630.648</td>
<td>21888.03</td>
<td>0</td>
<td>0.002</td>
<td>0.013</td>
<td>0.078</td>
<td>1.41</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8781.99</td>
<td>121 x 180 x 892</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1767.01</td>
<td>7476.05</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>30002.3</td>
<td>83 x 90 x 96</td>
<td>0</td>
<td>0.0026</td>
<td>0.05</td>
<td>0.091</td>
<td>1.15</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta_1 + \Delta_2 =$</td>
<td>54 x 543 x 816</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Al</td>
<td>27.10</td>
<td>5</td>
<td>112.15</td>
<td>23884</td>
<td>0.021</td>
<td>0.012</td>
<td>0.21</td>
<td>0.079</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1754</td>
<td>66 x 361 x 879</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ga</td>
<td>69.9</td>
<td>3</td>
<td>826.10</td>
<td>27715</td>
<td>0.026</td>
<td>0.011</td>
<td>0.10</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>13498</td>
<td>77 x 359 x 93</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>In</td>
<td>114.8</td>
<td>5</td>
<td>2212.38</td>
<td>28593.88</td>
<td>0.01</td>
<td>0.017</td>
<td>0.25</td>
<td>0.147</td>
</tr>
<tr>
<td></td>
<td></td>
<td>37684</td>
<td>79 x 361 x 947</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tl</td>
<td>204.04</td>
<td>5</td>
<td>7792.39</td>
<td>32223.62</td>
<td>0.002</td>
<td>0.012</td>
<td>0.03</td>
<td>0.263</td>
</tr>
<tr>
<td></td>
<td></td>
<td>134154</td>
<td>89 x 362 x 636</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sc</td>
<td>44.1</td>
<td>5</td>
<td>320.80</td>
<td>36714</td>
<td>0.051</td>
<td>0.014</td>
<td>0.10</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7140</td>
<td>101 x 361 x 714</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>6404</td>
<td>91 x 361 x 89</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>He</td>
<td>3.99</td>
<td>5</td>
<td>1.007</td>
<td>20860</td>
<td>0.035</td>
<td>0.018</td>
<td>5.0</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td></td>
<td>33.377</td>
<td>58 x 361 x 45</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table I. (continued).

|    | W. | $v$, $10^8 \Delta$. | $qne^3$, m$.3$ | $\pm 1.3$ | $-\xi$. | $dr$. | $-z$. | $361.8$+.
|---|---|---|---|---|---|---|---|
| O  | 16 | 3.65                              | 6692    | 2.11 | 0.03 | 0 | 0.01 | 5
|    |    | 171.3 | $O''$ | 184 x 361.79 | 1.13 | 100 |    |        |
|    |    | 2.03 | | 3714.9 | | | |    |
|    |    | 95 | | 41 x 90.20 | | | |    |
|    |    | 62 | | 148 | | | |    |
|    | 34 | | | | | |    |
| S  | 32.07 | 17.96 | 10150.85 | 10 | 0.1 | | 34 |    |
|    |    | 1044 | 28 x 362.14 | 0.35 | 0.019 | 20.14 | 0.226 | 3
|    |    | 651 | 6329.7 | | | | |    |
|    |    | $\Delta_1 + \Delta_2$ = | 35 x 180.67 | 0.21 | 0.009 | 10.05 | 0.113 | 1.5
|    |    | | 91 x 181.1 | | | | |    |
| Se | 79.2 | 103.70 | 10192 | 0.057 | 0.034 | 3.5 | 0.092 | 3
|    |    | 6392 | 28 x 364.00 | | | | |    |
|    |    | 44.68 | 4363.4 | | | | |    |
|    |    | 2737 | 12 x 363.61 | 0.057 | 0.04 | 4.08 | 0.092 | 3
|    |    | | 161 x 90.40 | | | | |    |
|    | $\Delta_1 + \Delta_2$ = | | | | | | |    |

Data on which the Table is based.

Na. The limit is 24476.11. It is the limit found in [I.] corrected by the result of Zickendraht’s measurements of the high orders of NaS. The value of $v$ adopted is that deduced from Fabry and Perot’s interferometer measurements of the D-lines using 9-figure logarithms. Consequently, the results are more reliable than would otherwise be expected from such a low atomic weight. But on this point, see below (p. 331).

The limit for K is 21964.4—corrected from the value in [I.] by addition of 1.06 as indicated by Zickendraht’s observations. The value of $v$ is very ill-determined. A value of $v = 57.73$ would make $q = 361.83$. Saunders’ results for S (3) give $v = 57.75$, and K.R.’s for S (4) give $57.60 \pm 0.30$. The value in the table is that used in [I.] 57.87 ± 1. The limits for Rb and Cs are those given in [I.] for S ($\infty$), viz., 20869.73 and 19671.48.

Ca. S ($\infty$) = 31515.48.

Ag. D ($\infty$) = 30644.60, found from first three lines. Fabry and Perot have measured by the interferometer D11 (2) and D11 (2) and K.R.’s are extremely close to these. They have been taken as correct to 0.01 Å.U. The lines D12 and D11 are so close that their difference of wave number as given by Kayser and Runge are probably of the same order of exactness. We may regard, therefore, K. and R.’s D12 - D11 and F. and R.’s D21 - D11 as quite exact up to the second decimal place. This gives $v = 920.61$. It cannot be uncertain to more than a few units in the first decimal. A more correct value is obtained below (p. 404).

The limits of the 2nd and 3rd groups of elements are those given in [II.]. In the cases of Zn, Cd, Hg, the interferometer measurements of Fabry and Perot are used, except $v_3$ for Cd and Hg, with 9-figure logarithms in order to get an extra significant figure, their readings being reduced to Rowland’s scale by Hartmann’s factor 1.000034. In these cases the values of $v$ may be taken as practically correct to 0.01 Å.U.

For Se, see Appendix I.

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He. Limit given in [I.], which is practically exact. \( v = 1.007 \) is given by Paschen as a result of all his readings and is probably not more than \( 0.002 \) in error. Consequently, the numbers for He have weight in spite of its low atomic weight.

0. The limit is 23204.00. Although \( v_1 \), \( v_2 \) are known with fair accuracy, the possible proportional errors are considerable, so that the data have small weight. The limit for the doublet series is 21204 with \( v = 0.62 \). The values are still more indefinite.

S. Limit 20106. The D series give 20110. This gives a considerable range of uncertainty.

Se. Limit 19275.10. The atomic weights for O, S, and Se are those of the International Committee of 1910.

The table shows at once that the two groups which give doublet series agree in giving the \( \Delta \) as multiples of a number close to \( 361.8w^2 \). Group II., giving the triplet series, require in several cases multiples of \( 90w^2 \) or \( 180w^2 \). It is curious that the groups which first indicated this relation do not show it so markedly and with so little doubt as the doublet series, in which by themselves it would probably never have been noticed. There seems to be some kind of displacement with the middle lines of the triplets. If, for consideration, the values of \( \Delta_1 + \Delta_2 \) be taken, this irregularity disappears, and, moreover, with the larger observed quantities, the proportional errors will be less.

If we agree to look upon the 361 as the normal type, and for numerical comparison multiply the 90 by 4 and the 180 by 2, and, if further, the results are supposed to be weighted by the estimated limits of variation assigned in the last column of the table, the method of least squares gives for the value of \( q = \Delta/w^2 \)—

<table>
<thead>
<tr>
<th>Group</th>
<th>( q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>361.900</td>
</tr>
<tr>
<td>II.</td>
<td>361.720</td>
</tr>
<tr>
<td>III.</td>
<td>362.051</td>
</tr>
<tr>
<td>All three groups</td>
<td>361.890</td>
</tr>
</tbody>
</table>

In Groups II. and III. it is possible too much weight has been given to Hg, \( v_3 \), and Tl. We will take as the preliminary value for \( q \) that of silver, viz., 361.81, which is practically that of the general weighted mean. The true value cannot vary much from this—probably less than \( 0.2 \). With this, the subsidiary values become 180.90 ± 1 and 90.45 ± 0.5.

It is seen that in the doublet groups all the elements can come within this limit. In fact, with the exception of K and Ga, they come extremely close. Ga is spectroscopically uncertain as well as in its atomic weight, and the uncertainty of K is due to the uncertainty in its value of \( v \). In the triplet groups also, all calculated from \( \Delta_1 + \Delta_2 \) have possible variations which will bring them within, although the closeness is not so marked as for the doublet elements. The sequence formulae are well established in Groups I. and III., but there are uncertainties in Group II. which yet require clearing up. In this relation also, the table shows slight regular variations as, e.g., \( \Delta_1 \) and \( \Delta_2 \) err from the general mean in different directions, but in these cases
the values of $\Delta_1 + \Delta_2$ come much closer to it. The values of $(\Delta_1 + \Delta_2)/w^2$ are therefore added to the table. It is clear, however, that when the spectroscopic observations are good, the relation here established will enable very accurate measures of the atomic weight to be obtained. In fact, with the possible accuracy attainable in spectroscopic measurements, it may be hoped to obtain far more reliable values of these constants than by weighing, except in those cases where they are small. The table, for example, affords considerable support for Brauner's estimates, except, possibly, in the Mg group, where the irregularities are due to spectral causes. The case of Zn may be taken as an example here. Its spectral values are very good, it shows with $w = 65'40$ the multiple $543'357$ instead of $542'70 \pm 30$. If the excess is due to the value of the atomic weight, it should be 048 larger, which would be allowable within Brauner's estimates to bring it to the adopted value of $q$, i.e., $w = 65'448$. This is more fully considered below. The numbers for Se also seem to show that 79'2 is too small for its atomic weight. 79'40 would make $q$ for $\Delta_1 = 362'16$ and for $\Delta_2 = 361'77$, and the spectral uncertainties would account for the outstanding differences.

If $\delta_i$ is written for $\frac{1}{\xi}$, it may be noticed that the values of the $\Delta$ for the first of each sub-group may be written—

<table>
<thead>
<tr>
<th>i.</th>
<th>ii.</th>
<th>iii.</th>
<th>iv.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Na</td>
<td>Cu</td>
<td>Mg*</td>
<td>Zn</td>
</tr>
<tr>
<td>155$\delta_1$</td>
<td>50 $\times 4\delta_1$</td>
<td>32 $\times 5\delta_1$</td>
<td>31 $\times 6\delta_1$</td>
</tr>
</tbody>
</table>

and, moreover, the same multiples of $\delta_i$ recur in several of the same group, e.g., $\Delta_1 + \Delta_2$ for Zn, Cd, and Hg, and $\Delta_1$ for Eu are all multiples of $6\delta_i$, also the $5\delta_i$ occurs in Mg, Sr, Ba, and Ra. Analogy would lead to a corresponding $3\delta_i$ for Na. The values of the atomic weight and the doublet separations of Na are known with great accuracy, and no possible value given to $\xi$ could change the multiple from 155 to 156 or 153. The only loophole for an explanation may be that the value of $\nu$ as found by Fabry and Perot comes from the Principal series, and that VP (1) is not really $S(\infty)$. This latter point has been discussed in [I.] and also in [II., p. 38]. It is equivalent to a considerable change in $S(\infty)$. To obtain a value 156 or $32 \times 3$ would require an increase of '07 in $\nu$, i.e., to 17'25. Such a value would be quite well in consonance with the measures of Saunders and of K. and R. for other doublets, e.g., D (2) 17'30 (S), S (3) 17'22 $\pm$ 26 (K.R.), S (4) 17'05 $\pm$ 38 (K.R.), P (1) 17'20 (K.R.). But Fabry and Perot's values for P (1)—independently verified by Lord Rayleigh—would seem conclusive against this value, unless F. and P.'s apply only to VP (1) and 17'25 to $S(\infty)$. This would correspond to a lateral displacement of $\delta_i$ (see below) between VP (1) and $S(\infty)$.

S and Se both give 8 $\times 14\delta_1$, which falls in line with the other sub-groups. In fact,

* This is the value first deduced when the international system of atomic weights was used. It is $\delta_i$ more than that in the table. The question is considered below.
if \( n \) denote the order of the group, the sub-groups would be based on \((2n+1)A_1\) and \((2n+2)A_2\). This would leave \( A_1 \) and \( 2A_0 \) for group 0. He, as is seen, may be either.

The foregoing evidence is, I think, conclusive that the atomic weight terms are multiples of a quantity very close to one quarter of 361.8\( w^2 \). Before attempting on existing knowledge to obtain a closer value to this quantity, it will be desirable to consider certain other ways in which the atomic weight plays a part, and which will provide further data for its more exact determination. As it will be convenient to have a name for these quantities which seem to have a real existence, the word "oun" \((w_0)\)* is suggested.

The curious irregularities in the value of the oun noticeable in the elements of the 2nd group in connection with the separate \( \Delta_1 \) and \( \Delta_2 \) values, whilst the values found from \( \Delta_1 + \Delta_2 \) are normal is worth examining in closer detail. The values of \( \nu_1 + \nu_2 \) given in the table are deduced from the sums of \( \nu_1 \), \( \nu_2 \), each determined independently by least squares from the best observations. If the values of \( \nu_1 \) + \( \nu_2 \) are determined directly the values are slightly different, which is natural as they are found from selected pairs. The old values and the values thus found are collected here, and with them the values of \( \delta/\lambda^2 \).

<table>
<thead>
<tr>
<th></th>
<th>Mg.</th>
<th>Ca.</th>
<th>Sr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>New</td>
<td>60.79(36236)</td>
<td>158.01(36145)</td>
<td>581.21(36160)</td>
</tr>
<tr>
<td>Old</td>
<td>60.79(36236)</td>
<td>157.98(36139)</td>
<td>581.28(36164)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>New</td>
<td>1248.85(36156)</td>
<td>2882.26(36184)</td>
<td>578.998(36223)</td>
</tr>
<tr>
<td>Old</td>
<td>1248.54(36148)</td>
<td>2882.26(36184)</td>
<td>578.998(36223)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Cd.</th>
<th>Eu.</th>
<th>Hg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>New</td>
<td>1712.84(36241)</td>
<td>2634.5(36194)</td>
<td>6397.53(36246)</td>
</tr>
<tr>
<td>Old</td>
<td>1712.74(36239)</td>
<td>2634.5(36194)</td>
<td>6397.66(36257)</td>
</tr>
</tbody>
</table>

It will be shown later that spectroscopically Mg belongs rather to the Zn sub-group than to the Ca. The same tendency is exhibited here. The more probable values of \( \nu_1 + \nu_2 \) have brought the oun more closely to equality with 361.60\( w^2 \) for the Ca sub-group, and with 362.4 for the Mg and the Zn sub-group. The value of \( \nu_1 \) for Eu may be 2633.5 instead of 2630.5, and if so, its value of the oun would come to 362.34. If the variations in the value of the oun had been more irregularly distributed, it might have been natural to assign the variations (small as they are) to errors in the value of the atomic weight. But this does not seem justified unless there are chemical reasons whereby atomic weights in any particular group have a liability to be all over-estimated or all under-estimated. In view of the latter

* The pronunciation of oun will be the same in the chief European languages.
possibility it may be well to determine the amount of such error required to bring, 
say, the value 362'4 to 361'8, and 361'6 to 361'9, as it is probable the true value of 
the ratio lies between 361'8 and 361'9. The former requires an increase in atomic 
weight of 0.00, and the latter a decrease of 0.00 of the accepted values. The 
following would be the changes in atomic weight required:—

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Ca</td>
<td>Sr</td>
<td>Ba</td>
<td>Mg</td>
<td>Zn</td>
</tr>
<tr>
<td>-0.025</td>
<td>-0.036</td>
<td>-0.052</td>
<td>+0.04</td>
<td>+0.09</td>
</tr>
</tbody>
</table>

According to the estimates of accuracy given by Brauner the changes for Mg and 
Ca are quite impossible, for Zn just possible, and for the others possible. In the case 
of Mg and Ca, however, small errors in \( r_1 + r_2 \) are considerable proportional errors 
and the deviations may be caused by these. It is necessary to have these estimates 
before us. Notwithstanding them, the close agreement of the numbers in each set, 
and the difference between the two sets must produce the conviction that the 
differences are real, and are not due to errors either in the spectroscopic measure-
ments or the atomic weight determinations.

In the table the multiples given are those which give the own most closely. An 
inspection, however, shows that in each element there is some disturbing influence 
affecting the \( \Delta_1 \) and \( \Delta_2 \) in opposite directions. Moreover, the sum of the multiples 
chosen are in certain cases not the multiple taken for \( \Delta_1 + \Delta_2 \), and this should clearly 
be so. This happens in Cd, Eu, and Hg. There is apparent a general rule that \( r_1 \) is 
too small and \( r_2 \) is too large, the deviation increasing with the atomic weight. The 
discrepancy is equivalent to a transference from the true \( \Delta_1 \) to the true \( \Delta_2 \). 
Evidently the transfer in Cd, Eu, and Hg has been so large as to increase \( \Delta_2 \) by 
more than \( \delta_1 \), so that the closest multiple now appears to be too large by unity. If 
the multiples in \( \Delta_2 \) be diminished by unity, the sum is equal to that for \( \Delta_1 + \Delta_2 \) and 
the discrepancy between the own from \( \Delta_1 \) to \( \Delta_2 \) increases in a regular order. A 
similar change has occurred in Sr, only here while the multiple of \( \Delta_2 \) has apparently 
increased, that of \( \Delta_1 \) has apparently decreased. If the ratio \( \Delta_2 : \Delta_1 \) be taken as 
79:171 in place of 80:170 the discrepancy again falls into order with the others. 
With these changed ratios the values become

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Sr</td>
<td>Cd</td>
<td>Eu</td>
<td>Hg</td>
<td></td>
</tr>
<tr>
<td>. . . . 171 \times 90'068</td>
<td>203 \times 90'252</td>
<td>246 \times 90'205</td>
<td>242 \times 90'446</td>
<td></td>
</tr>
<tr>
<td>79 \times 91'144</td>
<td>90 \times 91'351</td>
<td>87 \times 91'27</td>
<td>82 \times 91'17</td>
<td></td>
</tr>
</tbody>
</table>

This transference must take place in the \( D_2(\infty) = S_2(\infty) \) term. The values are 
given in the following table in which the first column gives the value of \( \Delta_2 : \Delta_1 \), the 
second the value of the transference, the fourth the transference in \( r_1 \) to \( r_2 \), the fifth 
the new value of \( \Delta_2 \), and the third and sixth are as explained later :—
What is the nature of the modification? Perhaps the simplest explanation to test is that a fraction of $\delta$ is transferred. The third column gives the fraction of $\delta$, which is equal to the transfer. It is noticed at once that the two groups fall into two separate sets. With the exception of Sr, the fraction in the first is about 17. Mg and Ca can both fit in with this, for the values are so small that they depend on decimals in the value of $\Delta_1$, $\Delta_2$, and therefore beyond our significant figures. In Ca, indeed, evidence is given later that $\Delta_2$ is somewhat higher and would bring the ratio close to 18. But Sr is quite out of step with the others. Zn has no transfer, Cd and Eu are equal, but Hg is 5030. If the ratio in Hg be taken to be 82:243 however, the fraction agrees with those of Cd and Eu. The Hg oun is then 361'43m² in place of 362'54 and closer to the mean value, and as will be shown later there is evidence for the new value of $\Delta_2$ (see p. 397). If this explanation is valid it must be possible to bring Sr into the scheme with a transfer of 117, but it is difficult to see how this can be done. 639 is about four times too great, in other words, where the others are modified by a fraction of $\delta$, Sr is modified by the same fraction of $\delta$. The above arrangement brings Mg into the Ca group and upsets the law whereby its first $\Delta_1$ should be a multiple of 5$\delta$. As this law seems to have a considerable weight of evidence in its favour, and moreover, as will be seen shortly, Mg tends to go spectroscopically with the Zn group, it may be well to see the result of keeping $\Delta_1 = 40\delta$ and the ratio $\Delta_2: \Delta_1 = 19:40$. This will require a transfer of about 6'3 with a considerable uncertainty owing to the small values of $\Delta_2$ and $\Delta_1$, and $\Delta_2 = 408'7$. With this the fraction of $\delta$ is 1'1727. To bring to the same fraction as in Cd the transfer should be about 4, which the uncertainty in 6'3 is not great enough to permit. As the fraction 77 is of the order 1 - 215 it suggests that the modification is produced by adding $\delta$ to the atomic volume term in the sequence of the P series, viz. (atomic weight term $+ \delta$) $\left(1-\frac{215}{m}\right)$. The question must be left open at present. It has been noted that the arrangement which gives $\Delta_1 = 159\delta$, for Mg throws it out of the rule that the first members of the different groups are successive multiples of $\delta$. When the calculations were first made, the values of the
international atomic weights were used, and for Mg it is 24'32 in place of Brauner's 24'362. This clearly gave $\Delta_1 = 160\delta_1$ and $\Delta_1 + \Delta_2 = 237\delta_1$ with $\delta = (362'09 \pm 65) w^2$ the uncertainty '65 not including that of atomic weight and being chiefly due to uncertainty in $v_1 + v_2$. The transference required now is 2'70, and the fraction of $\delta_1$ is '502, again clearly not that of the Ca group, but when account is taken of the uncertainty in $v_1 + v_2$ quite possibly agreeing with that of the Zn group. The assumption that the international value of $w$ is more correct than Brauner's certainly gets over the difficulties mentioned above. But we are not justified in choosing the values from the particular systems which best suit our theories. The discrepancy between the international and Brauner's is very great—from 10 to 15 times Brauner's indication of his possible error.

Another suggestion as to a possible explanation may be given. There have been various indications in [I. and II.] that small variations in $N$ may occur. If so it is possible to produce the changes observed by a small change $dN$ in the middle line of the triplet. The necessary changes to do this are given in the sixth column. The changes clearly depend on the squares of the atomic weight, for if they are expressed in the form $xw^2$ they are

$$Sr \ldots \ 5'20 = 6'767w^2 = 4 \times 1'564w^2$$
$$Ba \ldots 3'34 = 1'777w^2$$
$$Ra \ldots 8'13 = 1'586w^2$$

 Cd \ldots 11'86 = 9'426w^2

Eu \ldots 21'76 = 9'426w^2

Hg \ldots 38'50 = 9'596w^2

in which it may be noticed that 9'426 = 6 × 1'571. Again multiples of a quantity depending on the square of the atomic weight enter, and it is especially interesting to note that the Zn group are affected with the multiple 6. If Ca and Zn show similar displacements, Ca would require $dN = 25$ in place of '16 and Zn 4'03. Zn is clearly 0, i.e., is unaffected, but considering the small numbers involved in Ca and consequently large proportional errors, Ca might well show 25 instead of '16. The question naturally arises, do these quantities depend in any way on the sum? Now any change in $N$ may be supposed to arise either as a real change in $N$ itself or an apparent change due to the introduction of a factor in connection with the $1/D^2$. In other words, the quantity $VD$ is

$$N \cdot \frac{1 + f}{(m + d)^2} \quad \text{or} \quad N \cdot \frac{(1 + f)^2}{(m + d)^2}.$$  

Looked at from this point of view, 9'426 $w^2$ requires $N (1 + 0'000859w^2)$ or $N (1 + 0'000429w^2)^2$. Now 5$\delta_1$ would give '000452 $w^2$, but if the present explanation is the true one, this is not a likely value since it will not include the alkaline earths. A value 6$\delta_1$ = '0005428 $w^2$ would be expected. The Ba value 1'777 would give (1'000088 $w^2$), or practically (1 + $\delta_1)^2$. It rather looks as if this explanation is a part of the truth. If more exact measures were at disposal it might be well to assume
these results as holding, recalculate for the denominators and discuss the rearrangement now required. It may be noted, however, that in the Zn sub-group a factor \((1 + x\delta)^2\) in \(D_i\) \((\infty)\) would reduce the calculated 

out below 362.462 and \((1 - y\delta)^2\) raise it in the Ca sub-group above 361.6046 and at the same time increase the factors in the numbers above towards \((1 + 6\delta)^2\) and \((1 + \delta)^2\). The factors may of course enter either as \((1 + x\delta)^2\) or \((1 - x\delta)^{-2}\).

Collaterals.

The first set in doublet or triplet S or P series is always the stronger. The others may be considered as receiving a sort of lateral displacement, by the atomic weight term, in the recognised way, and may be called collaterals. This kind of displacement is, however, not confined to the series generally recognised, but is of very common occurrence, and, indeed, depends not only on the \(\Delta\) but also on other multiples of \(\delta\). In fact, the doublet and triplet series are only special cases of a law of very wide application. Some evidence of its existence will be given below. It will be sufficient now only to refer to certain points connected with the law, and to a convenient notation to represent it. This kind of relation was first noted in the spectra of the alkaline earths, and as the lines are both numerous and at the same time strong and well defined, and, therefore, with very small observation errors, any arguments based on them must have special weight. Moreover, there are long series of step by step displacements involving large multiples of \(\Delta\) between initial and final lines, so that we may feel some certainty that these large multiples are real and not mere coincidences.

As a compact notation is desirable the following has been adopted. In general† the wave number of a line is determined by a formula of the form \(N/D_1^2 - N/D_m^2\), and lateral displacements may be produced by the addition (or subtraction) of multiples of \(\delta\), say \(x\delta\) or \(x\Delta\), to \(D_1\) or \(D_m\). This is indicated by writing \((x\delta)\) to the left of the symbol of the original line when it is added to \(D_1\), and to the right when added to \(D_m\). Thus \(\text{CaS}_1(2)\) is 6162.46. So far as numerical agreement goes 6439.36 is a collateral of this represented by \((2\Delta_1 + 10\Delta_2)\text{CaS}_1(2) (+\Delta_2)\). This means that whereas, see [II.],

\[
\text{Wave number of CaS}_1(2) = \frac{N}{(1'796470)^2} - \frac{N}{(2'484994)^2},
\]

\[
\text{Wave number of 6439.36} = \frac{N}{(1'796470 + 2\Delta_1 + 10\Delta_2)^2} - \frac{N}{(2'484994 + \Delta_2)^2},
\]

\[
= \frac{N}{(1'815732)^2} - \frac{N}{(2'486362)^2}.
\]

* A note on this relationship was given at the Portsmouth meeting of the British Association, see 'Report, B.A.' (1911), p. 342.

† Though not always, as I hope to show in a future communication.
Before going further it is desirable here to consider the nature of the cumulative effects produced by errors in the values of \( \delta \), or of the limits, in the course of a succession of step by step displacements. There may be a small error in the starting point, e.g., \( S(\infty) \) in the above example, or in the value adopted for \( \nu \). We will consider these separately, taking the case where the displacement is on the left, or the first term.

1. The limit correct, but \( \nu \) slightly too large.—Then \( \delta \) calculated from this is also slightly too large. It will, however, serve to identify a large series of steps in succession, i.e., to reproduce the successive differences of the wave numbers of the lines. But the errors will all be cumulative, and if the last line of a set be calculated direct from the first, its denominator is too large and its wave number too small. In this case a more correct value of \( \delta \) can be obtained by using these extreme lines, and this corrected value must satisfy all the other lines. In general a new correction will only affect an extra significant figure in the value of \( \delta \).

2. \( \delta \) correct, but limit wrong.—In this case a slight error in the limit will be of no importance unless the \( \delta \) and its multiples are considerable; and, as a rule, the limits are known with very considerable accuracy, except possibly in the alkaline earths and a few others. Let us suppose the limit adopted (say \( S(\infty) \)) is too large, that is, its denominator too small. If the second line is due to a positive displacement, its denominator is larger than that of the first, and the wave number less. Suppose \( D_1 \), \( D_3 \) the denominators for the two lines, \( D_1 > D_3 \) if the displacement is positive, the separation is \( N/D_1^2 - N/D_3^2 \). If the limit is chosen too large \( D_1 \) and \( D_3 \) are chosen too small, although \( D_3 - D_1 \) is correct since \( \delta \) is supposed correct. If \( D_1 \) becomes \( D_1 - x \), the error in the separation is \( 2Nx/D_1^3 - 2Nx/D_3^3 \), which is positive since \( D_3 \) is supposed \( > D_1 \), i.e., the calculated separation is too large. If the displacement is a negative one, \( D_3 < D_1 \), the true separation is now \( 2N/D_3^2 - 2N/D_1^2 \) and the error \( 2Nx/D_3^3 - 2Nx/D_1^3 \), which is now negative since \( D_3 < D_1 \). The effect would be that in any series of step by step displacements \( \delta \) would appear to require continual decreases, and at the end the "corrected values" would not at all fit the initial cases. If, then, it is found that when \( \delta \) is corrected as in Case 1 the corrections tend to alter the former corrected one, and not to produce additional significant figures only, it may be surmised that the limit has been wrongly chosen. It is clear, then, that where there are a number of successive collaterals with a large multiple of \( \delta \) between the extreme ones, we have at disposal a means whereby much more accurate values for \( \delta \) and the limits are obtainable. Cases are given below, e.g., in BaD.

For low atomic weights \( \delta_1 \) is always a small quantity and except for orders where \( m = 1 \) or 2, the alteration in wave number is small. For the present purpose which is to obtain proof of the existence of the displacements here indicated, no evidence can be admitted in which the change in wave number produced by a displacement \( \frac{1}{2} \delta_1 \), is comparable with the possible error of observation. The evidence, therefore, is of
greatest weight when derived from the spectra of elements of high atomic weight, or from cases in which the displacements are due to multiples of $\Delta$.

It is possible for a line to be simultaneously displaced to right and left, as for instance CaS$_1$ (2) given above. Such lines exist, but since there is a very considerable scope for adjustment of values by a proper choice of say $x$ and $y$ in $(x\delta)X(y\delta)$, and specially so in $y$ when $m > 2$, such cases cannot be considered as established unless $\delta$ is very large, or the $\Delta$ enter only, or unless there is independent evidence by the existence of intermediate steps.

When these collaterals were first found it was noticed that in general a positive displacement seemed in the majority of cases to increase the intensity of the lines, and a negative to decrease it. This is clear when the displacements considered are those from the 1st to the 2nd set of a doublet series where the displacement is a negative one and there is always a decrease in intensity. It is also evident in the satellites of the D series. Apparently, as will be shown, the typical line of the series is the satellite. The strong line is a positive collateral of this and always shows a great increase of intensity. Although these facts are obvious the connection was not recognised, until the relation showed itself first in a series of collaterals. It is, I think, safe to say that a positive displacement produces a tendency to increase of intensity; there may be other causes acting so as sometimes to mask the effect, but in general, where the rule appears to be broken, the suggested displacements should be regarded with some doubt. In so far as I have used this rule in the following, the results are biased and of course the evidence for the rule to that extent weakened.

It would be possible to give here long lists of collaterals. As, however, the present communication has reference chiefly to the discovery of general laws as a necessary preliminary to the more thorough examination of special spectra, it will be sufficient to refer for evidence to the cases which arise in the succeeding discussion. This seems, however, a natural place to refer to certain cases discussed in Parts I. and II., where unexpected deviations occurred between the calculated and observed position of a line in the middle of a series in which for the other lines the agreement was especially good. As special instances, the cases of TIS$_1$ (4) and CaS$_1$ (5) [II., p. 39] may be taken. The suggestion that TIS$_1$ (4) may be due to a transcription error is not valid, and was occasioned by an oversight in confounding $d\lambda$ with $dn$. If the normal line be denoted by TIS$_1$ (4), the observed is the collateral TIS$_1$ (4) ($15\delta$) giving $O-C = -01$ in place of $-121$. Similarly, the observed Ca line is CaS$_1$ (5) ($-6\Delta$) with $O-C = -03$ in place of '61. There are many examples of such sudden jumps which are certainly not due to errors of observation. Several instances will be found below in the D series.

The Diffuse Series.

To the question what is the positive criterion of a Diffuse series no clear answer up to the present has been given. We find in general three sets of series associated together. Two of these have the same limits, the other a limit peculiar to itself.
The latter is the Principal series, and the difference between the wave numbers of its first line and of its limit gives the limit of the other two. Of the other two series, one shows a Zeeman effect of the same nature as that in the Principal. This is called the Sharp series—or (by Kayser and Runge) the 2nd associated series. The third series is called the Diffuse—or the 1st associated series. It has in fact a negative kind of criterion. The preceding definitions apply to the three series in all elements, including such elements as Li, He, and others which show singlet series. When doublets and triplets appear, we have a simple physical criterion for the Principal series in that it is that series in which the doublets or triplets converge with increasing order. This criterion can be applied even when the 1st line has not been observed. In certain elements the constant separations are shown between satellites. In these cases the series is certainly a D-series, at least in those recognised up to the present—but further knowledge may show that in certain cases such satellites may appear in other series.* If, passing beyond the mere physical appearance of the series or their visible arrangement in the spectrum, we attempt to represent their wave numbers by formulae of the recognised types, we have further criteria for the Principal and Sharp, viz., that the 1st line of the Principal may also, very nearly at least, be calculated from the formula for the Sharp—or vice versa—and that the denominators in their formulae differ, roughly indeed but sufficiently closely for use as a criterion, by a number not far from 5. But when an attempt is made to deal in the same way with a line of the diffuse series, no general type of formula has, at least as yet, been found. In the alkali metals, as was seen in [I.] all the D-series take a positive value for $\alpha$—in other words, the fractional parts of the denominators decrease with increasing order, and the general conclusion might be drawn that this was a common feature of all diffuse series. But the opposite occurs in the triplet spectra of the 2nd group of elements, whilst a similar rule of a positive value of $\alpha$ recurs in the 3rd group. This suggests that the series giving doublets have $\alpha$ positive and triplets $\alpha$ negative, but this is contradicted by the triplet series of O, S and Se, which behave in the same way as the doublets of Groups 1 and 3. The question naturally arises, is there a typical D-sequence with $\alpha$ positive, and the diffuse series in the 2nd group do not really belong to this type, or is there no actual D-sequence, i.e., no regular type of formula to which the D-series conform. The difficulty of finding formulae to accurately represent any particular D-series would point to the latter supposition, a supposition also which is strengthened when we study comparatively the series of numerical values of the denominators found directly from observations as is done below. In the case of the alkalis the formulae given in [I.] (as well as those in $1/m^2$) do not reproduce well the high orders and are probably only within the limits of error because the lines are so diffuse that the observation errors are very large. In fact one of the few excessive deviations found in [I.] was that of NaD(6), in which it is

* $E\phi$, in ScS., see Appendix I.
not probable that the error is one of observation. In Group 2 the Zn sub-group can
be reproduced fairly well with a formula in \( \frac{\alpha}{(2m-1)} \) in which \( \alpha \) is negative. Mg
can also be reproduced within error limits by a formula of the same kind, but it is
impossible to do so for Ca, and Sr and Ba require additional terms in \( \frac{1}{m^2} \). In
Group 3 Al is quite intractable, and if really depending on a formula, appears to
require complicated algebraic or circular functions. In and Tl also are not amenable
to formulæ in \( \frac{\alpha}{m} \) only or \( \frac{\alpha}{m+\beta/m^2} \). Nevertheless, the general build of the series
is so similar to that of the others that it would seem probable that the wave numbers
should also be of the form \( S(\infty) - \frac{N}{(m+d_m)^2} \). If so it is possible to calculate \( d_m \)
from the observations and a comparative study may throw some light on the origin of
the different lines. The attempt to deal with these series from the formulæ point of
view, however, brought out the fact that the satellites are related to the strong lines
in a similar way to that in which the Principal line doublets are, viz., by a constant
difference in the denominators and that their differences probably depend on
multiples of the "own," as is the case in the Principal series. As the evidence
depends also on a comparison of the numerical values of \( d_m \), this point will also be
considered now.

The actual values of \( d_m \) will depend on the accuracy of the value \( S(\infty) \) (or \( D(\infty) \))
of the limit. In the calculations below the most probable value has been used (see
note under each element) and the true value has been taken to be that + \( \xi \). In order
to be free from mental bias these have been in general taken to be the same as \( S(\infty) \),
which involves the theorem that \( D(\infty) = S(\infty) \). But of this little doubt can be felt.
The true values of \( d_m \) can then be given in the form \( d_m + k\xi \) where \( k \) is small. For
high orders of \( m, k \) is comparatively large and can only be used when \( \xi \) is very small.
It is however generally the case that errors made in this way are only a fraction of
the observational errors.

As in the normal type where there are no satellites \( VD_1 = VD_9 = VD_{29} \) and where
there are satellites \( VD_{12} = VD_{23} \), \( VD_{13} = VD_{34} \), it is only necessary to tabulate the
values of \( d_m \) for the case of \( VD_1 \) or \( VD_{11}, VD_{12}, VD_{13} \) respectively. When this is done
certain regularities are clearly apparent, which can be made more exact by allowing
small observational errors and giving a small permissible value to \( \xi \). It would cumber
the space at disposal to give both sets of values, especially as it is possible to easily
indicate the differences on the one set of tables. Table II, then gives the values of \( D_m \)
with the modified value of \( \xi \), with the maximum errors attached in the usual way
in ( ), and the calculated value given as a correction to the selected value. Thus for
\( NaI(3), D(3) = 3'986626 (133) - 289\xi - 104, \) 3'986626 is the selected value, 133
possible change in last three digits in this, -289 is change for \( \xi = +1 \), and the observed
value is 104 less than the selected. The values of the errors of observed wave-length
over calculated (\( O - C \)), and of possible observed errors (O) are given in each case on
the right. The tables for the different elements are collected together and discussion
of each is given later when considering the ordinal relations of the denominators.
Na. NaP (∞) is given in [L] = 41446.76 ± 1.69, but Wood's measurements of the high orders require a value about 1.48 larger, say, close to 41448.24. Also Fabey and Buisson's interferometer measure of NaP(1) give, when referred to Hartmann's K scale, n = 16972.85. Whence

\[ VP(1) = 41448.24 - 16972.85 = 24475.39 \]

and this should be D(∞) and S(∞). Further, S(∞) is given in [L] as 24472.11 ± 3.84 and Zickendraht's measures of high order require about 3 or 4 more, or, say, 3.5, which is within allowable limits. This would give S(∞) = D(∞) = 24475.61. Thirdly, D(∞), calculated from \( n = 3, 4, 5 \), gives 24475.20. Zickendraht's measures, however, if exact, require about 2 larger. The three combined appear to point to a value close to 24475.40, and this was taken for calculation. In the modified table above, it was found better to take D(∞) about 1 larger, in the direction of Zickendraht's results, and the table is therefore based on 24476.40. For NaD(6) Zickendraht, as well as K.R., gives an abnormally great separation. Lehmann's value of \( D_1(2) \) gives 243 greater, making 1st ordinal difference = 3. K.R.'s value of D(6) gives 6.965755.

K. KP1(∞) from [L] = 35006.21 ± 1.55 agrees well with Bévan's measures of high orders, possibly slightly less. For P1(1) K.R. give \( n = 13041.77 \) and S 13042.96. These, then, give for D(∞) = S(∞) = 21964.44 or less (K.R.) and 21963.25, or less (S.). The value 21964 has been taken.
**Table II. (continued).**

Rb.

\[ \Delta = 12935. \quad \delta = 263.77. \]

<table>
<thead>
<tr>
<th></th>
<th>D.</th>
<th>O - C.</th>
<th>O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ran.</td>
<td>2.766216 - 96ξ</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>60669 = 2305</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S.</td>
<td>3.705547 - 232ξ + 0</td>
<td>.00</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>833</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&quot;</td>
<td>4.683718 (234) - 468ξ - 26</td>
<td>.05</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>386</td>
<td></td>
<td></td>
</tr>
<tr>
<td>K.R.</td>
<td>5.673688 (382) - 833ξ + 22</td>
<td>.00</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>135</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&quot;</td>
<td>6.668673 (576) - 1352ξ - 118</td>
<td>.01</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>155</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R. or S.</td>
<td>7.664713 - 205ξ - 68</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>145</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S.</td>
<td>8.661017 - 296ξ - 70</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>9.661017 - 411ξ + 391</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10.6428 - 55ξ</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&quot;</td>
<td>11.6464 - 72ξ</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&quot;</td>
<td>12.635 - 9ξ</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \text{Rb}_1 (\infty) = 33687.50 \pm 2 \text{ [I]}, \text{ and probably greater. Bevan's observations show slightly larger, say, about } \cdot 5, \text{ i.e., } P (\infty) = 33688. \text{ Using Saunders' for } P_1 (1), VP (1) = 20871.29 = D (\infty). \]

According to Saunders, RbD shows satellites for \( m = 3 \) and 4, giving \( D_{11} (3) - D_{12} (3) = 2.63 \) and \( D_{11} (4) - D_{12} (4) = 2.02 \) with uncertainties of 1. These give differences in the denominators respectively of 610 and 946, and 961 = 593, 1464 = 923.

Cs.

\[ \Delta = 32551. \quad \delta = 638.22. \]

<table>
<thead>
<tr>
<th></th>
<th>( D_{11} )</th>
<th>( D_{12} )</th>
<th>O - C.</th>
<th>O.</th>
<th>O - C.</th>
<th>O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>P.</td>
<td>2.554329 (228) - 76ξ - 43</td>
<td>461</td>
<td>546989 (226) - 97</td>
<td>.5</td>
<td>3</td>
<td>1.3</td>
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<tr>
<td>&quot;</td>
<td>535183 (200) - 201ξ + 40</td>
<td>541</td>
<td>526567 (200) + 9</td>
<td>.2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>S.</td>
<td>4.535388 (160) - 424ξ + 1</td>
<td>146</td>
<td>524635 - 161</td>
<td>.00</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>&quot;</td>
<td>5.533110 (400) - 76ξ + 22</td>
<td>146</td>
<td>524175 - 26</td>
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<tr>
<td>R.</td>
<td>6.532631 - 126ξ + 77</td>
<td>146</td>
<td>523696 - 428</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&quot;</td>
<td>7.532631 - 194ξ - 158</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>&quot;</td>
<td>8.532631 - 282ξ + 56</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&quot;</td>
<td>9.532411 - 397ξ + 36</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&quot;</td>
<td>10.52533 - 530ξ + 11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Since the publication of [L] RANDALL's measurement of $P(1)$ with considerable accuracy. This, with $P(2,3)$, gives $P_1(\infty) = 31401.78$, and RANDALL's value for $P_1(1)$ gives $VP(1) = 19673.12$. BEVAN's observations show $P_1(\infty)$ about 2 larger. Probably, however, this value for $VP(1)$ is close to the true value for $D(\infty)$, and the calculations are based on $D(\infty) = 19673.00$.

For $D_1(3)$ LEHMANN gives denominators 548 larger for $D_{11}$ and 548 less for $D_{12}$. If we allow $S \times 2$ the weight of $L$, the value of $O - C$ would come out about zero.

\[\Delta = 7311, \quad \delta = 146.22.\]

<table>
<thead>
<tr>
<th>$D_{11}$</th>
<th>$D_{12}$</th>
<th>O-C</th>
<th>O</th>
<th>O-C</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.979076(43) - 120\xi - 6$</td>
<td>$228_1 \cdot 978272 + 4$</td>
<td>0.01</td>
<td>0.10</td>
<td>-0.01</td>
<td>0.05</td>
</tr>
<tr>
<td>$3.984047(173) - 288\xi + 24$</td>
<td>$278_1 \cdot 983060 + 30$</td>
<td>-0.01</td>
<td>0.10</td>
<td>-0.01</td>
<td>0.20</td>
</tr>
<tr>
<td>$4.985509(1) - 565\xi + 35$</td>
<td></td>
<td>-0.01</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$D(\infty) = 31515.48$ found from K.R.'s value for $D(2,3,4)$.

\[\Delta = 27791, \quad \delta = 421.07.\]

<table>
<thead>
<tr>
<th>$D_{11}$</th>
<th>$D_{12}$</th>
<th>O-C</th>
<th>O</th>
<th>O-C</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.979583(19) - 120\xi + 3$</td>
<td>$238_1 \cdot 977150(19) + 12$</td>
<td>-0.01</td>
<td>0.05</td>
<td>-0.03</td>
<td>0.05</td>
</tr>
<tr>
<td>$4.983998(1967) - 288\xi + 23$</td>
<td>$106_1 \cdot 982891(175) - 27$</td>
<td>-0.01</td>
<td>1.00</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td>$5.987280(776) - 565\xi - 2$</td>
<td>$156_1 \cdot 985701(2170)$</td>
<td>0.00</td>
<td>2.00</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>$5.988130(3720) - 972\xi + 46$</td>
<td></td>
<td>0.00</td>
<td>5.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

† Calculated from $D_2 - v$.

$D(\infty)$ found from first three by formula = 30644.66, modified to 30644.76.
The limit uncertain, see special discussion (p. 403).
The observation errors after the first are so large that the satellite differences might be also $238_1$, or larger, as in Cu.

**Table II. (continued).**

**Mg.**

\[ \Delta_1 = 854. \quad \Delta_2 = 413. \quad \delta = 21'48. \]

<table>
<thead>
<tr>
<th></th>
<th>(D)</th>
<th>O - C.</th>
<th>O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>P.</td>
<td>(1 \cdot 822169 - 27 \cdot 5\xi + 12)</td>
<td>-1</td>
<td>1.5</td>
</tr>
<tr>
<td>K.R.</td>
<td>(2 \cdot 828774(20) - 103\xi - 12)</td>
<td>-0.02</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>(3 \cdot 831255(20) - 256\xi + 59)</td>
<td>-0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>*</td>
<td>(4 \cdot 83494(190) - 514\xi + 61)</td>
<td>-0.01</td>
<td>0.03</td>
</tr>
<tr>
<td>†</td>
<td>(5 \cdot 833320(1808) - 904\xi - 24)</td>
<td>0.00</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>(6 \cdot 833320(4210) - 1452\xi - 583)</td>
<td>0.02</td>
<td>0.20</td>
</tr>
</tbody>
</table>

* Calculated from \(D_{II} - \nu_1\).
† The observed value gives \(\nu_1 = 42'87\) in place of the normal value 40'92. If this be corrected to 40'92, giving equal weights to \(D_a D_b\), the value would be 6'833809.

\[ D(\infty) = S(\infty) = 39752 \cdot 83 \pm 2'73, \text{ as given in [II.], from the formula in } 1/m^2. \] This is modified in the above to 39751'08.

**Ca.**

\[ \Delta_1 = 2791. \quad \Delta_2 = 1369. \quad \delta = 58'14. \]

<table>
<thead>
<tr>
<th>(D_{11})</th>
<th>(D_{12})</th>
<th>(D_{13})</th>
<th>O - C.</th>
<th>O.</th>
<th>O - C.</th>
<th>O.</th>
<th>O - C.</th>
<th>O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1'947172(8) (- 33\xi - 4)</td>
<td>135 *046417(25) + 5</td>
<td>86 *045952(25) + 20</td>
<td>-0.05</td>
<td>1.0</td>
<td>-0.06</td>
<td>2.0</td>
<td>-0.04</td>
<td>3.0</td>
</tr>
<tr>
<td>3'08696(20) (- 133\xi + 14)</td>
<td>135 *081941(20) + 11</td>
<td>86 *081476(20) - 14</td>
<td>-0.02</td>
<td>0.03</td>
<td>-0.01</td>
<td>0.03</td>
<td>-0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>4'091707(104) (- 312\xi + 17)</td>
<td>145 *090893(104) - 17</td>
<td>*090428 - 21</td>
<td>0</td>
<td>0.05</td>
<td>0</td>
<td>0.05</td>
<td>0</td>
<td>0.05</td>
</tr>
<tr>
<td>†5'091707(538) (- 598\xi + 326)</td>
<td>0</td>
<td>0</td>
<td>-0.06</td>
<td>10.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>†6'091707(4856) (- 1012\xi - 1400)</td>
<td>0</td>
<td>0</td>
<td>0.14</td>
<td>50.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>†7'091707(7732) (- 1546\xi + 161)</td>
<td>0</td>
<td>0</td>
<td>-0.02</td>
<td>50.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Calculated from \(D_{21}\) and \(D_{31}\), treating each as of equal value.
† Collaterals (see text). The values calculated direct from the observations are respectively 5'082736, 6'056500, 6'976528.

\[ D(\infty) = 33981'85, \text{ being } 33983'45 \pm 5'8, \text{ as given for } S(\infty) \text{ in [II.], with } 1/m^2 \text{ modified by putting } \xi = -1'6. \]
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Table II. (continued).

Sr.

\[ \Delta_1 = 11835. \quad \Delta_2 = 5533. \quad \delta = 277.89. \]

<table>
<thead>
<tr>
<th>( D_{11} )</th>
<th>( D_{12} )</th>
<th>( D_{13} )</th>
<th>O-C.</th>
<th>O.</th>
<th>O-C.</th>
<th>O.</th>
<th>O-C.</th>
<th>O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.993184 (7) - 36( \xi +1 )</td>
<td>133</td>
<td>.989572 (8) - 7</td>
<td>85</td>
<td>* 987349 + 45</td>
<td>-3</td>
<td>1.5</td>
<td>1.7</td>
<td>2.0</td>
</tr>
<tr>
<td>( 653 )</td>
<td>( 3.174741 (17) - 146( \xi +15 )</td>
<td>128</td>
<td>.171407 (17) - 7</td>
<td>85</td>
<td>.169184 (29) + 15</td>
<td>-0.02</td>
<td>.03</td>
<td>.01</td>
</tr>
<tr>
<td>( 84 )</td>
<td>4.198084 (101) - 337( \xi +41 )</td>
<td>155</td>
<td>.193916 (100) - 82</td>
<td>85</td>
<td>.191693 (1) + 597</td>
<td>-0.02</td>
<td>.05</td>
<td>.04</td>
</tr>
<tr>
<td>( 195 )</td>
<td>5.203364 (965) - 642( \xi +112 )</td>
<td>155</td>
<td>† 199196 (482) + 300</td>
<td>85</td>
<td>† 196973 (1) - 84</td>
<td>-0.02</td>
<td>.20</td>
<td>-0.06</td>
</tr>
<tr>
<td>0</td>
<td>6.203919 (2539) - 1088( \xi -40 )</td>
<td>155</td>
<td>† 199751 (1) + 210</td>
<td>( \dagger )</td>
<td>.00</td>
<td>.30</td>
<td>-0.02</td>
<td>( \dagger )</td>
</tr>
<tr>
<td>0</td>
<td>7.203919 (2838) - 1702( \xi -1819 )</td>
<td>( \dagger )</td>
<td>.12</td>
<td>.20</td>
<td>( \dagger )</td>
<td>.12</td>
<td>.20</td>
<td>( \dagger )</td>
</tr>
<tr>
<td>0</td>
<td>8.203919 (20174) - 2522( \xi +5971 )</td>
<td>-3</td>
<td>1.0</td>
<td>( \dagger )</td>
<td>.12</td>
<td>.20</td>
<td>( \dagger )</td>
<td>.12</td>
</tr>
</tbody>
</table>

* Calculated from \( D_{31} - v_1 - v_2 \). The difference might be \( T_{4}^{35} = 31\delta \).
† Calculated from \( D_{21} - v_1 \) and \( D_{32} - v_1 \).
‡ Calculated from \( D_{31} - v_1 \).

\( D(\infty) = 31027 \cdot 25 \), being \( 31027 \cdot 65 \pm 4 \), as given for \( S(\infty) \) in [II.], with \( 1/m^2 \), modified by putting \( \xi = -0.4 \).

Ba.

\[ \Delta_1 = 29328. \quad \Delta_2 = 11976. \quad \delta = 683.2. \]

<table>
<thead>
<tr>
<th>( D_{11} )</th>
<th>( D_{12} )</th>
<th>( D_{13} )</th>
<th>O-C.</th>
<th>O.</th>
<th>O-C.</th>
<th>O.</th>
<th>O-C.</th>
<th>O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3.114613 (12) - 137( \xi -11 )</td>
<td>145</td>
<td>.105049 (20) + 19</td>
<td>115</td>
<td>.093194 (20) - 11</td>
<td>-0.02</td>
<td>.03</td>
<td>.04</td>
<td>.05</td>
</tr>
<tr>
<td>( 64 )</td>
<td>4.158338 (812) - 325( \xi +24 )</td>
<td>108</td>
<td>.151506 (325) - 154</td>
<td>295</td>
<td>.131694 (160) + 101</td>
<td>-0.01</td>
<td>.05</td>
<td>.09</td>
</tr>
<tr>
<td>( -145 )</td>
<td>5.148774 - 49</td>
<td>( \dagger )</td>
<td>( \dagger )</td>
<td>( \dagger )</td>
<td>( \dagger )</td>
<td>( \dagger )</td>
<td>( \dagger )</td>
<td>( \dagger )</td>
</tr>
<tr>
<td>6.1517</td>
<td>7.0414</td>
<td>( \dagger )</td>
<td>( \dagger )</td>
<td>( \dagger )</td>
<td>( \dagger )</td>
<td>( \dagger )</td>
<td>( \dagger )</td>
<td>( \dagger )</td>
</tr>
</tbody>
</table>

* Possibly not BaD (see text).

\( D(\infty) = 28610 \cdot 63 \), being \( 28612 \cdot 63 \pm 1 \), as given for \( S(\infty) \) in [II.], with \( 1/m \), modified by \( \xi = -32 \) as explained in text, p. 358.

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Table II. (continued).

Ra.

\[ \Delta_1 = 92658. \quad \delta = 1853.16. \]

<table>
<thead>
<tr>
<th>( D_{12} )</th>
<th>( D_{13} )</th>
<th>( D_{13} )</th>
<th>O - C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.065042 - 360( \xi )</td>
<td>2( \delta_1 ) - 064116 + 10</td>
<td>2( \Theta ) - 060410 - 46</td>
<td>-00</td>
</tr>
<tr>
<td>5.081257 - 598( \xi ) + 26</td>
<td>2( \delta_1 ) - 080331 + 115</td>
<td></td>
<td>-01</td>
</tr>
<tr>
<td>6.090986 - 1028( \xi ) + 22</td>
<td></td>
<td></td>
<td>00</td>
</tr>
</tbody>
</table>

\( D(\infty) = 22760.19 \), being 22760.09, as given for \( D(\infty) \) in [II.], modified by \( \xi = 1 \).
The numbers are calculated from the problematic data given in [II., p. 65].

Zn.

\[ \Delta_1 = 7204.42. \quad \Delta_2 = 3486.20. \quad \delta = 154.93. \]

<table>
<thead>
<tr>
<th>( D_{11} )</th>
<th>( D_{12} )</th>
<th>( D_{13} )</th>
<th>O - C.</th>
<th>O</th>
<th>O - C.</th>
<th>O</th>
<th>O - C.</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.905892 (28) - 112( \xi ) - 20</td>
<td>15( \delta_1 ) - 905311 (49) + 36</td>
<td>93( \Theta ) - 904963 (100) - 15</td>
<td>-02</td>
<td>-03</td>
<td>-03</td>
<td>0.05</td>
<td>0.00</td>
<td>0.10</td>
</tr>
<tr>
<td>3.907519 (108) - 272( \xi ) + 6</td>
<td>13( \delta_1 ) - 907015 (108) + 54</td>
<td>105( \Theta ) - 906627 (?) + 43</td>
<td>-00</td>
<td>-03</td>
<td>-01</td>
<td>0.03</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>4.908913 (404) - 539( \xi ) - 8</td>
<td></td>
<td></td>
<td>00</td>
<td>0.05</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.909843 (2961) - 940( \xi ) - 40</td>
<td></td>
<td></td>
<td>00</td>
<td>0.20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>*6.909593 (4068)</td>
<td></td>
<td></td>
<td></td>
<td>0.20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>*7.910072 (11320)</td>
<td></td>
<td></td>
<td></td>
<td>0.30</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Collaterals (see text).

\( D(\infty) = 42874.17 \) being \( S(\infty) = 42876.42 + 3.34 - 1.08 \), as given in [II.], modified by \( \xi = -2.25 \).

\( D_{13}, D_{12} \) are calculated from the more accurate \( D_{21}, D_{22} \) by the use of the exact value of \( r_1 = 388.905 \).
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TABLE II. (continued).

Cd.

\[ \Delta_1 = 23105.56, \quad \Delta_2 = 10368.54, \quad \delta = 455.28. \]

<table>
<thead>
<tr>
<th>( D_{11} )</th>
<th>( D_{12} )</th>
<th>( D_{13} )</th>
<th>O-C.</th>
<th>O.</th>
<th>O-C.</th>
<th>O.</th>
<th>O-C.</th>
<th>O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.902039(28) - 111( \xi ) - 13</td>
<td>( 18\delta )</td>
<td>-899990(44) + 7</td>
<td>( 11\delta )</td>
<td>-898748(83) + 15</td>
<td>-01</td>
<td>-03</td>
<td>-01</td>
<td>-02</td>
</tr>
<tr>
<td>3.910576(81) - 272( \xi ) - 11</td>
<td>( 19\delta )</td>
<td>-908413(630) - 22</td>
<td>( 15\delta )</td>
<td>-906706(1) + 2</td>
<td>-00</td>
<td>-03</td>
<td>-00</td>
<td>-00</td>
</tr>
<tr>
<td>4.914104(216) - 541( \xi ) - 19</td>
<td>( 19\delta )</td>
<td>-911941(708) + 31</td>
<td>( 15\delta )</td>
<td>-00</td>
<td>-03</td>
<td>-00</td>
<td>-00</td>
<td>-10</td>
</tr>
<tr>
<td>5.915470(2656) - 942( \xi ) - 29</td>
<td>( 11\delta )</td>
<td>-00</td>
<td>-20</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.920598(6648) - 1511( \xi ) - 60</td>
<td>( 11\delta )</td>
<td>-00</td>
<td>-30</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ D(\infty) = 40710.85 \text{ being } S(\infty) = 40710.60 + 2.15 \]

Eu.

\[ \Delta_1 = 51223, \quad \Delta_2 = 18329, \quad \delta = 833.04. \]

<table>
<thead>
<tr>
<th>( D_{11} )</th>
<th>( D_{12} )</th>
<th>( D_{13} )</th>
<th>O-C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.930707 - 114( \xi )</td>
<td>( 29\delta )</td>
<td>-924667 - 25</td>
<td>( 33\delta )</td>
</tr>
<tr>
<td>3.942578 - 279( \xi ) - 5</td>
<td>( 20\delta )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4.946742 - 552( \xi ) + 17</td>
<td>( 13\delta )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5.949450 - 960( \xi ) + 94</td>
<td>( 13\delta )</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

\[ D(\infty) = 40363.19 \text{ being } S(\infty) \text{ in } [II, p. 73]. \] No estimated possible errors given.

\[ 2 \times 2. \]
Table II. (continued).

Hg.

\[ \Delta_1 = 87814.99 \quad \Delta_2 = 30002.3 \quad \delta = 1451.49 \]

<table>
<thead>
<tr>
<th>( D_{11} )</th>
<th>( D_{12} )</th>
<th>( D_{13} )</th>
<th>( D_{14} )</th>
<th>O – C.</th>
<th>O</th>
<th>O – C.</th>
<th>O</th>
<th>O – C.</th>
<th>O</th>
<th>O – C.</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.932562 (25) – 115( \xi ) – 5 11( \delta_1 )</td>
<td>.928571 (38) + 6 19( \delta_1 )</td>
<td>.921671 (38) – 9 ( \delta_1 )</td>
<td>.921314 + 8</td>
<td>( \cdot 00 )</td>
<td>( \cdot 03 )</td>
<td>( \cdot 01 )</td>
<td>( \cdot 05 )</td>
<td>( \cdot 01 )</td>
<td>( \cdot 01 )</td>
<td>( \cdot 05 )</td>
<td></td>
</tr>
<tr>
<td>3.942723 (151) – 279( \xi ) + 37 17( \delta_1 )</td>
<td>* .936554 (197) + 5 18( \delta_1 )</td>
<td>.930022 (611) + 7 15( \delta_1 )</td>
<td>.924379 (418) + 52</td>
<td>( \cdot 01 )</td>
<td>( \cdot 05 )</td>
<td>( \cdot 00 )</td>
<td>( \cdot 20 )</td>
<td>( \cdot 00 )</td>
<td>( \cdot 20 )</td>
<td>( \cdot 01 )</td>
<td>( \cdot 15 )</td>
</tr>
<tr>
<td>4.945989 (1396) – 552( \xi ) – 36 16( \delta_1 )</td>
<td>.940183 – 12 17( \delta_1 )</td>
<td>.934014 – 105 27( \delta_1 )</td>
<td>.924216 – 57</td>
<td>( \cdot 00 )</td>
<td>( \cdot 20 )</td>
<td>( \cdot 00 )</td>
<td>( \cdot 01 )</td>
<td>( \cdot 01 )</td>
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</tr>
<tr>
<td>5.948130 (7) – 960( \xi ) + 89 22( \delta_1 )</td>
<td>.940147 + 85 51( \delta_1 )</td>
<td>.921641 + 75</td>
<td></td>
<td>( ? )</td>
<td>( \cdot 00 )</td>
<td>( \cdot 00 )</td>
<td>( ? )</td>
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<tr>
<td>6.950307 { } ( 18\delta_1 (?) )</td>
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<td></td>
<td>( \cdot 00 )</td>
<td>( \cdot 00 )</td>
<td>( ? )</td>
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<tr>
<td>7.948774</td>
<td>( 18\delta_1 (?) )</td>
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<td>8.956716</td>
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<td>15.949</td>
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</tr>
</tbody>
</table>

* Calculated from \( D_{21} \).

\[ D(\infty) = 40140.35, \text{ being } S(\infty) = 40139.55 + 5.41, \text{ as given in [II.], modified by } \xi = 0.8. \]
### Table II. (continued).

#### AL

\[ \Delta = 1754. \quad \delta = 26'57. \]

<table>
<thead>
<tr>
<th>( D_{11} )</th>
<th>( D_{12} )</th>
<th>O - C</th>
<th>O</th>
<th>O - C</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>2·631287 (25) - 83( \xi ) + 0</td>
<td>45( \cdot )631181 (25) + 12</td>
<td>.00</td>
<td>.03</td>
<td>.01</td>
<td>.03</td>
</tr>
<tr>
<td>!17( \Delta )</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>3·426069 (82) - 183( \xi ) - 5</td>
<td>30( \cdot )425272 (82) - 8</td>
<td>.00</td>
<td>.03</td>
<td>.00</td>
<td>.03</td>
</tr>
<tr>
<td>94( \Delta )</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>4·261194 (200) - 353( \xi ) + 20</td>
<td>52( \cdot )259812 (200) + 2</td>
<td>.00</td>
<td>.03</td>
<td>.00</td>
<td>.03</td>
</tr>
<tr>
<td>54( \Delta )</td>
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</tr>
<tr>
<td>5·166498 (620) - 629( \xi ) - 130</td>
<td>20( \cdot )29</td>
<td>.01</td>
<td>.05</td>
<td></td>
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<td>29( \Delta )</td>
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</tr>
<tr>
<td>6·115632 (2088) - 1044( \xi ) + 160</td>
<td>18( \Delta )</td>
<td>.00</td>
<td>.10</td>
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<tr>
<td>7( \Delta )</td>
<td></td>
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<tr>
<td>7·087558 (3431) - 1626( \xi ) + 666</td>
<td>7( \Delta )</td>
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<tr>
<td>8·0173 + 400</td>
<td>9( \cdot )0604</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>10·0523</td>
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</tr>
</tbody>
</table>

\( D (\infty) = 48164 \cdot 12 \) being \( S(\infty) = 48161 \cdot 46 \pm 2 \cdot 49 \), as given in [II.], modified by \( \xi = 2 \cdot 66 \).

#### In.

\[ \Delta = 37684. \quad \delta = 477'01. \]

<table>
<thead>
<tr>
<th>( D_{11} )</th>
<th>( D_{12} )</th>
<th>O - C</th>
<th>O</th>
<th>O - C</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>2·823978 (48) - 102( \xi ) + 10</td>
<td>55( \cdot )821553 (48) - 10</td>
<td>.01</td>
<td>.05</td>
<td>.01</td>
<td>.05</td>
</tr>
<tr>
<td>37( \Delta )</td>
<td>58( \Delta )</td>
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</tr>
<tr>
<td>3·806329 (167) - 251( \xi ) - 76</td>
<td>26( \cdot )793927 (167) - 154</td>
<td>.02</td>
<td>.05</td>
<td>.04</td>
<td>.05</td>
</tr>
<tr>
<td>62( \Delta )</td>
<td>62( \Delta )</td>
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</tr>
<tr>
<td>4·776755 (392) - 497( \xi ) - 20</td>
<td>26( \cdot )764335 (784) - 287</td>
<td>.00</td>
<td>.05</td>
<td>.03</td>
<td>.10</td>
</tr>
<tr>
<td>44( \Delta )</td>
<td>50( \Delta )</td>
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</tr>
<tr>
<td>5·755767 (2954) - 869( \xi ) - 63</td>
<td>32( \cdot )740503 (7343) - 98</td>
<td>.00</td>
<td>.20</td>
<td>.00</td>
<td>.50</td>
</tr>
<tr>
<td>50( \Delta )</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>6·696300 (4791) - 1369( \xi )</td>
<td>7( \cdot )716503 - 179</td>
<td>.20</td>
<td>.08</td>
<td>.30</td>
<td></td>
</tr>
<tr>
<td>7( \cdot )</td>
<td>.715387</td>
<td></td>
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</tr>
<tr>
<td>8( \cdot )</td>
<td>.717621</td>
<td></td>
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<td></td>
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<tr>
<td>9( \cdot )</td>
<td>.717556</td>
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<tr>
<td>10( \cdot )</td>
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<tr>
<td>11( \cdot )</td>
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</tbody>
</table>

* Collaterals \((25\Delta) D_2(m)\).

\( D (\infty) = 44454 \cdot 76 \pm 2 \cdot 48 \) being \( S(\infty) \) of [II.].
DR. W. M. HICKS: A CRITICAL STUDY OF SPECTRAL SERIES.

Table II. (continued).

Ti.

\[ \Delta = 184154. \quad \delta = 1507.34. \]

<table>
<thead>
<tr>
<th>( D_{II} )</th>
<th>( D_{IIs} )</th>
<th>O–C</th>
<th>O</th>
<th>O–C</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.897392</td>
<td>*2.899520 (80) – 111( \xi ) + 10</td>
<td>*888344</td>
<td>.01</td>
<td>.03</td>
<td>.02</td>
</tr>
<tr>
<td>2( \delta_1 )</td>
<td>24( \delta_1 )</td>
<td>890476 – 16</td>
<td>.01</td>
<td>.03</td>
<td>-.02</td>
</tr>
<tr>
<td>3.898764 (89) – 270( \xi ) – 30</td>
<td>24( \delta_1 )</td>
<td>*888590 + 53</td>
<td>.00</td>
<td>.03</td>
<td>.00</td>
</tr>
<tr>
<td>2( \delta_1 )</td>
<td>887459 + 53</td>
<td>3( \delta_1 )</td>
<td>.00</td>
<td>.03</td>
<td>.01</td>
</tr>
<tr>
<td>4.898100 (213) – 535( \xi ) – 26</td>
<td>28( \delta_1 )</td>
<td>886329 – 116</td>
<td>.01</td>
<td>.10</td>
<td>.00</td>
</tr>
<tr>
<td>3( \delta_1 )</td>
<td>885199 + 39</td>
<td>.03</td>
<td>.10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.896880 (414) – 935( \xi ) + 1</td>
<td>28( \delta_1 )</td>
<td>-.03</td>
<td>.20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3( \delta_1 )</td>
<td>-.02</td>
<td>.20</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>6.895750 (2287) – 1495( \xi ) – 220</td>
<td>28( \delta_1 )</td>
<td>-.04</td>
<td>.30</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Collaterals (see text).

\[ D (\infty) = 41470.53 \pm 1.72 \] being \( S (\infty) \) of [II.] modified by \( \xi = .3. \)

O.

\[ \Delta_1 = 172. \quad \Delta_2 = 95. \quad \delta = 9.33. \]

<table>
<thead>
<tr>
<th>( D'' )</th>
<th>O–C</th>
<th>O</th>
<th>( D' )</th>
<th>O–C</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.972467 (40) – 120( \xi ) + 16</td>
<td>-.12</td>
<td>.3</td>
<td>2.980383 (93) – 121( \xi ) – 3</td>
<td>.03</td>
<td>1.00</td>
</tr>
<tr>
<td>9( \Delta_1 )</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>3.966621 (14) – 284( \xi ) – 12</td>
<td>.015</td>
<td>.018</td>
<td>3.978835 (72) – 287( \xi ) + 7</td>
<td>-.026</td>
<td>.26</td>
</tr>
<tr>
<td>9( \Delta_1 )</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>4.964213 (18) – 558( \xi ) + 15</td>
<td>-.008</td>
<td>.01</td>
<td>4.977287 (33) – 562( \xi ) + 28</td>
<td>-.017</td>
<td>.02</td>
</tr>
<tr>
<td>3( \Delta_1 )</td>
<td></td>
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<tr>
<td>5.962837 (38) – 966( \xi ) + 20</td>
<td>-.005</td>
<td>.01</td>
<td>5.976771 (68) – 974( \xi ) – 33</td>
<td>.010</td>
<td>.02</td>
</tr>
<tr>
<td>5( \Delta_1 )</td>
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<tr>
<td>6.961977 (67) – 1538( \xi ) – 15</td>
<td>.005</td>
<td>.01</td>
<td>6.976599 (215) – 1547( \xi ) + 38</td>
<td>-.007</td>
<td>.04</td>
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<tr>
<td>2( \Delta_1 )</td>
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<tr>
<td>7.961633 (309) – 2301( \xi ) + 20</td>
<td>-.001</td>
<td>.03</td>
<td>7.986756 (626) – 2321( \xi )</td>
<td>-.007</td>
<td>.07</td>
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<td>2( \Delta_1 )</td>
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<tr>
<td>8.961285 (551) – 3281( \xi ) – 63</td>
<td>.003</td>
<td>.06</td>
<td>9.976599 (7) – 4523( \xi ) + 1042</td>
<td>-.06</td>
<td>1</td>
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<tr>
<td>9( \Delta_1 )</td>
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</tbody>
</table>

\[ D'' (\infty) = 23204.1. \]

\[ D' (\infty) = 21204.2. \]

The observed \( D' \) corrected to give \( \nu = .72 \), treating the observed line as the mean of the doublet.
Table II. (continued).

S.

\[ \Delta_1 = 1044. \quad \Delta_2 = 651.7. \quad \delta = 37.28. \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \lambda_n )</th>
<th>( \Delta )</th>
<th>O - C.</th>
<th>O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4 \cdot 553453 )</td>
<td>( 430 \xi + 7 )</td>
<td>( 14 \Delta )</td>
<td>.01</td>
<td>.05</td>
</tr>
<tr>
<td>( 5 \cdot 544330 )</td>
<td>( 776 \xi - 7 )</td>
<td>( 5 \Delta )</td>
<td>0</td>
<td>.01</td>
</tr>
<tr>
<td>( 6 \cdot 539110 )</td>
<td>( 1273 \xi - 29 )</td>
<td>( 3 \Delta )</td>
<td>.007</td>
<td>.01</td>
</tr>
<tr>
<td>( 7 \cdot 535978 )</td>
<td>( 1944 \xi + 173 )</td>
<td>( 2 \Delta )</td>
<td>.02</td>
<td>.03</td>
</tr>
<tr>
<td>( 8 \cdot 533890 )</td>
<td>( 2828 \xi - 202 )</td>
<td>( 4 \Delta )</td>
<td>.02</td>
<td>.04</td>
</tr>
<tr>
<td>( 9 \cdot 529714 )</td>
<td>( 3936 \xi - 129 )</td>
<td>( 1 \Delta )</td>
<td>.01</td>
<td>.08</td>
</tr>
</tbody>
</table>

\[ D(\infty) = 20084.2. \]

Se.

\[ \Delta_1 = 6407. \quad \Delta_2 = 2722. \quad \delta = 226.8. \]

Calculated from Observed Values.

| \( 4 \cdot 629362 \) | \( 54 \xi - 452 \xi \) | \( \star \cdot 626133 \) | \( 108 \xi \) | \( 62297 \xi \) |
| \( 5 \cdot 621963 \) | \( 810 \xi \) | 617055 | 608778 |
| \( 6 \cdot 615643 \) | \( 1320 \xi \) | | |
| \( 7 \cdot 601450 \) | \( 2002 \xi \) | | |
| \( 8 \cdot 611058 \) | \( 2911 \xi \) | | |
| \( 9 \cdot 592507 \) | \( 4034 \xi \) | | |
| \( 10 \cdot 57831 \) | \( 540 \xi \) | | |

\[ D(\infty) = 19274. \]

Modified Table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \lambda_n )</th>
<th>( \Delta )</th>
<th>O - C.</th>
<th>O.</th>
<th>O - C.</th>
<th>O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4 \cdot 629285 )</td>
<td>( 22 )</td>
<td>( 55 \xi_1 )</td>
<td>( 626167 + 11 )</td>
<td>( -02 )</td>
<td>( .06 )</td>
<td>( -01 )</td>
</tr>
<tr>
<td>( 5 \cdot 624976 )</td>
<td>( 41 )</td>
<td>( 38 \xi_1 )</td>
<td>( 622822 - 56 )</td>
<td>( -02 )</td>
<td>( .03 )</td>
<td>( .03 )</td>
</tr>
<tr>
<td>( 6 \cdot 620667 )</td>
<td>( -5 )</td>
<td>( 55 \xi_1 )</td>
<td>( 617549 - 362 )</td>
<td>( .00 )</td>
<td>( .03 )</td>
<td>( .10 )</td>
</tr>
<tr>
<td>( 7 \cdot 616358 )</td>
<td>( 156 )</td>
<td>( 55 \xi_1 )</td>
<td>( 613240 - 531 )</td>
<td>( -02 )</td>
<td>( .04 )</td>
<td>( .09 )</td>
</tr>
<tr>
<td>( 8 \cdot 612049 )</td>
<td>( 700 )</td>
<td>( 196 \xi )</td>
<td></td>
<td>( .076 )</td>
<td>( .08 )</td>
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<tr>
<td>( 9 \cdot 607740 )</td>
<td>( 91 )</td>
<td>( \Delta_1 + \Delta_2 )</td>
<td></td>
<td>( .00 )</td>
<td>( .05 )</td>
<td></td>
</tr>
<tr>
<td>( 10 \cdot 59860 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( .15 )</td>
<td></td>
</tr>
</tbody>
</table>
The Satellite Separations.—As the values of the satellite differences are practically independent of the exact value of \( D(\infty) \), their consideration may be taken up at once and the details of the calculations respecting the tables postponed until the consideration of the order differences. An examination of the tables will show conclusively that these differences are multiples of the “oun.” Dealing first with the differences for the first lines, the following figures, contained implicitly in the tables, will show how closely this is the case. The nearest multiples of the oun are appended, as calculated from the first approximations of Table I. The possible errors are those of the \( D_{11} \) lines (except Zn).

<table>
<thead>
<tr>
<th>Element</th>
<th>Satellite Difference</th>
<th>oun</th>
<th>Calculated Difference</th>
<th>Possible Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cs</td>
<td>7394±228</td>
<td>46δ₁ = 7340</td>
<td>2029±28</td>
<td>18δ₁ = 2049</td>
</tr>
<tr>
<td>Cu</td>
<td>794±43</td>
<td>22δ₁ = 804</td>
<td>1234±44</td>
<td>11δ₁ = 1248</td>
</tr>
<tr>
<td>Ag</td>
<td>2424±19</td>
<td>23δ₁ = 2422</td>
<td>6065±7</td>
<td>29δ₁ = 6040</td>
</tr>
<tr>
<td>Ca</td>
<td>746±8</td>
<td>51δ₁ = 741</td>
<td>6822±7</td>
<td>33δ₁ = 6873</td>
</tr>
<tr>
<td>Sr</td>
<td>3620±7</td>
<td>52δ₁ = 3596</td>
<td>3980±25</td>
<td>11δ₁ = 3991</td>
</tr>
<tr>
<td>Ba</td>
<td>2170±?</td>
<td>31δ₁ = 2154</td>
<td>6909±38</td>
<td>19δ₁ = 6894</td>
</tr>
<tr>
<td>Zn</td>
<td>525±77</td>
<td>15δ₁ = 581</td>
<td>346±38</td>
<td>δ₁ = 363</td>
</tr>
<tr>
<td></td>
<td>369±28</td>
<td>9δ₁ = 350</td>
<td>581±9</td>
<td>14δ₁ = 93</td>
</tr>
</tbody>
</table>

The only case of “failure” is the first difference for Sr in which the estimated possible error is extremely small. If the possible error be the sum of those of each line, the value is 15 in place of 7, and if δ₁ be calculated from the most probable value of the oun it should be about \( \frac{9}{40} \) greater, i.e., \( 52δ₁ = 3600 \). It will be noted that where triplets enter, the two satellite differences, and consequently the two satellite separations, are extremely close to the ratio 5:3. This ratio seems to persist also in Hg where the separations are in reverse order, and we find a ratio of 3:5 in place of 5:3. The law for this ratio is in fact much more closely obeyed than the corresponding one for the ratio 2:1 for the triplet separations. It is therefore of great assistance in searching for the lines of F series whose limits are \( VD(2) \), and which consequently possess constant triplet separations in this ratio. Its explanation should be expected to throw some valuable light on the constitution of the atom. The general dependence of the differences on the small “oun” δ₁ should also be noted.

Passing now to the consideration of the satellite differences for orders beyond the first \( (m>2) \), it is seen that they still depend on multiples of the oun, but different from those of the first order. In a large number of cases the multiples are the same.
for different orders within limits of errors, especially in the doublets and differences of the second and third satellites. Thus we find Cs, 14δ; Ca, 13δ, 8δ; Sr, 12δ for 
$m = 3$ and 15δ for $m > 3$ for first separations, and 8δ for all orders in the second; Ba 
and Zn show too few for comparison (see discussion below); Cd, 19δ, 15δ; Hg is 
irregular, Al is anomalous; In 26δ, Tl, 27δ, for $m = 2$, and 28δ, for $m > 3$; Se, as 
amended later, shows 55δ, for $m = 4$, 6, 7, and 38δ, for $m = 5$, the lines for $m < 3$
being outside region of observation. The evidence points to a normal rule that the 
differences for the orders beyond the first in any spectrum are the same, but different 
from—in general greater than—that of the first.

The Order Differences.—The order differences change very considerably with a 
change in the value taken for the limit, i.e., in the value given to $\xi$. No doubt with 
unlimited choice of $\xi$ it would be possible to arrange a set of denominator differences 
all multiples of the own within error limits, for a series of values of $\xi$ could be found 
making the first difference a multiple. Out of these one or two would probably give 
the second such a multiple. After the second the error limits as a rule come to be 
very large, in fact larger than half the own itself, except in case of very high atomic 
weight. No conclusions could be drawn from any such arrangement. But in the 
present cases the choice of $\xi$ is bounded by very narrow limits, for the relation 
$D(\infty) = S(\infty)$ is supposed to hold, and, as a rule, the values of $S(\infty)$ are known 
with very considerable accuracy, and the possible limits of variation are known. 
They were given in [I.] and [II.]. Before proceeding to draw general conclusions 
from Table II., it will be well to consider in more detail the data for the different 
elements on which the table is based.

Na.

Although the readings for NaD are very inexact, the peculiarity of the large 
depression shown for $m = 6$, as well as the large recovery afterwards to mantissae 
close to unity, must be real effects. It is of course possible that NaD$_{1}(6)$ is a 
collateral from the normal type. If D$_{1}(6)$ be calculated from D$_{2}(6)−\nu$, the mantissae 
becomes 989054, in other words, the D$_{2}$ line begins to show the rise to the large 
final value at $m = 6$, whilst D$_{1}$ does not do so until $m = 7$. The D$_{3}$ lines would seem 
zu succumb to the disturbing effects sooner than the D$_{1}$. It was pointed out in 
[I., p. 83] that in the Na the D series apparently belongs to the F type, in which 
the mantissa is 998613. It would almost seem that the peculiar rise shown is due to 
the fact that it reverts to this F sequence. Here, as we shall see in other cases, 
the values of the first members of these series appear to be subject to some kind of 
displacement which affects their (supposedly) normal relations to other lines. If now 
the first mantissa be supposed normally to be this 998613, it is 9691 above that in 
the modified table, and this is $13\Delta + 32$, thus completing the order differences as 
multiples of $\Delta$. But in any case the data for Na are of small use for the present 
purpose, as the errors are so large, and $\Delta$ so small. The arrangement in the table 

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gives the values of \( O - C \) least. If \( \xi \) be taken about \( '6 \), the order differences can run 
\[ 3\Delta, 4\Delta, 4\Delta \] 
within error limits.

K.

It is seen that the numbers, with the exception as in other cases of the first 
difference, fall into multiples of \( \Delta \) quite naturally without a change in \( D(\infty) \), though 
possibly a small change in \( \xi \) might make the values of \( O - C \) less. \( \Delta \) is so large, that 
the theory of the dependence on multiples receives considerable support. The first 
also is close to \( 20\Delta \). This element is one in which the value given in the first 
discussion above for the \( O - C \) is \( 362968''99r \), which is presumably too large by '8 to '9. 
If it be \( 361'8 \), \( \Delta \) should be \( 4\frac{1}{10} \) less and = 2933. This would scarcely effect the other 
intervals, but it would make the first one = \( 20\Delta - 704 \). Again there is a sudden fall 
(at \( n = 8 \)). It doubtless corresponds to a real effect, for SAUNDERS as well as 
LIVEING and DEWAR make \( \nu \) anomalous here. S. observes \( v = 61'25 \) and L.D. 59'15, 
but both give \( D_2(8) \) the same. If this be taken as correct and the normal \( D_2(8) \) 
found from \( D_2 - 57'87 \), the mantissa is 733756, giving the same difference \( 2\Delta \). This 
shows that the \( D_2 \) set have not participated in the sudden fall—at least to the same 
degree as \( D_1 \)—a result analogous to what happens in Na.

Rb.

In Rb there is some doubt whether a satellite series exists. The question has 
already been discussed in [I., pp. 71, 86]. SAUNDERS has given for \( D(3) \) lines whose 
wave numbers are 12883'93, 12886'56, and 13121'19, with normal separations 237'26, 
and satellite separations 2'63. Also \( D(4) \) is a doublet having a separation 235'52, 
which certainly points to an unobserved satellite about 2'4 ± '1. But RANDALL'S 
observations for \( D(2) \) show only a doublet of normal separation—that is clearly with 
no satellite. Moreover the \( F \) series, which depends on \( D_2(2) \) and \( D_2(2) \) for its 
limits, is a singlet series and not a double one. In the table the series is taken as if 
it were without a satellite, the reading for \( D_1(4) \) being corrected from \( D_2(4) - \nu \). In 
other words it represents the satellite lines if they actually do exist. In the latter 
case the strong lines would show denominators about 609, and 1100 above those in 
the table for \( D_2(3) \) and \( D_2(4) \). The first is about \( 2\delta \), and the second of the order of 
\( 4\delta \), whilst if normal, judging from other cases, they should be equal. Moreover, in 
all other cases (In and Ti excepted) the satellite separations are practically the same 
multiples of the own in the same group. Cs shows a difference of 14\( \delta \), so that the 
supposed ones here are far too small, as well as irregular. The observed separations, 
moreover, are equal within errors of observation, which would rather point to an 
alteration in \( D_1(\infty) \). Now a lateral displacement of \( 2\delta \) on \( D(\infty) \) would produce a 
separation of 2'41, which is about the observed value.

The table shows a stationary point at \( m = 8 \) and 9 and then the large fall shown 
in the other elements. They could be accounted for by a lateral displacement.
(2δi)D(∞), the mantissa of m = 8 being at the same time subject to the fall of multiples of δ which would scarcely stop at 14δ. In the table, however, the errors are inserted on the supposition that the mantisse remain constant.

Cs.

The mantissæ appear to run down by equal intervals from m = 4 to 6, are equal for 6, 7, 8, then a large drop of about 11δ to the same value for 9, 10, and return to the value at m = 8 for m = 11. The possible errors are so large that the regularity is curious. It is possible that they might run down at equal intervals of 3δt to the last one for m = 10. Or, if there are very small observational errors, the drop for 9, 10 may be due to a lateral displacement, about (+7δ)D(∞). It should be noticed that with Cs all the order differences but one are multiples of 3δi, or the group multiple.

Cu.

The two first doublets of CuD are strong. The third is much weaker than would be expected. Moreover, it gives a separation between D11 and D12 of 252·14, whereas it should be somewhat less than ν = 248·28. This (λ = 3688·6) can therefore scarcely be the normal chief line of this doublet. Now Éder and Valenta give a spark line at 3687·75 which gives a separation with D20 of 245·91, leaving a satellite separation of 2·37, which is within limits in fair order with the corresponding separations in the two previous doublets, viz., 6·60 and 3·39. Moreover, the satellite separation of 2·37 gives a satellite difference of 1317 and 9δ = 1315, so that the normal satellite differences would run 22δi, 27δi, 36δi. This, then, would seem to be the wanting normal chief line, and it is then interesting to note that the line usually accepted as D11(4) is a collateral of this. The denominator difference of the two is 2474 and 17δ = 2484 (δ = 146). Hence the K.R. line 3688 is the collateral D11(4) (−17δ), and apparently the small intensity is due to the usual decrease produced by a negative lateral displacement. The modified table is taken on ξ = −·08. It is remarkable how close the observed differences come to multiples, but little reliance must be placed on deductions from them both on account of the large possible errors and the smallness of δi.

Ag.

Unfortunately the D series in Silver is poorly developed—only the first satellite has been observed, and the three chief lines after the first have very large possible observation errors. Nothing, therefore, can be learnt as to the march of the satellite differences beyond the fact that the observed—2436—is very close to 23δi = 2421. In the modified table ξ = 1, the differences are very close as is seen to multiples of δi, but there can be no certainty with such large possible errors.
Mg.

In the D series of Magnesium, as arranged by Kayser and Runge and as generally accepted, there are clearly certain abnormalities. \( D_1(4) \) is more intense than we should expect, and its separation from \( D_2(4) \) is 45'39 in place of 40'92, whilst that of \( D_3(4) \) and \( D_4(4) \) is very close to the exact value. This cannot be due to observational error, for this is very small ("03). Either, therefore, the true line is hidden by this bright one, which can scarcely be the case, or it is a collateral. In the former case the true line ought to be that found by deducting \( v_1 \) and \( v_1 + v_2 \) from the satellites. In the second case it would require the addition of 19\( \delta \), to the denominator of \( D(\infty) \), and the addition would explain the increased intensity. The two results agree, the wave numbers resulting being respectively 35054'80 and '71. The former would give a denominator '832041 in place of that in the table, but its observational error would be that of \( D_2 \), viz., 945, while that of the collateral depends on the observed \( D_4 \), and is 190. \( D(5) \) gives normal separations within limits. \( D(6) \) gives \( v = 46'87 \) and 22'15, but normal within the observation errors (2'8) [see Note 1 at end].

But there is another question which arises in connection with Mg. In the Ca sub-group the first lines have a denominator about 1'9, i.e., with \( m = 1 \). In the Zn sub-group the lowest value of \( m \) is 2. In the MgD series, as generally accepted, the first line is \( \lambda = 3838 \), which requires \( m = 2 \). If there is a line corresponding to \( m = 1 \) it should be in the neighbourhood of 14900. Now Paschen has observed a strong line at \( \lambda = 14877'1 \), but there is no triplet, which would be decisive against the allocation if we could be certain all the lines must exist. But there are cases where normal lines are observed weaker than we should expect, or are not seen at all. The well-known case of KD(3) is one example, and it is curious that if 14877 be taken as MgD(1) the denominator comes out as given in the table in a very natural order with the other denominators. The question is considered later under the F series, and the evidence there adduced is rather against the present suggestion (p. 398).

Ca.

The value of \( \delta \) is calculated from \( \Delta_2 \) as 58'14, \( \Delta_1 + \Delta_2 \) would give 58'18, practically the same. To bring the differences of the first three denominators to multiples of \( \delta \) it is necessary to diminish the limit given from the consideration of the S series by 1'6 (variation limits given in [II.] = 5'8). The values can then be arranged as in the table. One result of this is to increase the value of \( \Delta_1 \) (for the given \( v_1 = 105'89 \)) to 2793 from 2791, which gives \( \delta = 361'30m^2 \) in place of 361'17, and thus closer to the adopted value 361'80. A noticeable peculiarity in this series is the very rapid falling down of the denominators after \( m = 4 \). It is so large and at the same time so irregular that they cannot be brought into line with the others without diminishing the limit by a large amount and by different amounts. It clearly points to the existence
of collaterals, formed by the addition of oums to the limit $D(\infty)$. As such increase tends to increase intensity it may account for these surviving when the typical ones are either too faint or are destroyed to form the collaterals. It is useless to attempt to determine these multiples, because the observation errors are so large themselves as to be a large multiple of the oum, and at the same time we have no knowledge of what the typical $VD(m)$ should be. In general in the 2nd group the successive denominators are formed by the successive addition of smaller and smaller multiples of the oum until probably a constant value is reached. In the present case, with the quantity $155\delta$, that limit is certainly not reached. But it may be instructive, in order to illustrate the nature of the suggestion, to find what the collaterals ought to be if the denominators of $VD(m)$ remain the same for $m > 4$, viz., $091707$. The multiples are found to be $7\delta, 15\delta, 33\delta$. The series of the observed $D_{12}$ lines may then be exhibited by the following scheme, where $d$ stands for $091707$:

\[
\begin{align*}
D(\infty) & - VD(2+d-99\Delta_2-155\delta), \\
D(\infty) & - VD(3+d-155\delta), \\
D(\infty) & - VD(4+d), \\
D(\infty)(+7\delta) & - VD(5+d), \\
D(\infty)(+15\delta) & - VD(6+d), \\
D(\infty)(+33\delta) & - VD(7+d).
\end{align*}
\]

The line $D_{12}(2)$ is interesting. PASCHEN gave it as 19859'9 with the remark "Wahrscheinlich doppelt 19856'9, 19864'6," and he allotted 19864'6 to $D_{12}$ and 19856'9 to the Principal series. But in [II, p. 56] reasons were given against the latter allocation. In fact the line is very close (probably within observation errors) to the collateral formed by adding one oum to $D_{12}$. The wave-length of such would be $19857'8 = D_{12}(2)(+\delta)$ [see Note 2 at end].

Sr.

The value of $\delta$ is calculated as 277'89 from $\Delta_1 + \Delta_2 = 125 \times \delta_2$, which gives $\delta = 361'64\mu\lambda$. The differences as shown in the table are extremely close to multiples of $\delta$. Moreover, the limits of variation for the first two are so small that the variations of ROWLAND's standards from the correct values for his scale may become of importance. For $D(3)$ the values should be 2 less, whilst for $D(2)$, failing direct observations for reduction to vacuo, recourse must be had to extrapolation on KAYSER and RUNGE's formula,* which has been done. In order to bring the differences for $D_{11}(3)$ and $D_{11}(4)$ and of $D_{12}(4)$ and $D_{12}(4)$ to multiples of $\delta$ within error limits, it is necessary to take $\xi$ about $-4$ or $D(\infty) = 31027'25$. When this is done the denominators can be arranged as in the table. The difference of the two

* RANDALL appears to have done this for $D_{11}$ but not for $D_{12}$, which also makes his value of $v_1 = 392'6$ instead of 394'42, which is close to the true value.
first denominators = 181557 = 653 x 278\textquoteright 03. It is possible the real errors attached to $D(2)$ by Randall may be greater and the difference slightly less; but if we suppose 278\textquoteright 03 to be the real value of $\delta$ it makes $\delta = 361\textquoteright 82\omega^{2}$, and, therefore, very close to the adopted value. It would appear that $D_{13}(4)$ has been displaced from its normal value, judging from the irregularity introduced into the separations. If so the separations might be $14\delta$ in place of $15\delta$.

Ba.

Starting with the uncertain value of $S(\infty) = 28642\textquoteright 63$, as given in [II.], the value of $\delta$ as calculated from $\Delta_{1}^{2} + \Delta_{3}$ is 68270, and from $\Delta_{3}$ is 68434, both being near the most probable values but on opposite sides. The value 683 is taken at first as a rough approximation. Apparently, the first set of lines have not been observed. Randall\* has observed two lines, 29223\textquoteright 4, 23254\textquoteright 8, which give a separation 878\textquoteright 27, which is $\nu_{1}$, but no signs of satellites—or, rather, if there are satellites, the separation observed should be much smaller. If, however, the satellites have gone here, and this pair denote the first two lines of the first triplet, they depend on $VD_{13}$ and the value of the denominator is 2085331,† which would range well with those of Ca and Sr, viz., 1946, 1987, but the second lines of these give 3082, 3169, and of Ba 3093, which would rather point to a less value than 2085 for the first line. But if $VD(2)$ is larger than $D(\infty)$, the lines would be -23254\textquoteright 8 for the first and -29223\textquoteright 4 for the second, giving a denominator for the first of 1825551. The differences of the denominators of the $D_{13}$ lines for $m = 2, 3, 4$, will then be 267632, 38612 instead of 6528, 38612, and are therefore more in agreement with the type of the other elements of this group. Moreover, the former, as we shall see, is a multiple of $\delta$, whilst the other (6528) is as far out as it can be. Both values, however, are inserted in the tables (see also p. 389).

The satellite differences for $D(3)$ are 9492±32 and 7516±40. The values of $14\delta$ and $11\delta$ are respectively 9562±5 and 7513±4, and hence the first cannot be 14\delta within limits of error, although it is so close as to produce a conviction that it really is so. Now for small variations of the limit $D(\infty)$ the separation differences are scarcely affected, but, as we saw in [II.], there was evidence to show that the limit $S(\infty)$ was considerably less than that found, and, in that case, the separation differences would be slightly changed. A decrease of $D(\infty)$ would increase those differences. If, however, it is so large as to bring up the first to 14\delta, the second is increased so much that it is not 11\delta within limits. Consequently, the two conditions confine the choice of $D(\infty)$ within very narrow limits. It is found to be close to a decrease of 32, i.e., $D(\infty) = 28610\textquoteright 63$. This again changes the values of $\Delta_{1}$, $\Delta_{3}$, with the given values of $\nu_{1}$ and $\nu_{3}$ to 29379 and 11997, giving from $\Delta_{1}$, $\delta = 683\textquoteright 2$. The table is

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† The values in this and in the table are calculated from the limit as modified below.
drawn up on this basis. With these data, the suggested allocations for D₁(2) give the following differences between their denominators and that of D₁₃(3), viz.:

\[
\begin{align*}
\text{With } D₁₃(2) &= 29223 \\
D₁₃(2) &= -23254
\end{align*}
\]

\[
6528 = 9.5 \times 683 \\
267632 = 392 \times 683.
\]

The first, therefore, cannot be a multiple within error limits.

The values shown for D(4) agree very well, but the regularity is upset. Also the actual lines have changed in appearance, and their intensities are not normal. The intensities for D₁(3) are 10, 6, 4, those for D₁(4) 4, 4, 6. We should expect D₁(4) to be stronger, and certainly D₁₃(4) to be much weaker. It would seem that collaterals must displace the normal lines. We have, in the foregoing pages, been led to expect that an addition of an own increases the intensity and a deduction diminishes it. If so, we should expect a deduction in D₁₁ and an addition in D₁₃. To bring D₁₁ 14\(\delta\) above D₁₃ requires the deduction to be made in VD. This would make the typical value of the denominator greater by 2733, viz., 4'161071. In the case of D₁₃, to bring it closer to D₁₂, i.e., distant 11\(\delta\), the addition would have to be to D(\(\infty\)), and if so, the value of 29\(\delta\), given in the modified table, would have been a mere coincidence. But no such addition of a multiple of \(\delta\) (nor of \(\delta\)) will do this. If, however, 2\(\delta\) be added to the denominator in D(\(\infty\)), it is brought to separation of 10\(\delta\)+36, giving an error in \(\lambda\) for D₁₃(4) of -0.02 in place of -0.06. If, then, \(\delta\) be also deducted from VD, the separations will be 11\(\delta\). The separations would then take the form

\[
\begin{align*}
423\delta &
66\delta \\
14\delta &
14\delta
\end{align*}
\]

This arrangement is to be preferred in that (1) it explains the abnormal intensities of D(4), (2) brings the separations into line with other elements. The arrangement suggested may be stated thus: if D₁₁(4) and D₁₃(4) represent the typical lines, the observed D₁₁ is D₁₁(4) (-2\(\delta\)) with decreased intensity, and D₁₃ is (+2\(\delta\) D₁₃(4) (-\(\delta\)), the increased intensity due to the +2\(\delta\) in D(\(\infty\)) being greater than the decrease due to -\(\delta\) in VD.

The results for orders > 4 are similar to those of the other elements of their groups, probably collaterals of additions to D(\(\infty\)).

Zn.

On account of the small values of \(\delta₁\), and the considerable observation errors, the satellite separations in Zinc do not give decisive results. If we take the observed values for the D₁₂ and D₁₃ sets, the denominator differences are 489, 419 for \(m = 2\), and 584, 399 for \(m = 3\). Now the second sets, the D₂, have much smaller observation
errors than the satellites of the first—see remark in the table. If the latter be calculated from them, using the accurate value of \( v_1 = 388.90 \), the differences come out to be 525, 369 and 456, 399, quite reversing the order of magnitude for the first satellite of the lines for \( m = 2, 3 \), and, moreover, their differences are larger than \( \delta_i \).

The differences best consonant with the measures—using the derived values from \( D_x \)—are \( 13\delta_1 = 503 \) and \( 10\delta_1 = 388 \), giving a ratio of \( 1:30 \). Those entered in the table, however, are \( 15\delta_1 = 581, 9\delta_1 = 348 \), both within the limits of observation, and adopted because they give a ratio \( 5:3 \), the same as the other elements of Group 2. The satellite separations for \( m = 3 \) may be the same as the latter within limits, but not necessarily so.

The order differences do not work in well with the above when \( \xi = 0 \). If, however, \( \xi \) be put \(-2.25 \), they fall into line with extreme accuracy, as shown in the table. It is of interest to notice that the differences are multiples of \( \delta_n = 6\delta_1 \), which seems to be specially associated with zinc.

The denominators for \( m = 6, 7 \) are now 6'915855 (4068) and 7'924179 (11320), showing too large a rise to be due to error observations. Treated as collaterals with \(-2\delta_1 \) and \(-3\delta_1 \) they become 6'909593, 7'910070, clearly near the probable limit.

Judging from analogy, we should not expect the differences to stop at \( 4\delta_n \). The series \( (7, 6, 4) \delta_n \) would probably be continued, but the errors are too large to settle the question. If the series were continued, e.g., by \( 2\delta_n, 0 \), the denominators would be 6'910288 - 698 and 7'910288 - 218. But the best agreement is to take them as they are. This would also be in analogy with Ca, in which \( D(\infty) \) begins to change when \( VD \) stops changing.

Cd.

In the table \( \xi \) is taken 25 above \( S(\infty) \), as given in [II.]. It is seen that the arrangement fits in with great accuracy, and as \( \delta_i \) as large as 114, the arrangement may be considered to have some weight. The denominators calculated from the \( D_3 \) and \( D_2 \) lines (more accurately determined) do not agree with the observed \( D_3 \) satellites. It would therefore appear that the second and third members of the triplets may also be subject to special displacements. Here, for instance, the lines of order 3 are brought into line if the observed \( VD_3(3) \) is \( VD_3(3)(-\delta_i) \). The value of \( D_3(4) \), calculated from K.R.'s \( D_3(4) \), cannot be the normal one, even when his extreme possible error is allowed. This shows again that \( VD_3(4) \) must be displaced from \( VD_3(4) \), in this case by \( 10\delta_i \).

The denominator for \( m = 6 \) shows the sudden large rise after a slow change which Zn also exhibits, but it cannot be explained, as in that case, by treating it as a collateral due to a modification of \( D(\infty) \) alone. The closest collateral of this type would be due to \( \delta_i \) and this changes \( D(\infty) \) by 5'66 producing in the denominator a change too far in the opposite direction. In fact it becomes 6'912000, 3471 below.
that of \( m = 5 \) in place of 5060 above. But the observation error in this line is so large that as it stands it may correspond really to a denominator equal to that of \( m = 5 \). If further the series of differences \( 75\delta_1, 31\delta_1, 12\delta_1 \), be continued diminishing further as looks probable, it would come more nearly in line. For instance, as each term is about '4 the previous, suppose the next is \( 5\delta_1 = 569 \). The value of \( D_{13} \) as thus calculated differs from the observed by '19 while the possible error is '30. This would make the constancy of the denominator begin at an order one higher than in Zn. The case of Hg will be seen to support this tendency.

There are two sets of doublets, 364974, 350009, and 300553, 290324 with separations 117113 and 117195 which are clearly associated with the D series. If we write down the observed satellite separations in the \( D_1(2) \) and the \( D_1(3) \) lines and of \( D_{13} \) with the above we get the following scheme: 1823, 1110, 26713 and 798, 618, 26240. At first sight it makes the new lines appear as collaterals by the change of about \( \frac{1}{2}\Delta \), on \( D(\infty) \), but this cannot be the case, because a change of this amount would very considerably diminish the doublet separations below 1171. If the \( D_{13} - D_{13} \) separation be deducted from that of the lines in question there results 26713–1110 = 25603 and 26240–618 = 25622. Now the separations 1110, 618 depend, as is seen above, on \( 11\delta_1, 15\delta_1 \), so that the new lines may be written \( *D(\infty) - VD_{13}(2)(-11\delta_1) - \Delta, D(\infty) - VD_{13}(3)(-15\delta_1) - \Delta \), where \( \Delta \) is a constant which on more accurate calculation is found to be 25580±2. In other words, the \( VD \) of the new lines is derived from \( VD_{13} \) in the same way as that is from \( VD_{12} \). This formula is of a type of which there has been no example hitherto. If it remained there the evidence, in spite of the curious connection with the other satellites, would scarcely be weighty enough to cause the introduction of a new departure. I hope however to show in a future communication that this expresses a very common relation between sets of lines, the constant \( \Delta \) being in reality a composite one. The question naturally arises do the new terms give rise to an \( F \) series in the same way as the ordinary \( D \) ? It should be at a distance about 26713 in wave number above that of \( F \). The line 157133 \((n = 636223)\) is 26690 above that of the line 164015 \((n = 609535)\) which is allotted by PASCHEN to \( F_3(3) \) and is clearly the first of the lines in question. There is an \( F(4) \) line at 116308 \((n = 859557)\) and another at 112684 \((n = 887201)\) is 26644 above this. This may be the corresponding line sought for, but if so the line 11630 must be \( F_3(4) \) and the lines \( F_1(4), F_2(4) \) would then be absent. These lines were at first assigned by PASCHEN to a new doublet set of series, but later to combinations of his new singlet series with \( D_1(\infty), D_2(\infty) \). This question will be considered as a whole later, but the suggested explanation given above points rather to the fact that we have to do with a triplet series in which the third number is too faint to be observed.

\* \( D(\infty) \) stands as usual for \( D_1(\infty) \) or \( D_2(\infty) \).

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The D series of Mercury shows a marked divergence from those of the other elements so far considered in that (1) the separations of the satellites increase as they go from the chief line, (2) the satellites do not seen to correspond in the different orders, and (3) there are a larger number. The increased tendency which this element has shown to break up into collaterals appears also here. One is led to infer that with varying conditions of the production of the light different collaterals appear.

The dependence on the sun is, however, here clearly shown, and the evidence is all the stronger because the magnitude of \( \delta \), itself is large (363) and because all the apparently unconnected differences come within close multiples of \( \delta \). This is clearly seen in the following table where the denominator differences are exhibited together:

\[
\begin{align*}
3980 &= 11\delta_1 - 11 & 6909 &= 19\delta_1 - 15 & 346 &= \delta_1 - 17 \\
6201 &= 17\delta_1 + 32 & 6530 &= 18\delta_1 + 2 & 5398 &= 15\delta_1 - 45 \\
5782 &= 16\delta_1 - 20 & 6262 &= 17\delta_1 + 93 & 9750 &= 27\delta_1 - 48 \\
7989 &= 22\delta_1 + 6 & 18516 &= 51\delta_1 - 10.
\end{align*}
\]

It is still possible within errors that the differences for the first satellites shall be the same as for the second and succeeding, viz., 17\( \delta \), but it is very unlikely. For the second satellites this cannot hold. It is clear that the regular law is not contradicted, but is upset by the formation of new configurations or aggregations in the oscillators.

The table is drawn on \( \xi = 8 \). This brings in the best agreement and, moreover, brings \( S(\infty) \)—supposing it and \( D(\infty) \) are the same—closer to the value found from the first three lines. The value given in [II.] being one modified slightly to bring all calculated (even for \( m = 1 \)) within limits. The agreement is seen to be remarkably close. It is to be expected that the differences for the satellites of the same lines will be more accurate than the differences between the chief lines themselves, and this is exemplified in the table. The observation errors after \( m = 6 \) are too considerable to draw certain conclusions from. Apparently the denominators increase by small multiples of \( \delta \) to about \( m = 8 \) and then remain constant.

Al.

In Al, the satellite differences deviate from the ordinary rule in that they increase with increasing order for \( m = 2, 3, 4 \). They are 94, 800 and 1380, and by no stretching to the extreme possible errors can the two last be made equal. The inequality is certain. Moreover, the observed differences are very close to multiples of \( \delta \). If the first satellite position be calculated from \( D_n \), its difference is 110, the observed is 94 and 4\( \delta = 106 \), 94 can be 106 within limits. \( D_3 \) gives in the same way 800 for \( m = 3 \), the same as observed and 1468 for \( m = 4 \) instead of 1380. The last may be the same as the observed within limits, but as 52\( \delta = 1381.6 \) and 55\( \delta = 1462 \),
it is possible there may be this difference and \( VD_{21}(4) \) is not \( VD_{12}(4) \) as in the typical cases. The real difference may be any multiple between \( 52\delta \) and \( 55\delta \). In fact \( \delta \) is so small that there is not absolute certainty.

AID has proved itself the most intractable series to bring into any simple formula of the ordinary kind. It was, in fact, the difficulty with this element which first led me to seek another solution—on the lines now being considered.

It will be seen that it lends strong support to the theory suggested. The table is arranged with \( \xi = 2'66 \). The exactness of the relations there shown is very remarkable, and when it is remembered that \( \Delta \) is a large number like 1754, the practically exact multiples referring to the first five lines must carry very great weight in the argument that AID at least is subject to a modification of successive denominators by multiples of certain units. The objections to the arrangement are two: (1) that \( \xi = 2'66 \) is outside the error limits of \( S(\infty) \), and (2) the denominators appear to go on diminishing without reaching a limit. A slight alteration, however, in \( \Delta \) will get over the first. For instance, if \( \Delta = 1754-5 \), \( \xi \) would be about '5 less, \( D(\infty) \) would be within limits of \( S(\infty) \), and the same arrangement would also hold, but it could not be much more diminished because with \( m = 5 \) and \( 6 \) the changes introduced into the denominators by \( \xi \) would upset the multiples 54 and 29. A change of \( \Delta \) by \(-5 \) would change the ratio to \( w^2 \) from 361'88 to 361'78. \( D(\infty) \) is, therefore, probably very close to 48163'62.

If \( \xi = -10 \), the denominators tend to a limit about '107 for \( m = 7 \) and beyond. But this is far outside permissible limits of \( D(\infty) \), and, moreover, the striking arrangement with multiples of \( \Delta \) is quite upset. We must therefore conclude either that the limit is not reached until an order \( m = 10 \), or beyond, or collaterals enter.

If the former, multiples of \( \Delta \) can enter, but the observation errors are too large to give certainty. If collaterals entered based on \((9\delta)D(\infty)\) are used with \( \xi = 2'16 \) or \( D(\infty) = 48163'62 \) the mantissa for 7, 8, 9 come respectively to '113569, '113590, '113956, and for 10 for a \( VD_{11} = VD_{21}, 113700 \). The separation observed for \( m = 10 \) is 107'96 instead of 112'15, either an observation error or a displacement of \( D_{11} \) or \( D_{21} \). A displacement of \( D_{11} \) by \(-2\delta \) on \( D(\infty) \) brings the separation very nearly correct, although the observation error in the wave number of these two lines is as large as 4'4, it is probable their difference is much more exact and that the defect shown by \( v = 107 \) is real. (For \( m = 10, D_{11} \) is practically \( D_{12} \)) The results therefore go to show that the \( D_{1} \) and \( D_{2} \) lines for \( m = 7, 8, 9, 10 \) are collaterals, \((9\delta)D(\infty)\), except that for \( D(10) \) an extra displacement of 2\( \delta \) is added. Although the numbers above are so nearly equal we must not place too much reliance on them, as the observation errors have a very large effect on the denominators for such high orders. If the suggested arrangement is correct it must mean that K. and R.'s measurements must have been of a very high order of exactness, which would further mean that the measures for the \( D_{2} \) lines would not be so exact since the observed values of the \( \nu \) for \( m = 7, 8, 9 \) are respectively \(-68 \) (possibly real on \( D_{3} \), '7, '13 in error.
I am inclined therefore to think that the exact equality for \((+9\delta)\) is a coincidence, especially as the difference for \(m = 6\) and \(7\) is not a multiple of \(\Delta\) like the others. Taking the corrected \(D(\infty) = 48163'62\), the mantissa for \(m = 6\) is '116154, giving with '113569 a difference \(2585 = (1 + \frac{1}{2})\Delta - 46\), which is as far as it can be from a multiple of \(\Delta\).

If \(+7\delta\) be taken for the collateral, and \(5\Delta\) for the difference, the limiting denominator is '107384 and the corresponding \(O-C\) are -'03, +'03, '07. Now these make, as against the observed \(D_2\) lines, the values of \(v_1\) about correct, which gives a certain amount of weight to this arrangement of collaterals based on \((+7\delta)\).

In the foregoing the conclusions up to \(m = 6\) may be taken as well based. No definite answer can be given to the question of what happens beyond \(m = 6\), although the balance of evidence perhaps points to the last, viz., collaterals based on \((+7\delta)\), and this is strengthened by considerations which follow.

MANNING* has recently observed under diminished pressure certain groups of lines which by their look suggest doublets and satellites related to the \(D\) type. The strongest set, apparently a \(D_{11}\) and \(D_2\) doublet, are 4260'05, 4241'25, giving a separation of 104'05 ('3). If 4260'05 be treated as having the same limiting term as the \(D\) series, the denominator of the \(VD\) part comes out to be 2'107364 (21). Now this mantissa has the limiting value according to the supposition of the preceding paragraph, viz., 107384. If this is not a mere coincidence, the connection should throw a great deal of light on the relations of these series, and would warrant a more searching discussion.

This would, however, lead too far from the immediate point at issue. It will be sufficient merely to indicate somewhat more clearly the connection. With the limit \((7\delta)\ D(\infty)\) the mantissa of \(D_1(7)\) is 299\(\Delta\) below that of \(m = 2\) (see Table II.), and it should therefore be \((\text{with } D(\infty) = 48163'62)\) 631328 (25) \(-83\xi-299\Delta\), whilst that of MANNING’s 4260 is 107364 (21) \(-42\xi\). If these are the same

\[
299\Delta = 523964 (46) - 41\xi
\]

\[
\Delta = 1752'388 - '14\xi '15,
\]

and this gives \(\delta = 361'55u^2\), which is too small. But in this \(D(7)\) of the accepted series is referred to \((7\delta)\ D(\infty)\), whereas 4260 is referred to \(D(\infty)\). If it be referred also to \((7\delta)\ D(\infty)\), its mantissa is 107868 (21) \(-42\xi\)

\[
299\Delta = 524468 (45) - 41\xi
\]

\[
\Delta = 1754'073 '15 - '14\xi,
\]

and therefore of the right order with \(\delta = 361'89u^2\).

MANNING’s new lines suggest, on a superficial glance, a series of bands, but there can be little doubt of their connection with the Diffuse series. Their relations to one another can be discussed either on the basis of the old \(D(\infty)\) or of \((7\delta) D(\infty)\). With

4260.05 as \( D_{11} \) (intensity 10) goes 4241.25 as \( D_{21} \) (intensity 9). \( D_{21} - 112.15 \) should give \( D_{12} = 4261.53 \). This has not been observed, but it is \( D_{11}(1)(-13\delta) \). In fact, the error between this calculated value and that deduced from \( D_{21} \) is only \( \Delta \lambda = 0.02 \), and a satellite difference of \( 13\delta \) is more in accordance with that of other elements than the small one of \( 14\delta \) in the accepted series. Amongst the other lines are the collaterals 4280.4 (intensity 9) = (\( \Delta \)) (4260) with \( O-C = -0.04 \), and 4363.7 (intensity 2) = (5\( \Delta \)) (4260) with \( 0-C = 1 \).

In.

As in the case of Al, so In shows an increase of satellite differences with the order. The first three, 5\( \delta \), 26\( \delta \), 265 may be considered as certain, but the next, 32\( \delta \), although it is close to the observation, may, as in the case of aluminium, be the same as the others (26\( \delta \)) within error limits, owing to the large error in \( D_{12} \). K.R. gives the difference in wave-lengths as \( 1.04 \) A.U., whilst Hartley and Adeney in the spark give it as \( 4 \), i.e., closer. In the table it is entered as 32\( \delta \) as being closer to the observations, but if it really is 26\( \delta \), the \( O-C \) is +0.20 against \( O = .50 \). It is possible that many cases of diffuseness may be due to the simultaneous existence of several collaterals based on differences of \( \delta \), which for lines where \( m \) is large or for small wave-lengths give differences in \( \lambda \) too small to resolve. In this case, for instance, with \( m = 5 \) a displacement by \( \delta \) produces collaterals differing by about 0.006 A.U., and several would give the impression of a nebulous line, broadened on one side or the other. For \( m = 6 \) there is clearly some collateral change different in \( D_{12} \) and \( D_{21} \). For if \( D_{12} \) be calculated from \( D_{21} \) it gives a position for \( D_{12} \) of longer wave-length than \( D_{11} \), or the inverse of the typical order. No conclusions therefore can be drawn as to the satellite differences for \( m = 6 \), except that \( D_{11} \) is probably of the form \((5\delta)D_{1} \). Beyond this it is curious that the \( D_{2} \) lines persist while the \( D_{1} \) lines do not, which may be accounted for by their being also like \( m = 6 \) additive collaterals.

Again, also, the order differences show themselves as close multiples of \( \delta \). The table is based on \( \xi = 0 \), but it may be brought into still closer agreement by taking \( \xi \) a small negative number, about \( -0.2 \) to \( -0.4 \). The difference between 5 and 6 becoming suddenly so large (59463 order 124\( \delta \)) and the entrance of the peculiarity mentioned above, suggest that some collateral influence comes in. Further, if we regard the denominators of \( D_{21} \) or of \( D_{12} \) calculated from \( D_{21} - \nu \), after a small difference of 8824, the differences begin again to increase. This has always in the previous cases pointed to a collateral displacement in \( D(\infty) \). The first object is to see by what displacement the denominators may be brought to a limiting uniform value. If \( \xi \) be put \( -13 \), the denominators for \( m > 5 \) become 715818, 714394, 716198, 715743, 7222, 7319. Omitting the values for \( m = 10 \) and 11, in which the probable errors are very large, it is clear that, allowing for quite reasonable observation errors, the denominators are in the neighbourhood of a limiting value. Now, a collateral of \( +2\delta \) in \( D(\infty) \) produces a displacement of \(-13.48 \), and this makes the denominator
for \( m = 6 \) to be \( 716474 \), and this is 23931 below the observed denominator for \( D, (5) \). Now, \( 50\delta = 23850 \), which is the same as the difference for \( m = 4 \) and 5 instead of being less, as analogy with others would lead us to expect. But the observation errors are large (maximum of order \( 16\delta \), about) so that room is allowed for this. There would then be one step from \( m = 5 \) to 6 of something less than \( 50\delta \) combined with a collateral displacement of \((+2\delta) \) on \( D (\infty) \). To indicate the explanation this is entered in the table, a difference of \( 50\delta \) making the denominator for \( D (6) = 6716653 \). The observed abnormality as between \( D, (6) \) and \( D, (10) = D, (6) \) is that the wave number of \( D, (6) \) is 1.26 less than \( D, (10) \), whereas it should be 8 or 9 greater, with a denominator about \( 32\delta \) greater instead of 1721 less. There is, in fact, a further defect beyond the normal value of about \( 36\delta \). Thus, if the difference for \( D, (5, 6) \) is \( x\delta \), that of \( D, (5, 6) \) is \((x+36)\delta \). The lines would then be represented as follows:

\[
d_1 = 755767, \quad d_2 = 740503, \\
D, (6) = (2\delta_i) D (\infty) - N/\{6+d_1-(x+36)\delta\}^2, \\
D, (m) = (2\delta_i) D (\infty) - N/\{(m+d_2-x\delta)\}^2.
\]

The collateral addition intensifies \( D, (2) \) and explains its continuance, but in \( D, (1) \) the increase, owing to addition of \( 2\delta_i \) to \( D (\infty) \), is overweighted by the diminution of the excess \( 36\delta \) in the \( VD \) part, and so, after the first, the rest are too faint to observe. At least, that is a suggestion of a possible explanation.

In Thallium the satellite separations appear to be the following multiples of \( \delta_i \): 24, 27, 28, 28, 28. But so far as limits of error permit, they might be 24, 27, 27, 27, 27. A peculiarity appears in the \( D \) lines in that the doublet separation for the first set is 7795'08 (48), whereas the normal value is very close to 7792'39. The difference is therefore real and not attributable to observation errors. Moreover, the next four show a gradual diminution, although still remaining normal within extreme permissible errors. The doublet values beyond this depend for the measurements of the second line on measurements of Cornu. They give separations 15 to 20 less than normal, but little reliance can be placed on deductions from them, for Cornu's results may err possibly by several units in the first decimal place, and with these small wave-lengths any error in \( \lambda \) is multiplied by 22 to 23 in the wave numbers. I have, therefore, not brought them into the discussion.

The table is based on \( f = .3 \), though no attempt has been made to find the best value. The mantissa of the first line is abnormal, since it is less than the second instead of greater, and, moreover, its difference from it cannot be a multiple even of \( \delta_i \). Since the satellite difference is very close to such a multiple \( (9048 = 6\delta + 4) \) it is probable that the abnormality affects both in the same manner. Now the arrangement may be made normal by regarding the first line as a collateral \((\delta_i) D, (2) \). The
addition of $\delta_1$ to the denominator of $D(\infty)$ produces a change $-19'19$, and this changes the denominator of $VD_z$ to those given in the table, and as is seen now, produces a difference of $2\delta_1$ between it and the next. Now, this alteration in $D(\infty)$ diminishes the value of $\nu$ by $5'65$, whilst, as we have seen above, it is apparently $69$ too much—or $VD_z(2)$ is $5'65+69=6'34$ below the value of $VD_x(2)$. Now, this is just the change made by deducting $2\delta_1$ from the denominator of $VD_x$. The exact value is $6'74$, which is within the limits. The way in which, with the considerable numbers involved ($\delta_1 = 377$), all the different abnormalities are simultaneously made to fit in with a normal scheme gives some confidence that this is the real explanation.

The scheme of actual lines may be represented thus:—

$$
\begin{align*}
\text{Actual } D_{11}(2) &= (\pm \delta_1) D_{11}(2), \\
D_{12}(2) &= (\pm \delta_1) D_{11}(-2\delta_1), \\
D_{21}(2) &= (\pm \delta_1) D_{21}(-2\delta_1).
\end{align*}
$$

Contrary to the case in other elements the successive differences are equal after the first, and the limiting value of the denominator is reached at $m = 7$. They can all from 7 to 14 be, within limits, equal; but there is an apparent rise with the high orders.

O.

Two series—one of doublets and one of triplets—have been recognised in oxygen. The table shows that the D lines of both sets fall into line quite naturally with multiples of $\Delta$, closely except in the case in the doublet sets of $m = 7$, 8. In both these cases the denominators are equal within limits, but much larger than those for $m = 6$ instead of being less, and the deviation is real since the difference is more than 15 times the probable error for $m = 7$, and $1'5$ times that for $m = 6$, which latter has a very considerable probable error $5$ as against $07$.* The divergence for $m = 8$ can be accounted for, as it is probable that there are two close lines here due to different series, viz., that for this series $m = 8$ and the other for a parallel series for which $m = 5$, and may therefore be stronger. As it throws some light on the subject it may be well to say a few words about it here. Runge and Paschen give three lines at $6264'78$, $6261'68$, $6256'81$, with separations $7'83$, $12'43$, and intensities $1$, $3$, $1$, so that the centre is the strongest. There is a corresponding set at $5410'97$, $5408'80$, $5405'08$, with the same separations within error limits and intensities $3$, $4$, $3$, again with the centre strongest. The strongest lines of these two triplets form a series with the observed value of $D''(8)$. They are of a diffuse type and in fact

* These are not to be confounded with K.R.'s possible errors. The possible errors are probably larger.
come between the D'' and D' series. The limit taken is 22926.11 and the scheme is as follows:

\[
\begin{align*}
3.969545 - 8 \quad & \quad 0.01 \quad 0.03 \\
6\Delta_1 & \\
4.968512 + 7 \quad & \quad 0 \quad 0.04 \\
6\Delta_1 & \\
5.967480 + 24 \quad & \quad 0 \quad 0.05
\end{align*}
\]

Moreover, the difference between the first denominator and the corresponding one for D'' is 2924, and this is 17\Delta_1. It is of course understood that the digits 11 in the limit have been chosen so that the 17\Delta_1, 6\Delta_1, come very close. The argument depends on the possibility of doing this. In fact Rydberg's tables give the limit 22926, so that the modification by 11 is extremely slight. Thus the observed line is the line corresponding to \( m = 5 \) of this series, and it probably hides the weaker line of D'' (8). This accounts for the deviation noted above between calculated and observed in D'' (8). I have no explanation to offer for the corresponding deviation for \( m = 7 \). All the others come so close that it is difficult to imagine that this does not fall in with the rule. It is equivalent to an error in \( \lambda \) of about 1'2 A.U. The doublet separation for D'' is 62 very closely, and the corresponding doublet difference is 1561 = \( \Delta \) say. A lateral displacement of 7\Delta on the limit would just make the change, but that explanation seems out of place here. The separations 7'83, 12'43 of the new lines require denominator differences in the limit of 373 and 473, and 4\Delta_1 = 380 and 5\Delta_2 = 475. There is another line at 6267'06, showing a separation of 5'31 (3). If this has the same VD as 6261 it requires a denominator difference in the limit of 277 and 3\Delta_2 = 283. The four lines are therefore (−5\Delta_1) (6261), (−3\Delta_2) (6261), 6261, and (+5\Delta_2) (6261).

S.

If Runge and Paschen's estimates of their errors are valid the value of the limit of the S series is determinable very accurately. It is 20085'46 (1'34), but to bring \( m = 7 \) as calculated within limits it is necessary to take S (\( \infty \)) more than 1 less. Accordingly the D lines have been calculated on the supposition that D (\( \infty \)) = 20084'5, and it cannot be far from this. To bring the differences within multiples it has been necessary to diminish this limit by putting \( \xi = -3 \). The multiples then come in partly as multiples of \( \Delta_2 \) and partly of \( \Delta_1 \). The value of \( \Delta_2 \) given in the first part of the paper is 651, but this gives \( \Delta_2 = 35 \times 180'67 m^2 \), whereas it should be, if the rule there established is correct, in the neighbourhood of 180'9, or 3'60 larger, say 651'7. This value has been adopted in the table, although the old one can be made to fit in though not so well. The agreement is good, especially when it is remembered that K. and P.'s estimates are less than possible errors.
The lines allotted by Runge and Paschen to the D series present a quite different appearance from the normal, although there can be little doubt but that they form the SeD. The weak satellite lines after the first appear on the violet side of the strong lines, whereas in all other yet known cases they lie on the red side. Moreover, the strong lines instead of standing by themselves are each the first members of complete triplets for \( m = 4, 5, 6, 7 \) (\( m = 4 \) is the first set observed). The numbers in the table are calculated with \( D(\infty) = 19274 \). The value of \( S(\infty) \) calculated from \( (4, 5, 6) \) is \( 19275'10 \) (2'4), but it requires, as in \( S \), a further diminution of over 1 (i.e., within error limits) to bring in the fifth line. Hence \( D(\infty) \) cannot be far from 19274. The value of \( \delta \) is calculated from \( \Delta_1 + \Delta_2 = 161 \times 90'40w^2 \), and consequently must be considered very exact. \( \Delta_1 \) and \( \Delta_2 \) are calculated by transferring \( 15w^2 \) from calculated \( \Delta_2 \) to \( \Delta_1 \), making the values \( 28\frac{1}{2} \times 361'62w^2 \) and \( 12 \times 361'61w^2 \). The numbers calculated from the observed values are given in a separate list. A glance shows that the usual regularity is here quite upset, and one feels convinced that some disturbing influence must have been at work. If we examine the wave numbers of the first four sets as exhibited in the following table, we notice that for the first

| \( m = 4 \) | \{ (14149'27)? \} (5) 14156'19 | (3) 14252'84 | (3) 14300'31 |
| \( m = 5 \) | \{ (6) 15803'95 \} (1) 15804'95 | (3) 15907'80 | (1) 15908'67 | (4) 15953'89 |
| \( m = 6 \) | \{ (5) 16768'10 \} (1) 16769'17 | (1) 16872'15 | (1) 16872'67 | (3) 16917'33 |
| \( m = 7 \) | \{ (7) 17375'92 \} (2) 17379'58 | (1) 17482'33 | (3) 17523'15 | (7) 17483'00 | (4) 17527'20 |

set, we should expect a weak satellite about 6'92 behind 14156, which is not likely to have been observed in that region far in the red. Its difference, 3129, is close to \( 55\delta_1 = 3118 \), and provisionally we may regard this as normal. The next two triplets \( (m = 5, 6) \) give separations respectively 1'00 and 1'07, corresponding to a lateral displacement in \( D(\infty) \) of \( \delta_1 \) (\( \delta_1 \) actually gives 925 displacement). For \( m = 7 \) the separation is 3'66 corresponding to a displacement of \( \delta \) (4\( \delta \), gives 3'7). Moreover, for this line the intensity has increased from 5 to 7 and gives suspicion of a displacement by addition. If we suppose that the chief lines have a lateral displacement \( (+\delta)D(\infty) \) it means adding 3'70 to their wave numbers, i.e., they are now 15807'65 and 16771'80, and they come 2'7, 2'63 in front of their satellites, which, allowing for errors in observation, is in fair order with the first separation 6'95. For \( m = 7 \), not only is the strong line abnormally more intense than for \( m = 6 \), but the faint line is so
also—which suggests they are both displaced—a suspicion increased by the abnormal increase of the difference shown in the table of denominators between 6 and 7. Provisionally the least change is to suppose the faint line displaced by $\delta$ and the strong line by $2\delta$, as it must, as was noted above, be $\delta$ more than the faint line.

For $m = 8$ it is curious that only one line occurs and no triplets. This suggests that there is no intensification by lateral displacement, and that provisionally it should be taken as normal. The table of difference shows an abnormal increase instead of a decrease, but this may be due to observation errors. If we now calculate the denominators for $m = 4, 5, 6, 7$, on the above suppositions, displacing the lines for $m = 9, 10$ also by $\delta$, we get

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\text{Denominator}$</th>
<th>$\text{Difference}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>629262 (54)</td>
<td>3129</td>
</tr>
<tr>
<td></td>
<td>4326</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>624936 (56)</td>
<td>2139</td>
</tr>
<tr>
<td></td>
<td>4406</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>620530 (105)</td>
<td>3475</td>
</tr>
<tr>
<td></td>
<td>4216</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>616314 (240)</td>
<td>3058</td>
</tr>
<tr>
<td></td>
<td>5256</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>611058 (728)</td>
<td>3629</td>
</tr>
<tr>
<td></td>
<td>3629</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>607429 (643)</td>
<td>9090</td>
</tr>
<tr>
<td></td>
<td>10:59833</td>
<td></td>
</tr>
</tbody>
</table>

Thus the changes indicated by the appearance and arrangement of the lines have brought the denominators and satellites into greater accordance with the general rule. The practical constancy of denominator differences is exhibited also in Tl. The only outstanding irregularity appears to be the satellite difference for $m = 5$. A lateral displacement of $-\delta$ in $D(\infty)$ would decrease the denominator by 743, and increase the difference from 2139 to 2882. It is better to leave the difference without an attempt of explanation at present.

The second list has been drawn up on this basis, taking $\xi = -\cdot1$ as the errors are somewhat smaller with this value. The denominator for $m = 10$ is left without further change. Another displacement of $2\delta$, would bring it also $19\delta$ below that for $m = 9$, but the observation errors render any deductions quite unreliable. The
suggested scheme of actual lines may therefore be represented as follows where $D_{11}, D_{12}$ stand for the normal type, and $D_{13}(m) = D_{11}(m)(-55\delta_{1})$:

\[
\begin{align*}
D_{11}(4), & \quad D_{12}(4) \\
(+\delta)D_{11}(5), & \quad D_{12}(5) \\
(+\delta)D_{11}(6), & \quad D_{12}(6) \\
(+2\delta)D_{11}(7), & \quad (+\delta)D_{12}(7) \\
D_{11}(8), & \\
(+\delta)D_{11}(9), & \\
(+\delta \text{ or } +5\delta_{1})D_{11}(10).
\end{align*}
\]

The order (4) of the first line is so large that the error limits are too wide for absolute certainty. In fact better agreement on the whole for the satellites would be obtained by taking the difference as $56\delta_{1}$, i.e., $4\delta_{14}, \delta_{14}$ being specially associated with this group (see p. 331). The line 6269.28 is separated from 6266.36 by 6.44, and is therefore possibly the lateral $(+2\delta)D_{11}(5)$.

The table shows of course the known essential difference between the behaviour of the elements of Group 2 and that of Groups 1, 3, 6, signified by the signs of $\alpha$ in the formula. It consists in the fact that in Group 2 the orders are formed in succession by the addition of multiples, whilst in the others it is by subtraction, with the exception that Cu and Ag of Group 1 are additive. But there are certain other features which appear between the different sub-groups when higher orders are looked at. The alkalis all show a gradually decreasing decrement with a sudden dive. Na then shows a sudden rise continued for several lines, and Cs has a similar indication. Cu and Ag with only a few lines observed show decreasing increments. The alkaline earths show decreasing increments and a sudden dive (Mg excepted). The Zn sub-group shows decreasing increments and then a sudden ascent. The Al Sub-group 3 shows decreasing decrements (Sc decreasing increments). O with S and Se show decreasing increments. In fact, were it not for the very clear behaviour of Zn, Cd, and Hg, the evidence would rather point to the conclusion that in each group, the low melting-point sub-group show subtraction ($\alpha$ positive) and the high melting-point addition ($\alpha$ negative). If this series depends on a formula sequence, it is difficult to see how it can be any simple algebraic one—the mantissa would rather seem to depend on a term similar to sin $ma$ or tan $ma$. In the detailed discussion above, however, it is seen how these changes of direction can be explained by lateral displacements. It is noticeable that where the irregularity observed in the first lines as compared with the others in the satellite differences appears, a similar irregularity exists also in connection with the first order differences. This is evident especially in the alkalis, where the first differences are so close to exact multiples of $\Delta$ or $\delta$ as to cause the conviction that they really are so.
It is a remarkable fact also, and one which will probably be of importance in throwing light on molecular constitution, that all those elements which do not exhibit satellites have order differences depending on multiples of \( \Delta \), whereas all the others (Al excepted) depend on multiples of the oun, \( \delta \) or \( \delta_i \). The elements without satellites are Na, K, Mg, possibly Al, both series of O, and S. All these depend on multiples of \( \Delta_n \) or \( \Delta'_n \). None of the others do so, and it may be regarded as an argument in favour of Rb possessing satellite series that its differences do not depend on \( \Delta \) directly. It would appear that Rb only begins to show them for \( m = 3 \). For \( m = 2 \) the line is not split up into a chief line and satellite, the doublet separation is normal, and it is instructive to observe that the order separation between the first set and second line is close to \( 5\Delta \), and only deviates from it in the same way that is mentioned in the previous paragraph. Also Ba seems to have in the same way no satellite for \( m = 2 \), the separation is quite normal, and this also shows a first order difference very close to \( \Delta_n \).* But Ca, on the contrary, which has a first difference \( = 99\Delta_n \), possesses satellites.

It is noticeable also that the high atomic weight elements appear to follow more regular and simple rules. Thus both Cs and Tl show descent by equal steps in both cases \( = 3\delta \).

The result of the discussion would seem to be that there can be no doubt but that satellite differences as well as the doublet and triplet differences depend on multiples of the oun. For the other supposition, viz., that in the Diffuse series the order differences also depend on differences of the oun, it can only be said that a case has been made out. The supposition in all cases fits conditions, but the conditions are not all sufficiently definite to give certainty. After the first two or three orders the observation errors are larger than the \( \delta \), and even for these the value of \( \delta_i \) for the low atomic weights is comparable with the errors. In some of these cases, however, multiples of \( \Delta \) which is much larger enter and strengthen the argument. The strongest examples are those of the alkaline earths (small errors and large \( \Delta \) or \( \delta_i \)), first lines of Cd, and Hg, Al (series in \( \Delta \)), In, Tl, and the \( \Delta \) series of O and S.

*The D (2) Term.—If the foregoing theory of the constitution of the Diffuse series is correct, it is further necessary, in order to complete the discussion, to determine the origin of the first term. The apparently close relation of the F series to the D series, and the several cases of collaterals of the former which had been noted with large multiples of \( \Delta \), suggested a trial to see if the denominators were multiples of this quantity. As in cases where satellites are present, the separations depend on them and not on the strong line, it is natural to expect that the satellite is a normal line and the strong line a collateral, and this is found to be justified by the calculations on this theory. In Table III. the first column of figures gives the value of the denominator taken from Table II. The second column gives the factors together with possible variations. Thus the denominator of KD\(_{11} \) (2) = 853302. This has

* But see discussion of BaF below.
**Table III.**

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Na</td>
<td>$D_{11} \cdot 988656$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>K</td>
<td>$D_{11} \cdot 853302 = 291 (2932 \cdot 27 \pm 130 - 364\xi) = 291\Delta$</td>
<td>361.944</td>
<td>362.68</td>
</tr>
<tr>
<td>Rb</td>
<td>$\uparrow D_{12} \cdot 766216 (f) = 88 = 59 (12965 \cdot 84 + 2 \cdot 74 - 1 \cdot 44\xi) = 59\Delta$</td>
<td>361.991</td>
<td>361.40</td>
</tr>
<tr>
<td></td>
<td>$-10\Delta = 59 (12942 \cdot 06)$</td>
<td>361.746</td>
<td></td>
</tr>
<tr>
<td>Cs</td>
<td>$D_{12} \cdot 546989 = 857 (638 \cdot 260 \pm 233 - 0887\xi) = 857\xi$</td>
<td>361.785</td>
<td>361.74</td>
</tr>
<tr>
<td>Cu</td>
<td>$D_{11} \cdot 554286 = 17 (32605 \cdot 06 + 13 \cdot 41 - 4 \cdot 47\xi) = 17\Delta - 10\Delta$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ag</td>
<td>Theory of constitution uncertain.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mg</td>
<td>$D_{1} \cdot 828688$</td>
<td>362.36</td>
<td></td>
</tr>
<tr>
<td>Ca</td>
<td>$D_{12} \cdot 945972 = 691 (1368 \cdot 99 \pm 03 - 047\xi) = 691\Delta_{2}$</td>
<td>361.84</td>
<td>361.84</td>
</tr>
<tr>
<td>Sr</td>
<td>$D_{12} \cdot 987349$, not a multiple.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$D_{12} \cdot 989572 = 178 (5559 \cdot 39 \pm 25 - 20\xi) = 178\Delta_{2}$</td>
<td>361.738</td>
<td>360.02</td>
</tr>
<tr>
<td>Ba</td>
<td>$D_{12} \cdot 825511 = 69 (11963 \cdot 9 \pm 1) = 69\Delta_{2}$</td>
<td>361.968</td>
<td>362.34</td>
</tr>
<tr>
<td></td>
<td>$^*D_{12} \cdot 041954 = 87 (11976 \cdot 4) = 87\Delta_{2}$</td>
<td>362.352</td>
<td></td>
</tr>
<tr>
<td>Ra</td>
<td>Not observed. $= 31\Delta_{2}, \uparrow$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Zn</td>
<td>$D_{12} \cdot 904978 = 260 (3480 \cdot 68 \pm 38 - 430\xi) = 260\Delta_{2}$</td>
<td>361.682</td>
<td>362.25</td>
</tr>
<tr>
<td>Cd</td>
<td>$D_{12} \cdot 902039 = 87 (10368 \cdot 26 \pm 26 - 1 \cdot 276\xi) = 87\Delta_{2}$</td>
<td>361.382</td>
<td>361.392</td>
</tr>
<tr>
<td>Eu</td>
<td>$D_{13} \cdot 917794 = 50 (18355 \cdot 88 \pm ? - 2 \cdot 28\xi) = 50\Delta_{2}$</td>
<td>361.44</td>
<td>360.93</td>
</tr>
<tr>
<td>Hg</td>
<td>$D_{13} \cdot 921662 = 31 (29731 \cdot 0 \pm 1 \cdot 22 - 3 \cdot 80\xi) = 31\Delta_{2}$</td>
<td>361.50</td>
<td>362.46</td>
</tr>
<tr>
<td>Al</td>
<td>$D_{12} \cdot 631287 = 360 (1753 \cdot 575 \pm 069 - 230\xi) = 360\Delta$</td>
<td>361.777</td>
<td>361.879</td>
</tr>
<tr>
<td></td>
<td>$D_{12} \cdot 631181 = 360 (1753 \cdot 280 \pm 069 - 230\xi) = 360\Delta$</td>
<td>361.717</td>
<td></td>
</tr>
<tr>
<td>In</td>
<td>$D_{12} + 16\Delta = 22 (37676 \pm 2 \cdot 18 - 4 \cdot 619\xi) = 22\Delta$</td>
<td>361.871</td>
<td>361.947</td>
</tr>
<tr>
<td>Tl</td>
<td>$D_{12} \cdot 888344 = 590 (1505 \cdot 667 \pm 136 - 1 \cdot 881\xi) = 590\delta$</td>
<td>361.650</td>
<td>362.063</td>
</tr>
<tr>
<td></td>
<td>$D_{11}$ coll. = 899520 = 597 (1506 \cdot 73 + 136 - 1 \cdot 881\xi) = 597\delta</td>
<td>361.913</td>
<td></td>
</tr>
<tr>
<td>O</td>
<td>$D'' \cdot 972483 (40) - 120\xi$</td>
<td>362.46 $\pm$</td>
<td>361.79</td>
</tr>
<tr>
<td></td>
<td>$46 (171 \cdot 66 \pm ?) = 46\Delta_{1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$D'' \cdot 980380 (22) - 121\xi$</td>
<td>363.308</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$63 (172 \cdot 063) = 63\Delta_{1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>New D \cdot 969543</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>$D_{1}(4) \cdot 553446 = 530 (1044 \cdot 24 \pm 088 \pm 811\xi) = 530\Delta_{1}$</td>
<td>362.113</td>
<td>362.14</td>
</tr>
<tr>
<td>Se</td>
<td>$D_{1}(4)$ observed \cdot 629262 = 99 (6356 \cdot 18 \pm 54 - 4 \cdot 565\xi) = 99\Delta_{1}$</td>
<td>361.893</td>
<td>364.00</td>
</tr>
</tbody>
</table>

* See below under discussion of BaF—two different triplets in question.

† See below under discussion of RaF.
observation errors and also possible error due to incorrectness of \( D(\infty) \), i.e., \( \xi \). These give the denominator as \( 291(2932'27 \pm 130 - 364\xi) \). Now 2932'27 is very close to 2939, which is given as the approximate value of \( \Delta \) in Table I. The denominator is then written 291\( \Delta \), and with this new value of \( \Delta \) the corresponding value of the oum is calculated as 361'944\( w^2 \) instead of the old value 362'68\( w^2 \), which for comparison is entered next to it.

**Notes on the Tables.**

**Na.** \( \Delta \) is so small that several multiples of it might be taken for \( D \) within limits. 

**Rb.** \( D_1 \) does not give a multiple of \( \Delta \) although close to it. If, however, Rb has satellites, the denominator for \( D_1 \) will be a few multiples of \( \delta \) less than that of \( D_11 \). That for Cs is 11\( \frac{1}{2} \)\( \delta \) less in the corresponding case. The values in the table are given for 8\( \delta \) and 10\( \delta \). Judging from the value of the oum it is probably near 8\( \delta \). In any case the multiple would be 59\( \Delta \). This is a very strong argument that Rb does possess satellites.

**Cs.** Neither \( D_11 \) or \( D_12 \) give multiples of \( \Delta \).

**Mg.** As in Na, \( \Delta_2 \) is too small, and the denominator too large to give anything definite.

**Sr.** \( D_{13} \) gives the oum clearly too small, although better than in the original table. \( D_{12} \), however, gives a value 361'738 quite close to the probable value. A similar result is shown also by Cd which occupies an analogous position in the next sub-group. If \( D_{13} \) behaves in what appears to be the normal manner, it would appear necessary to take the atomic weight to be 1'0 less than BRAUNER'S value, viz., 87'56 in place of 87'66, which is probably too large a change to be acceptable.

**Ba.** In Barium the first set is doubtful. That taken above shows no satellites. The denominator is therefore that for \( D_{13} \), and this is a multiple which gives a value of the oum much nearer the probable value than that in Table I. Evidence will be given later however under BaF that there is a normal satellite triplet, outside the region of observation, where \( D_{13} \) has the denominator 2'041954, which, from analogy with the other elements of this group, has a "mantissa" 1'041954, and this is again a multiple of \( \Delta_2 \). This, therefore, is probably the correct value, and the other set will be collaterally displaced from this by 18\( \Delta_2 \).

**Ra.** The first line should be far in the ultra red, and has not been observed. The multiple 31\( \Delta_2 \) is determined indirectly (see BaF below).

**Cd.** This element shows the same irregularity as in Sr, in that \( D_{11} \) does not give an exact multiple of \( \Delta_2 \), although one close to it. Here, however, we have to go back to \( D_{11} \) before finding the exact multiple, and \( D_{11} \) gives almost the precise value of the oum as in Table I, which was itself very exact.

**Eu.** \( D_9 \) is 917794, which is 50\( \Delta_2 \)+1344, \( \Delta_2 \) having the value 18329 of Table I. If it is 50\( \Delta_2 \) exactly, \( \Delta_2 \) would be 18355'88−2'28\( \xi \), making the oum (361'46−0'04\( \xi \)) in place of 360'93 of Table I, a great improvement.
Hg. \(D_{13}\) gives denominator = 31 \times 29731'0, and the latter factor is
\[7410'87 w^2 = 82 \times 90'375 w^2,\]
which is much closer to the probable value of the om, and moreover 82 is the correct multiple to give 54 \times 543'816w^2 for \(\Delta_1 + \Delta_2\), which has been taken as a basis for \(\delta\). This value of \(\Delta\) is supported by the discussion of the F series below.

Al. As the order differences are all multiples of \(\Delta\), and there may therefore be some doubt as to the real existence of satellites the values for \(D_{11}\) and \(D_{12}\) are inserted. The denominators for the two only differ by 4\(\delta\) = 108, or the observed by 96. As the \(\Delta\) differences can only refer to the \(D_{11}\) set, it would seem that these should be taken as the normal lines giving 361'777 as the value of the om.

In. Neither \(D_{11}\) nor \(D_{12}\) are exact multiples of \(\Delta\) although they are very close to 22\(\Delta\). \(D_{12}\) is 1722 \times 477'11, or 1723 \times 476'83. If these be taken as multiples of the om, they give the om as 362'01 and 361'80 in place of 361'94 of Table I., but the multiples are too large to found any conclusions upon. It would rather seem that there is some displacement from a typical multiple. Using \(\Delta\) as given in Table I., viz., 37684, 22\(\Delta\) = 829048. So that \(D_{12} = 22\Delta - 7455\) and 7455 = 16\(\delta\) - 177. If it is 22\(\Delta\) - 16\(\delta\) exactly, \(\Delta\) becomes 8'04 less and the om 361'871\(w^2\) in place of 361'947. The value of \(D_{12} + 16\delta\) is therefore inserted. If the typical term were 16\(\delta\) higher, the order differences would run 72\(\delta\), 62\(\delta\), 50\(\delta\), 50\(\delta\), in place of 58\(\delta\), 62\(\delta\), &c., and hence more in line with others.

Tl. Neither the observed nor the supposed collaterals are multiples of \(\Delta\). They are expressed as multiples of \(\delta\). Although they are large multiples, their values are quite definite provided we know \(\alpha\) \textit{a priori} that the denominators are multiples as a fact. If the multiple be altered by unity, the resultant quotient cannot come within the limiting values of the om.

If the normal \(D_{13} (2) = 7\Delta = 939078\), the order difference over \(D_{12} (3)\) would be 939078 - 888643 (89) = 50435 \pm 89, and 33\(\frac{1}{2}\)\(\delta\) = 134\(\delta\) = 50495, so that the order difference would come out as usual a close multiple of \(\delta_1\). All this group seem to show the same kind of irregularity.

O. There are three separate series, see data for Table II., differing by multiples of \(\Delta_1\), just as in the order differences. \(\Delta_1\) is too small to test the multiples of the denominators themselves.

S. The \(D (2)\), \(D (3)\) lines for S and Se are beyond observed regions. Sulphur however shows no satellites, and we may surmise therefore in analogy with others that the differences for \(D (2), D (3), D (4)\) are like the others multiples of \(\Delta_1\) or \(\Delta_2\). As a fact, \(D_1 (4)\) is a clear multiple of \(\Delta_1\), and the surmise is justified so far as \(\Delta_1\) is concerned. The value of \(\xi\) is not very certain.

Se. Se apparently has satellites, and the order differences are only multiples of \(\delta\). It should not therefore be expected that \(D_1 (4)\) or \(D_2 (4)\) should be a multiple of \(\Delta_1\). Nevertheless \(D_1 (4)\) is clearly such a multiple and is entered in the table.
The table shows that where triplets occur the multiples are those of \( \Delta_2 \) and not of \( \Delta_1 \), except in the case of the oxygen group of elements, in which \( \Delta_1 \) clearly takes the place of \( \Delta_2 \). If the law of multiples is correct, the values of the \( \Delta \) obtained in this way must clearly be far more exact than those obtained direct from the separations. A glance at the deduced values of the oum compared with the former values shows how much closer to the mean value 361.9 the new ones are than the old, and to some extent this adds to the weight of the evidence. The cases where the multiples do not appear to enter are those of Rb, Cs, In and Tl. The case of Rb has been considered above and a natural explanation offered. Cs, In and Tl have all large values of \( \delta \), in which case we have already seen a tendency for the spectra to depend on smaller multiples of the oum than the \( \Delta \). In the case of Cs, the oum is smaller than the multiple and it can give no evidence nor data for the oum. The case is different however for Tl. If the oum enters, the multiple can be no other than that given, and as is seen the value of the oum is improved. All the elements of the Al group show a deviation from the normal type in that the first satellite separations are much smaller for the first order lines than for the second, and seem to point to some displacement. As the Al orders differ by multiples of \( \Delta \), any irregularity in the multiple between the first and second orders does not alter the dependence of the denominator on the multiple of \( \Delta \). In In and Tl, however, the differences go by multiples of \( \delta \) or \( \delta_n \), and any irregularity on them will throw out the dependence of the first denominator on a multiple of \( \Delta \). As was shown above the addition of 16\( \delta \) in In not only produces the multiple, but at the same time shows a more usual march of differences for the orders. In Tl the observed denominator for \( D_{12} (2) \) is less than that for \( D_{12} (3) \) and quite abnormal. The other anomalies occur in that in Sr, \( D_{12} \) appears to take the place of \( D_{31} \), and in Cd, \( D_1 \). RaD (2) is in the ultra red and has not been observed. The elements Na and Mg must be left out of account because the ratio denom./\( \Delta \) must be so large that a number of multiples can be found all giving \( \Delta \) within observation limits. Cu shows a multiple, but the theory of the constitution of the series of Cu and Ag is doubtful and must also be left out.

With the above doubtful cases the values for K, Ca, Sr, Ba, Zn, Cd, Eu, Hg, Al, S and Se, are clearly exact multiples, and the large values of \( \Delta \) in Ba, Cd, Eu and Hg show that these multiples are real. This rule, exhibited as it is in so many cases, and in by far the majority of the elements comparable, must correspond to a real relation and cannot be due to mere coincidence. Against the reality of the relation is the antecedent improbability that those elements with the smallest value of \( \Delta \) should have the largest values of the denominator, as e.g., in the case of Na and Mg. A possible explanation is that the mantissa is the nearest multiple of \( \Delta_2 \) to some group constant. But see also under discussion of the F series. It might however have been expected on this ground that the denominators would be of the form \( 1 - M (\Delta) \). But the case of Na is clearly against this. Its denominator \( 988656 = 1 - 011344 \) and 11344 is 15.26\( \Delta \) and cannot be a multiple. It would seem conclusive that the
denominators of the extreme satellites of the first line are multiples of $\Delta_1$ or $\Delta_2$ and that explanations should be sought for apparent exceptions.

**The S and P Series.**

The relationships between the doublet and triplet sets of the P series and between the S and P series were discussed in [II., p. 51] by comparing the differences between the corresponding denominators. It is now possible to see how, if at all, these differences are related to the ion.

**The P Series.**—In the alkalies the differences between the corresponding denominators of the two sets were found to be constant within error limits and of course equal to $\Delta$. In the other elements in which the P series have been allocated, there was always a drop in the difference, which in several cases then remained constant for the succeeding orders. The values were given on [II., pp. 51–53]. They are reproduced here, and it is seen at once how they proceed on quite analogous lines with successive satellite differences of the D series considered above. The possible errors of the single lines from which they are deduced are given in brackets. Thus ZnP(1) are 1'599352 (2), 1'592143 (3), 1'588669 (4), and the differences are given as 7209 (2, 3), 3474 (3, 4). The higher orders, in which the possible errors are so large as to be themselves multiples of $\delta_1$, are not included. The value of $\delta_1$ is given with the symbol for the element.

It will be noticed that the more accurate the observations the closer are the differences to the multiples of the ion. But the observed variations from true multiples in the case of the large separations would seem to point to a difference in the $a$ as well as in the $\mu$. In any case it would seem that $\mu$ must alter per saltum from order to order, unless the sequence formula is a complicated function of $m$.

<table>
<thead>
<tr>
<th>Zn ($\delta_1 = 38'75$)</th>
<th>Cd ($\delta_1 = 113'8$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7209 (2, 3)</td>
<td>3474 (3, 4)</td>
</tr>
<tr>
<td>125−5</td>
<td>65+20</td>
</tr>
<tr>
<td>5355 (19, 28)</td>
<td>2525 (28, 36)</td>
</tr>
<tr>
<td>$\delta + 9$</td>
<td>38,−5</td>
</tr>
<tr>
<td>5191 (9, 9)</td>
<td>2414 (9, 9)</td>
</tr>
<tr>
<td>$\delta_1$−1</td>
<td>$\delta_1$</td>
</tr>
<tr>
<td>5154 (30, 30)</td>
<td>2375 (30, 30)</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>7209 (2, 3)</td>
<td>23109 (4, 5)</td>
</tr>
<tr>
<td>125−5</td>
<td>505−5</td>
</tr>
<tr>
<td>5355 (19, 28)</td>
<td>17423 (25, 25)</td>
</tr>
<tr>
<td>$\delta + 9$</td>
<td>18269 (922)</td>
</tr>
<tr>
<td>5191 (9, 9)</td>
<td>$\left{ 3\delta_1 + 27 \right.</td>
</tr>
<tr>
<td>$\delta_1$−1</td>
<td>$\left. 6461 (1837) \right}</td>
</tr>
<tr>
<td>5154 (30, 30)</td>
<td>17109 (26, 28)</td>
</tr>
<tr>
<td></td>
<td>$\left. 5\delta_1 + 4 \right}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hg ($\delta_1 = 362'87$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>87815 (1)</td>
</tr>
<tr>
<td>115−118</td>
</tr>
<tr>
<td>71967 (1)</td>
</tr>
<tr>
<td>235−122</td>
</tr>
<tr>
<td>71364 (1)</td>
</tr>
</tbody>
</table>

**Vol. CCXIII.—A.**

3 c
Note how Zn still affects $\delta_6$. The variations from multiples in Hg seem to have relation to the transference properly noted above, viz., from $\Delta_1$ to $\Delta_2$.

Al ($\delta_i = 26.57$).

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1751 (1)</td>
<td>134154 (1)</td>
</tr>
<tr>
<td>$16\delta_i - 3$</td>
<td>$53\delta_i + 191$</td>
</tr>
<tr>
<td>1329 (78, 78)</td>
<td>94018 (1)</td>
</tr>
<tr>
<td>$-2\delta_i + 5$</td>
<td>$27 - 191$</td>
</tr>
<tr>
<td>1377 (23, 23)</td>
<td>91195 (1)</td>
</tr>
<tr>
<td>0</td>
<td>$2\delta_i - 173$</td>
</tr>
<tr>
<td>1380 (48, 48)</td>
<td>90615 (1)</td>
</tr>
</tbody>
</table>

Tl ($\delta_i = 376.835$).

In Al the value after the first is 1381 within limits of error for all.

The S and P Connections.—The differences of the corresponding denominators in the S and P are also given in [II.]. The values are, however, subject to uncertainties due to uncertain limits in both S and P, in which the $\xi$ are not the same for both. In the case of the alkalies there seems a very clear connection with the $\Delta$, except in Cs, where as often before $\delta$ enters. In the other elements it was shown that the sequences are inverted and the differences are to be taken between the first of the S and the second of the P. In Al, Tl, and Zn, there is again a clear relation, but it is now to the denominator differences of the $P_1(2)$ and $P_2(2)$, or 1329, 94018, 2525 respectively, say $\Delta'$ for each. In the case of Cd and Hg no clear relation is apparent, although they behave approximately like Al and Tl. This want of exact agreement may be due to the effect of the transference inequalities considered above (p. 333) in connection with the own. The relations indicated above are shown in the following scheme, in which the differences for the S and P are taken from [II., p. 51–53].

```
Na ... '490162 = '5−13\Delta,
K ... '464597 = '5−12\Delta,
Rb ... '487501 = '5−\Delta,
Cs ... '491944 = '5−14\delta (roughly).
Al ... '489330 = '5−8×1334 = '5−8\Delta',
Tl ... '594887 = '5+94887 = '5+\Delta'.
Zn ... '528306 = '5+11×2573 = '5+11\Delta',
Cd ... '526358 = '5+26358,
Hg ... '603628 = '5+103628.
```
The F Series.

In Part I. the symbol F was used to denote the series whose limit depends on the values of \( VD(2) \) in a similar way to that in which the limits of the S and D series depend on \( VP(1) \). Where the D series show satellites the F series in consequence consist of doublet or triplet series with constant separations. They comprise some of the strongest lines in the respective spectra, but as in general they occur in the ultra-red region they have not received the same attention as the other better known ones. In the alkaline earths, however, they come well within the visible regions, and show strong sharply defined lines. They are related also to other strong lines by collateral and other displacements depending on considerable multiples of \( \Delta \), and so naturally come under discussion in the present communication. As will be seen later, the discussion gives the means of obtaining very accurate determinations of the \( \Delta \)—and consequently of the \( \alpha \)—as well as of settling other questions. I propose, however, not to attempt an exhaustive discussion in the present communication, partly because the main object now is only to illustrate the influence of the \( \alpha \), and partly because it would seem that a large number of lines which clearly belong to the F cycle are related in a manner neither ordinal nor collateral, nor according to Ritz’s combining theory.*

For convenience of reference the wave-lengths of these lines are given in the Appendix, together with short historical notes.

The Alkalies.—The table below gives the denominators for the two first lines in each as calculated from Paschen’s and from Randall’s results. Bergmann’s measurements for other lines are too much in error for the present objects. The limits used are the calculated values of \( VD \), using the limits \( D(\infty) \) given in Table II. above and the values of \( D(2) \) in the Appendix.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3·997919(169)−219(\xi)</td>
<td>3·992817(252)−290(\xi)</td>
<td>3·987849(433)−289(\xi)</td>
<td>3·977334(146)−287(\xi)</td>
</tr>
<tr>
<td>4·997267(2845)−569(\xi)</td>
<td>4·989237(696)−566(\xi)</td>
<td>4·983697(846)−564(\xi)</td>
<td>4·9698</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5·9710</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>6·9642</td>
</tr>
</tbody>
</table>

The question that first arises is, do these refer to actually the first lines of the series? If, like D, the lowest value of \( m \) were 2, the wave numbers of the lines would be somewhat above, Na, 0; K, 1200; Rb, 2960; Cs, 4450. For Na it would be outside, but the others come within regions observed by Paschen.†

He gave for K lines at wave numbers 1346·3 and 1182·9. The former requires a

* The relation is extremely common in certain types of spectra, e.g., the rare gases other than He. I hope to return to this in a future communication.

denominator $3'007542$ and the latter $2'987479$, the one apparently too large and the other too small to fit in with the progression of the lines for $m = 3, 4$. But the mantissa for the latter is within limits $2\delta$ below that for $m = 3$ [see Note 3 at end].

In Rb there is a line at wave number 2129'0 which would require a denominator $2'997805$, well in step with the other two. Paschen identifies it as $D_{33}(3) - P_2(4)$, assuming the existence of satellites in RbD. It would seem to be more probably the $F(2)$ sought for. There is another line given as 2156'1 or 2164'4. If the former is the more correct it gives denominator $3'001138$. In Cs no line appears with wave number near 4450. There are two lines, however, with wave numbers 3409'93 and 3321'37 which differ by $88'36 \pm 6$, and certainly suggest the doublet series depending on the $D(2)$ satellite. This requires a separation of 97'96, and if they belong to the $F$ series there must be a satellite with a separation $9'40 \pm 6$ which we should not expect to observe as being too faint. The lines give a mantissa $2'851708$ with a satellite difference 1003. The latter may be, within limits, $2\delta$, a value which in the alkaline earths seems to be closely associated with $F$ satellites. But the mantissa is less than that for $m = 3$, when a larger value should be expected. Even if not $F(2)$ itself it may be related to the $F$ cycle in a similar way to certain displacements found in the alkaline earths (see pp. 383, 413), and it should be noted that if so there seem to be lines in corresponding positions in K and Rb. They are (in wave numbers) the 1182'9 referred to above for K and 1911'05 in Rb. The latter requires a denominator $2'971391$. In this connection it is interesting to note that Paschen makes the remark that this line at times shows itself double. The separation calculated from his numbers is $1'12$, giving a denominator difference of 107 for $F_1(\infty)$ and $F_2(\infty)$, i.e., for $V D_{11}(2)$ and $V D_{12}(2)$. This would indicate a sort of incipient satellite in RbD. These considerations seem to show that there is some likelihood that $m = 3$ does not give the first line of the $F$ series, and they will be felt to have greater weight when the curious irregularity in the $F(2)$ of the alkaline earths to be noticed immediately is taken into account. The question is further discussed on p. 397 in connection with the other elements.

The next question is, is there any indication of $F$ satellites in the accepted lines? If so we should only expect to find it in Cs. Now Randall gives weak lines 8080'9 close to 8083'1, $F_1(4)$, and 8018'9 close to 8020'6, $F_2(4)$. They look like satellites only on the wrong side. The first changes the denominator by 2000, which is within easy limits of 1914 = $3\delta$. It will be shown that this is a common satellite difference in the alkaline earths. Further, it makes the denominator $4'9718$, thus bringing the values for $m = 3, 4, 5, 6$ in order, which is not the case in the table above. There is, therefore, something to be said in favour of taking the normal $F(4)$ doublet to be at 8080'9, 8018'9, and that that is then collaterally displaced by $3\delta$ to the stronger lines 8083'1, 8020'6. It is also quite in keeping with analogy in the alkaline earths that a similar displacement is not shown in the case of the first lines $F(3)$ (if $F(3)$ are the first lines).
Group II. The Alkaline Earths.—The series are most fully and regularly developed in Ca and Sr. In Ba and Ra the configurations which give rise to the normal type seem to be so modified that displaced lines become common, and in cases the normal line has disappeared. On the other hand, Mg seems to range itself with the Zn sub-group. It will be best therefore to deal with Ca and Sr first, and as they are built on a precisely similar plan to consider them together.

The following table gives the wave numbers of the series together with certain others which are clearly similarly related in the different elements. The separations are indicated by thick figures. The wave-lengths are given in Appendix II.

<table>
<thead>
<tr>
<th></th>
<th>Ca</th>
<th>Sr</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>16203·40 21·75 16225·15</td>
<td>14801·57 101·73 14903·30</td>
</tr>
<tr>
<td></td>
<td>16204·72</td>
<td>15046·98 61·70 15108·68</td>
</tr>
<tr>
<td></td>
<td>21799·02 21·13 21820·15 13·58 21833·73</td>
<td>20432·18 100·64 20532·82</td>
</tr>
<tr>
<td></td>
<td>24391·49 21·50 24412·99 13·66 24426·65</td>
<td>23045·78 99·26 23145·04 58·54 23203·58</td>
</tr>
<tr>
<td></td>
<td>25793·67 21·64 25815·31 13·55 25828·86</td>
<td>24457·06 100·05 24557·11 59·25 24616·36</td>
</tr>
<tr>
<td></td>
<td>26634·00 22·43 26656·43 14·29 26670·72</td>
<td>25303·32 100·47 25403·79 58·61 25462·40</td>
</tr>
<tr>
<td></td>
<td>27177·76 21·58 27199·34 15·09 27214·43</td>
<td>25850·61</td>
</tr>
</tbody>
</table>

Analogous Sets in Ca and Sr.

<table>
<thead>
<tr>
<th></th>
<th>17847·46 21·94 17869·40 13·86 17883·26</th>
<th>18061·87 100·20 18162·07 56·61 18218·68</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>17887·55 21·81 17909·34</td>
<td>18239·35 100·62 18339·97</td>
</tr>
<tr>
<td></td>
<td>18968·53 21·58 18990·10 14·01 19004·11</td>
<td>19016·64 100·34 19116·98 59·75 19176·73</td>
</tr>
</tbody>
</table>

Ba.

<table>
<thead>
<tr>
<th></th>
<th>13089·79 260·60 13350·39</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>13471·69 259·76 13731·45 157·55 13889·00</td>
</tr>
<tr>
<td></td>
<td>13477·44</td>
</tr>
<tr>
<td></td>
<td>18686·80 255·17 18941·97</td>
</tr>
<tr>
<td></td>
<td>17300·80 699·93 18000·73</td>
</tr>
<tr>
<td></td>
<td>21308·19 252·51 21560·70 148·32 21709·02</td>
</tr>
<tr>
<td></td>
<td>21530·22 701·89 22052·81</td>
</tr>
<tr>
<td></td>
<td>22706·84ab 22979·57</td>
</tr>
<tr>
<td></td>
<td>23667·07ab 23919·27am 23995·83</td>
</tr>
</tbody>
</table>

\( F_{12} (5) (\Delta_2) \) \( F_1 (6) (9\Delta_0) \) \( F_2 (6) (8\Delta_0) \)
It is clear that these lines also show satellites. Also, it is curious that the first sets of triplets apparently have the lines corresponding to the second separation displaced below those forming the first. Thus in Ca the second set (giving \( \nu_2 = 13.5 \)) have not been observed, in Sr the two \( (\nu_1 = 101, \nu_2 = 60) \) are separated by a gap of 143, and a similar effect will be found later in Ba. Owing to this fact, the formula constants are calculated from the 2nd, 3rd, and 4th sets. They give for \( F_{11} \):

\[
\begin{align*}
\text{Ca} & : \quad 28934'93 - N \left/ \left( m + 891511 + \frac{086641}{m} \right) \right.^2, \\
\text{Sr} & : \quad 27612'37 - N \left/ \left( m + 875560 + \frac{100548}{m} \right) \right.^2.
\end{align*}
\]

These give the following values of \( \text{O-C} \):

<table>
<thead>
<tr>
<th>( m )</th>
<th>2</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{Ca}</td>
<td>-1'18</td>
<td>'18</td>
<td>'22</td>
</tr>
<tr>
<td>\text{Sr}</td>
<td>-'42</td>
<td>'05</td>
<td>'02</td>
</tr>
</tbody>
</table>

The agreement is good, except for \( m = 2 \), and in this case the agreement is sufficient to show that the allocation for \( m = 2 \) is correct.

The limits are close to those of \( VD_{11} (2) \), which is not known with great exactness because the values of \( S (\infty) \) or \( D (\infty) \) given in [II.] for the second group are subject to possible errors of some units. With formulae in \( 1/m \) the values of \( D (\infty) = S (\infty) \) are given [II., p. 36] as 33994'85 for Ca and 31037'27 for Sr, whilst with formulae in \( 1/m^2 \), the respective limits are 33983'45, 31027'64. The values of \( VD_{11} \) deduced from these are respectively 28939'93, 27615'65 with \( 1/m \) and 28928'53, 27606'02 with \( 1/m^2 \). The limits, therefore, found above for \( F_{11} (\infty) \) lie each between their corresponding values as deduced from the D series direct. Assuming that the \( F (\infty) \) are more accurate, the values of \( D (\infty) \) deduced from them are 33989'85 for Ca and 31033'79 for Sr, in both cases close to the mean of those in [II.]. If the series depend on formula sequences, these limits may be taken as close to the correct values. If, however, the different orders proceed by multiples of \( \delta \) or \( \Delta \) in the way illustrated in Table II. for the D series, the limits may require modification by a few units.

As the separations of the F series depend on the separations of the satellites of the first lines of the D series, and these depend on displacements by definite multiples of \( \delta \), as given in Table II., it is possible to calculate the values of the former with extreme accuracy. Table II. gives 13\( \delta \) and 8\( \delta \) as the multiples in question for both the elements Ca and Sr. Using the values of \( \delta \) and of the denominators of \( D_{11} \) there given, the separations in question calculate out to 22'49, 13'75 for Ca and 100'34, 62'01 for Sr. These may be regarded as exact to the 2nd decimal place and independent of any possible variation of \( \xi \).
In the case of Ca the observed lines, with differences somewhat less than 22'49 and 13'75, seem to indicate the presence of close satellites. If 16203'40 is really $F_{12}(2)$, the separation of the first satellite is 1'32, with possible errors (26 ± 26), which form a very considerable proportion of the total amount. A displacement of 3δ produces a separation of 1'51 and it may be this. But 16203'40 has an excessive intensity for a satellite line, viz., 6, as against 4 for $F_{11}$, and, moreover, it may possibly be the collateral $S_{1}(2) (-\Delta \delta)$ which gives O–C = '03 with O = '10. If the latter allocation is correct, it would hide $F_{13}$, which should be $16225'15 (26) - 22'49 = 16202'66 (26)$, giving a separation of 2'06 ('52) due to 4δ which gives 2'02. The same considerations applied to the second set give a separation of 1'36 for the first satellite, in which again 4δ gives 1'26, and 22 for the second, δ giving 32. The separations are so small that no certain conclusions can be drawn as to their origin. The actually observed numbers may be due to 4δ and δ, but 3δ and 2δ are just possible [but see Note 4].

For Sr the first doublet is useless, as the line is due to the early measurements of LEHMANN, which are affected with considerable errors. The observed separation for $F_{11}$ and $F_{12}$ gives 101'73 instead of something less than 100'34. The second triplet gives 2'92 for the separation of the second satellite from the first and 2'06 for the separation of the satellite of the second line of the triplet, and from analogy with other satellite series, this would be the separation of the first and second satellites of the first line. Differences of 3δ and 2δ give separations of 3'06 and 2'04, so that it may be concluded that the satellites depend on these differences, a conclusion supported by the fact that a similar result is indicated as possible for Ca.

Returning to the curious fact noticed above that the first triplets of the series seem to be dislocated, the second fragment in Sr is found at a distance 143'68 below its normal position. For the present we note this can be explained by one of two possible collateral displacements, viz. ($-18\frac{1}{2}\delta$) $F(2)$ or $F(2) (3\Delta \delta)$, where $F$ stands for the normal $F_2$ or $F_3$. The case of Ba below will give evidence in favour of the latter explanation.

In addition to the lines of the series itself, there are two sets of triplets and a doublet which are clearly analogous in the two elements. They are given in the list above, following the series lines. The first triplets in each are curious as having the middle line the strongest.* They are also related to others in the way indicated in the following scheme:

\[
\begin{align*}
(8) & \quad 17847'46 \\
& \quad 21'94 \\
Ca & \quad 17842'52 \quad 13'98 \\
(8) & \quad 17856'50 \quad 12'90 \quad (10) \quad 17869'40 \\
& \quad 13'86 \\
& \quad (8) \quad 17883'26
\end{align*}
\]

* A similar peculiarity has already been noted in the associated OD series.
The first lines of the triplets give as denominators, supposing the true limits to be \( F_1(\infty) + \xi \) —

\[
\begin{align*}
\text{Ca} & \quad 3'145123 (22) - 141'8\xi \\
\text{Sr} & \quad 3'388848 (28) - 177'4\xi
\end{align*}
\]

1st triplet \[ 172177 + 30p - 22q - 24'6\xi \]

2nd \[ 183160 + 23p - 28q - 30'4\xi \]

\[ = 126 \times (1366'48 + 24p - 17q - 20\xi) \]

\[ = 126 \times (1368'78 - 20\xi) - 5\delta \]

where \( p, q \) lie between \( \pm 1 \). Clearly the differences are the multiples 126\( \Delta_2 \), 33\( \Delta_2 \), for the two elements respectively.

The first lines of the doublets give for Ca \( 3'150824 (22) - 142'6\xi \) and for Sr \( 3'420693 (18) - 182'5\xi \).

These differ from the denominator of \( F_n (3) \) by

\[
\begin{align*}
\text{Ca} & \quad 769567 + 123p - 22q - 132'6\xi = 562 \times (1369'33 + 22p - 04q - 236\xi) \\
\text{Sr} & \quad 488383 + 32p - 18q - 90\xi = 88 \times (5549'81 + 36p - 20q - 1'02\xi).
\end{align*}
\]

That is, they apparently differ by 562\( \Delta_2 \), 88\( \Delta_2 \) respectively. We shall see shortly that the best value for \( \xi \) makes the relation for Sr very exact, whilst that for Ca is more doubtful.
The following list contains the wave numbers of certain lines related to the F series in the two elements with the denominators appended for the chief lines:

<table>
<thead>
<tr>
<th>Element</th>
<th>Ca.</th>
<th>Sr.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-31237.30</td>
<td>1.363986 - 11.56ɛ</td>
</tr>
<tr>
<td></td>
<td>10084</td>
<td></td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>-31338.14</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-31345.01</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6893</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-31413.94</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9999</td>
<td></td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>-31513.93</td>
<td>1.361954 - 11.51ɛ</td>
</tr>
<tr>
<td>-15380.80</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1392</td>
<td></td>
</tr>
<tr>
<td>-15394.72</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-15447.35</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2175</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A -15469.10</td>
<td>1.571602 - 17.7ɛ</td>
<td></td>
</tr>
<tr>
<td>B - 5011.98</td>
<td>2.141121</td>
<td>8.89466 - 64.7ɛ</td>
</tr>
<tr>
<td></td>
<td>2271</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.141244 - 44.7ɛ</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5034.69</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10.06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5044.75</td>
<td></td>
</tr>
<tr>
<td>C 6171.18</td>
<td>2.194987 - 48.2ɛ</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2112</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6192.30</td>
<td></td>
</tr>
<tr>
<td>D 13133.01</td>
<td>2.634565 - 23.3ɛ</td>
<td>15271.72</td>
</tr>
<tr>
<td></td>
<td>20.04</td>
<td>2.981157</td>
</tr>
<tr>
<td></td>
<td>13153.05</td>
<td>99.09</td>
</tr>
<tr>
<td></td>
<td>15370.81</td>
<td></td>
</tr>
<tr>
<td>E 15580.59</td>
<td>2.865778 - 1077ɛ</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20.66</td>
<td></td>
</tr>
<tr>
<td></td>
<td>15601.25</td>
<td></td>
</tr>
</tbody>
</table>
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Ca.          Sr.
---          ---
F 17847.46   3'145123-141'8ξ  18061'89  3'388848-177'4ξ
  21'94        100'18
17869'40     18162'07  56'61
13'86        18218'68
17883'26
G 17887.55   3'150824-142'6ξ  18239'35  3'420693-182'5ξ
  21'81        100'62
17909'34     18339'97
H 18968.53   3'317300- ?  19016'64  3'572008-207'8ξ
  21'57        100'34
18990'10     19116'98  59'75
14'01
19004'11     19176'73
I ........................................ 21022'14  4'048761
.......................  59'91
........................................  21082'05  1'499227

From these we find the following differences, m denoting the mantissa only:—

<table>
<thead>
<tr>
<th></th>
<th>Ca.</th>
<th>Sr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_2 - \alpha_1 )</td>
<td></td>
<td>3( \delta ) within limits</td>
</tr>
<tr>
<td>( m \text{ of } F - m \text{ of } \alpha_2 )</td>
<td></td>
<td>5 ( (5378'80-33'2\xi) )</td>
</tr>
<tr>
<td>( m \text{ of } D - m \text{ of } A )</td>
<td>46 ( (1368'75-1'43\xi) )</td>
<td></td>
</tr>
<tr>
<td>( m \text{ of } G - m \text{ of } B )</td>
<td>7 ( (1368'71-14\xi) )</td>
<td>68-118( \xi ) = 0</td>
</tr>
<tr>
<td>( m \text{ of } F_{13}(3) - m \text{ of } F )</td>
<td>562 ( (1368'82-236\xi) )</td>
<td>88 ( (5549'81-1'02\xi) )</td>
</tr>
<tr>
<td>H-F</td>
<td>126 ( (1366'48-20\xi) )</td>
<td>33 ( (5550'30-91\xi) )</td>
</tr>
<tr>
<td></td>
<td>126 ( (1368'78-20\xi) - 5\delta )</td>
<td></td>
</tr>
<tr>
<td>D-C</td>
<td>321 ( (1369'40+51\delta\alpha-110\xi) )*</td>
<td></td>
</tr>
<tr>
<td>G-F_{13}(2)\dagger</td>
<td>158 ( (1368'89-173\xi) )</td>
<td>89 ( (5558'90-77\xi) )</td>
</tr>
<tr>
<td>G-F_{12}(2)\ddagger</td>
<td>158 ( (1368'16-173\xi) )</td>
<td>89 ( (5552'66-77\xi) )</td>
</tr>
<tr>
<td>E-D</td>
<td>169 ( (1368'12-144\xi) )</td>
<td></td>
</tr>
</tbody>
</table>

Also the first triplet A in Ca shows the same kind of dislocation as in F(2) of the other elements. The dislocation is 52'63, corresponding to a denominator difference of 931, and 16\( \delta \) is 930.

* \( d\alpha \) on 13133 may be >1.
\dagger F_{13}(2) as calculated from the formula.
\ddagger Allowing 2\( \delta \) for the satellite difference \( F_{12} - F_{13} \).
The number of the cases where multiples of \( \Delta_2 \) enter, as well as their appearance in the corresponding position in the two elements where corresponding lines are observed, must produce a conviction that they represent real and not chance relations. In the case of Ca it makes \( \Delta_2 \) close to 1368.7 corresponding to \( \delta = 361.77 \) \( w^2 \). If \( \xi \) has any but a very small value, the first two multiples are upset, but these may be due to chance. If \( \xi \) be made \(-6.5\) as suggested below, \( \Delta_2 \) will be about 1370.2 with \( \delta = 362.15 \) \( w^2 \), which is considerably greater than the most probable value. The probability is that \( \xi \) can only be a small \( \pm \) quantity, \( \xi = \pm 1 \) changing \( \delta/w^2 \) by \( \pm 0.6 \). A similar reasoning applied to Sr rather tends to show that here the value \(-6.5\) is to be preferred. It makes the first multiple = \( 5 \times 5594.60 \), and the other values of \( \Delta_2 \) become \( 5556.44 \), \( 56.21 \), \( 57.66 \), and \( 5556 \) gives \( \delta = (361.52 \pm 24) \) \( w^2 \), the uncertainty of this being due to a possible error in the atomic weight of \( 87.66 \), with \( \xi = -5.5 \) the first relation gives \( \Delta_2 = 5561.40 \).

Again the most probable values of the denominators of \( F_{13} \) (2) are

\[
\begin{align*}
\text{Ca} & = 934539 - 115.2\xi = 937277 - 115.2\xi - 2\Delta_2, \\
\text{Sr} & = 925946 + 114.0\xi = 937060 + 114.0\xi - 2\Delta_2,
\end{align*}
\]

The numbers on the right are practically equal. If analogous relations are found in Ba and Ra it points to the existence of a group constant about 937300. On the other hand it would seem that the denominators of \( VF_{11} \) (2) are, like those of \( VD_{13} \) (2) multiples of \( \Delta_2 \), also, for

\[
\begin{align*}
\text{denominator of CaF}_{11} \text{ (2)} & = 934539 + 5\delta = 683 (1368.71 - 16\xi), \\
\text{"} \text{SrF}_{11} \text{ (2)} & = 925946 + 5\delta = 167 (5552.79 - 68\xi),
\end{align*}
\]

and \( \xi = -6.5 \) in Sr makes \( \Delta_2 = 5557.21 \) in line with those above.

The denominator of \( \text{CaF}_{11} \) (2) is \( 8\Delta_2 \) less than that of \( \text{CaD}_{13} \) (2),

\[
\begin{align*}
\text{"} \text{SrF}_{11} \text{ (2)} \ " 11\Delta_2 \ " \ " \ " \text{SrD} \text{ (2)}.
\end{align*}
\]

Which of these two interpretations is the more likely must be left until the cases of Ba and Ra are considered. It should however be noted that there may be some uncertainty as to what lines really represent \( F_{11} \), \( F_{13} \), or \( F_{13} \), \( i.e. \), as to which of them the multiple law is to be attached.

There remains to consider the question of the real limits. The reasons for supposing them to be \( D_{11} \), \( D_{12} \), \( D_{13} \) are so strong that it is necessary to see whether the values obtained direct from the \( F \) series, and those required in Table II. cannot be brought into agreement.

If the \( F \) series possess what has been called in [IL] a formula sequence, the values obtained for \( F(\infty) \) above cannot be more than a few units in error, and in this case it

* Calculated from formula.
† The observed is probably \( F_{13} \) (2) since the separation with \( F_2 \) (2) is the full value.
must be possible to raise the limits for \( D(\infty) \) to agree with those calculated from \( F(\infty) \). That is, to raise that for Ca from \( 33981.85 \) to \( 33989.85 \) and for Sr from \( 31027.25 \) to \( 31033.99 \), or Ca by \( 8.00 \) and Sr by \( 6.74 \) or thereabouts. It may probably be possible to find numbers near those which would still make the order differences of Ca and Sr multiples of \( \delta \), but only by supposing that the successive mantissa-differences in the D series after rising begin to decrease with higher orders, which is against the rule in other cases. This is so far an argument against this way of reconciling the different values of the limits. If however the order differences in the F series behave in a similar manner to that considered above for the D, i.e., by multiples of \( \delta \) or \( \Delta \), the exactness of the \( F(\infty) \) found by means of a formula is no longer so close, and the question becomes one of seeing if, when they are made 8 less for Ca and 6.74 for Sr, it becomes possible to arrange the denominators in the same way.

If the attempt be made to reduce \( F(\infty) \) by 8 in CaF, a similar objection to that raised above will enter, viz., the successive mantissa-differences after falling begin to rise after \( m = 5 \). If however a reduction of about 6.5 be made, reducing the limit to that found in [II.] for \( S(\infty) \), the order mantissae differ successively within observation limits by \( 10\Delta_2, 4\Delta_2, \Delta_2, \Delta_2, 0 \). Further, in the case of Sr a fall of 6.75 produces a similar fall and rise in successive denominators. If however \( \xi \) be put \(-1.33\), the mantissa-differences become within limits \( 3\Delta_2, \Delta_2+9\delta, 16\delta, 11\delta, 4\delta \). If this is justified, it is curious that as in the D series where there are no satellites, the differences proceed by multiples of \( \Delta_2 \) the same rule should hold for CaF, where satellites are at least not certain. The difficulty can only be stated and the solution left open. It is possible that the order differences must be compared from the \( F_{11} \) of one line to the \( F_{11} \) of the next, for which there is evidence in Ba and Ra.

Barium.—In discussing Ba we start under the disadvantage that the lines belonging to D \( (2) \), with the corresponding satellite separations have not been observed, for the ultra-red doublet treated in the discussion on the D series does not seem to belong to the normal D \( (2) \). Moreover, the observed lines which are clearly related to the F series are so dispersed by collateral displacements that it is questionable whether it is possible to arrange a series proceeding by an algebraical sequence as in the other cases. The lines exhibited in the table above run on parallel lines with the corresponding lines in Ca and Sr, and are clearly closely related to the successive orders of the series, even if they are not the typical ones themselves. An attempt to obtain a formula from the first three gives a limit = 25906'7, and gives a value of the wave number for \( m = 5 \) of 22729'52 close to the strong line 22706'84. It is 250'05 behind the strong line 22979'57, which indicates that the last is probably the normal \( F_2(5) \), and makes the normal \( F_1(5) \) about 250 behind. This is in fair order with the march of the others. We may therefore feel justified in settling that the limit of \( F_1(\infty) \) is near 25906'7. The F separations are close to 260 and 157, they are therefore based on satellite differences in the D series of 15\( \delta \) and 9\( \delta \); \( \delta \) is so
large that there can be no doubt. These numbers are in analogy with the values for Ca and Sr, viz., 13δ and 8δ, are in the usual ratio 5:3, and stand to the observed values for BaD(3) given in Table II. in a similar relation to those in Sr. Now F3(∞) must be VD13(2), and if the general rule found above that the mantissa of VD13(2) is a multiple of Δ3 holds, it is possible to obtain a very accurate value. As a fact, with F1(∞) = 25906 the mantissa of F3(∞) is very nearly the multiple of 87Δ3. If it is made so exactly, taking Δ3 = 11960, then F1(∞) becomes about 25922. This value, with the D satellite differences of 13δ, 9δ, give

\[ F_1(\infty) = 25922 \]
\[ 260\cdot17 \]
\[ F_2(\infty) = 26182\cdot17 \]
\[ 157\cdot95 \]
\[ F_3(\infty) = 26340\cdot12 \]

and it is seen how close the separations come to those observed. If we put F1(∞) = 25922 + \(\xi\), the mantissa of

\[ D_{13}(2) = 1\cdot040539 - 38\cdot7\xi = 87 (11960\cdot2 - 45\xi) = 87\Delta_3. \]

The D(2) lines calculated from these and D1(∞) = 28610'63 found above give the following scheme in wave-lengths on Rowland's scale in vacuo:—

<table>
<thead>
<tr>
<th>D1</th>
<th>D2</th>
<th>D3</th>
</tr>
</thead>
<tbody>
<tr>
<td>44042'97</td>
<td>31758'94</td>
<td>28416'75</td>
</tr>
<tr>
<td>41178'36</td>
<td>30241'91</td>
<td></td>
</tr>
<tr>
<td>37193'67</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The last one only comes within the region in which Randall's ultra-red lines lie, his longest wave-length being 29223, belonging to the doublet treated as a possible D line above. If the rules employed are valid, these values can only err by a few units. The denominator of VF11(2) is 2'923500 - 113'9ξ. In the cases of Ca and Sr there is apparent the existence of satellites, viz., F13(2) = F11(2)(-3δ) and F15(2) = F11(2)(-5δ), and in both cases the mantissa of one of the F1(2) sets thus found are multiples of Δ3. If this is general the mantissa for BaF13(2) will be 920085 - 113'9ξ. This is 77 (11949'1 - 1'48ξ), sufficiently near to give some weight to the allocation. If ξ = -10 this is 77 × 11963'9 and the value for D13 given above becomes 87 × 11964'7, giving a value of Δ3 = 11964 within limits of error. Δ3 = 11964 makes the sum 683'66 = 361'98W2 with W = 137'43. The F satellites thus constituted would, if existing, have separations for m = 2 of 18'02 and 12'03, and for m = 3 of
7.63 and 5.09—but they have not been observed for F(2)—as indeed is the case in Ca and Sr the first set in which show separations of the full amount. The curious dislocation of the second half of the triplet from the first seen in Sr shows itself here also. The analogue appears to be shown in the triplet coming next in the list which appears to have kept its first member and satellite. The displacement is 381°06. This cannot be due to a displacement in F(∞), for if so the separations due to 15δ, 9δ, would be considerably larger. If it is treated as a displacement on VF the denominator difference is 43403 - 53δ, while 3Δ2 + 11δ = 43414. It is probably therefore this. The corresponding displacement in Sr was found to be 3Δ2. The separation between the first line and the satellite is 575, the satellite being due to Lehmann whose measures are not very accurate, it may well be 6.01 corresponding to a satellite difference of δ. The lines may therefore be represented

\[ F_{11} (2) (3\Delta_2 + 11\delta), \quad F_{22} (2) (3\Delta_2 + 11\delta), \quad F_{33} (2) (3\Delta_2 + 11\delta), \]

\[ F_{11} (2) (3\Delta_2 + 12\delta). \]

If the next two lines are correctly allocated, 18686 should have an unobserved satellite with a difference 2δ. This would make 18941'97 or F_{31} 260°26 ahead of the satellite, so that this supports the allocation. The line 21308'19 = F_{11} (4) corresponds to a satellite with 5δ. This makes, on the supposition of satellite differences of 2δ, 3δ, 21560'70 or F_{22} (4) 260°39 ahead of the satellite F_{13} (4), the satellite F_{13} (4) being absent. The line for \( m = 5 \) appears to be displaced to 22706. The value calculated from the rough formula gives a line 250 behind the strong line 22979'57, clearly showing that the latter is a F_{3} (5) line, and 22706 is very close to a displacement of Δ\(_2\) on the calculated. If this be made exact the undisplaced line would be at 22719'97, or 259°60 behind 22979'57. This is within error limits of 260°17. Hence F_{12} (5) has been altogether displaced to 22706'84 = F_{12} (5) (Δ\(_2\)), and 22979'57 is F_{3} (5). For \( m = 6 \) the formula gives F_{1} (6) = 23582'83. There is a doublet at 23667'07 ("28), 23919'27 ("14) with a separation 252°20, and no others in the neighbourhood. If these are the displaced F (6), the normal F (6) would be 23595'82 and 23855'93 and the observed lines 23667'07 = F_{1} (6) (9Δ\(_2\)) and 23919'27 = F_{3} (6) (8Δ\(_2\)). The calculated normal lines have separation 260°11, or practically 260°17. A line at 23995'83 is 413 = 260 + 133 ahead of the calculated F_{1} (6). It is therefore the undisplaced F_{3} (6).

There are a large number of other lines clearly related to the F type. Their complete discussion would require a more searching investigation than can be given now. Several sets are related in a manner which is quite common in spark and rich arc spectra, indicated by the fact that a number of lines may differ in succession by nearly the same separation—a kind of relation which cannot be due to collateral displacement by equal denominator differences. There are a few also which seem to be attached parasitically to S and D lines. There may be uncertainty also as to
whether the separations shown which differ from 260 and 158, differ through a satellite effect, or by successive collaterals of 15δ and 9δ. For instance, putting

\[ 15\delta = \Delta', \quad F_1'(\infty) - F_1(\infty)(\Delta') = 264'10, \quad F_2(\infty)(-\Delta') - F_2'(\infty) = 256'39, \]

\[ F_2(\infty)(-2\Delta') - F_2'(\infty)(-\Delta') = 252'35, \]

all which separations occur. In the lines now to be referred to, however, the separations will be supposed to owe their defect from 260 to the satellite effect, and thus treated it is clearly seen what an important rôle the \( \Delta_2 \) term plays.

Amongst the ultra-red lines observed by RANDALL* appear the following in wave numbers:

\[
\begin{align*}
(70) & \quad 9387'95 (4'4) & 159'93 \\
(60) & \quad 9547'98 (1'82) & 257'70 \\
(60) & \quad 9771'54 (9') & 159'76 \\
(5) & \quad 9804'787 (1'5) & \\
(60) & \quad 9964'527 (3) & 
\end{align*}
\]

The figures in brackets before the numbers give intensities and those after the estimated maximum errors. They clearly belong to the \( F \) cycle, and show within error limits the normal separations. The run of the intensities would point to negative values, with the first four respectively for \( f_{31}, f_{21}, f_{11}, f_{12} \), but also the 2nd, 4th and 5th might be \( f_1, f_2, f_3 \), whereby 9771 would not come in and the small intensity for \( f_2 \) would be abnormal. On the first supposition, \( f_{11} = -9771 \) gives a denominator \( 1'752908 (23) - 24'5\delta \) and \( f_{12} = -9804 = f_{11}(-\delta) \). On the second, \( f_{11} = 9547 \) gives a denominator \( 2'588000 (143) - 79\delta \). Now

\[ 752908 (23) - 24'5\delta = 63 (11950'3 \pm 36 - 36\delta) = 63 (11961'1 \pm 36 - 36\delta) - \delta, \]

or denominator of \( f_{12} = 63 (11961'1 \pm 36 - 36\delta) - 2\delta = 63 \Delta_2 - 2\delta. \)

In the following the wave numbers of some sets of lines with their separations are given. The low frequencies have been observed by LEHMANN and by HERMANN and HOELLER. LEHMANN gives many weaker ones not observed by HERMANN and vice versa. LEHMANN's observations were earlier but are not nearly so accurate as those of the others.

2. 12636'36 260'17 12896'53
3. 14141'15 260'94 14402'09 158'63 14560'72
4. 14903'26 254'37 15157'63 157'88 15315'51
   14934'33
5. 16669'61 259'69 16929'30
   16921'93 156'03 17077'96

The following numbers give the corresponding denominators calculated from $F_1(\infty) = 25922$. Where the separation differs from 260, the satellite value—or, which is the same thing, the denominator for $F_{21}$—is inserted as well, but, in order to distinguish it, it is printed further to the right—the changes due to $\xi$ being the same for both.

\[
\begin{align*}
(2) & \quad F_{11}(2) = 2'873178 - 108'1\xi \\
F'_{11}(2) & \quad 2'923500 - 113'9\xi \\
F'_{12}(2) & \quad 2'967999 - 119'2\xi \\
\delta + 3 & \\
(3) & \quad 3'051164 - 129'5\xi \\
\quad 051267 & \\
\quad 154087 & \\
\quad 5\delta_1 - 46 & \\
(4) & \quad 3'154916 - 143'1\xi \\
\quad 26\delta_1 + 18 & \\
\quad 3'159873 - 143'7\xi & \\
\quad 441229 & \\
\quad 2\delta_1 + 13 & \\
(5) & \quad 441588 & \\
\quad 2\delta + 5 & \\
\quad 3'442954 - 186'0\xi & \\
\quad 3'549987 - 203'9\xi & \\
\quad 3\delta_1 + 13 & \\
(6) & \quad 550517 & \\
\quad 3'610577 - 214'5\xi & \\
\quad \delta_1 + 43 & \\
(7) & \quad 610791 & \\
\end{align*}
\]

\[
\begin{align*}
& \quad 95506 = 8(11938'3 - 1'39\xi) \\
& \quad 11952 \\
& \quad 83165 = 7(11880'7 - 1'49\xi) \\
& \quad 11895 (11969) \\
& \quad = 16(11960'7 - 1'54\xi) \\
& \quad 12039 (11982) \\
& \quad 108209 = 9(12023'2 - 1'58\xi) \\
& \quad 11962 \\
& \quad 28667 = 24(11944'6 - 1'76\xi) \\
& \quad 11962 \\
& \quad 107563 = 9(11951'7 - 1'99\xi) \\
& \quad 11971'6 \\
& \quad 60060 = 5(12012'2 - 2'08\xi) + 2\delta, \\
& \quad 11964'6 \\
\end{align*}
\]
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\[
\begin{align*}
(8) & \quad 8'63911 \\
& \quad 2\delta + 14 \\
& \quad 3'864437 - 263\xi \\
\end{align*}
\]

\[14463 = 11929 + 15\delta - 28 - 3'1\xi \quad 11932\]

\[
\begin{align*}
(9) & \quad 8'65855 \\
& \quad 3'866436 - 263\xi \\
\end{align*}
\]

\[14495 = 11933 + 15\delta - 2'9\xi \quad 11962\]

\[
\begin{align*}
(10) & \quad 8'79299 \\
& \quad 3'878900 - 266'1\xi \\
\end{align*}
\]

\[
\begin{align*}
F_{12}(3) & = 8'92050 \\
F_{13}(3) & = 3'893395 - 269'0\xi \\
F_{12}(4) & = 8'71538 \\
F_{13}(4) & = 4'875553 - 528'3\xi \\
F_{12}(5)(-\Delta_2) & = 5'840534 - 908'3\xi \\
\Delta_2 - 8 - 14 & = 976927 = 82(11913'7 - 4'63\xi) \quad 11960 \\
? F_{12}(5) & = 5'851797 - 908'3\xi \\
\end{align*}
\]

\[
\begin{align*}
\Delta_2 & = 81(11957'4 - 1'91\xi) = 81(11940'6 - 1'91\xi) + 2\delta, \quad 11959'7; \\
F_{11}(2), F_{12}(3) & = 81(11957'4 - 1'91\xi) = 81(11940'6 - 1'91\xi) + 2\delta, \quad 11959'7; \\
F_{11}(3), F_{12}(4) & = 82(11928'5 - 3'16\xi) = 11960'1, \\
F_{11}(4), F_{11}(5) & = 82(11913'7 - 4'63\xi) = 11960'0 + 3'6, \\
\end{align*}
\]

The differences are given in thick figures. The last column gives the corresponding value of \(\Delta_2\) without regard to observation errors when \(\xi = -10\). In the case of (3) if the differences be referred to a hypothetical \(F_{11}\), displaced 3\(\delta\) from 14141, the two abnormal values come to 11969 and 11982. It will be remembered that we had an indication, above of \(\xi = -10\) with \(\Delta_2 = 11964\) in treating both \(V_{D13}(2)\) and \(VF_{13}(2)\) as depending on multiples of \(\Delta_2\). It would seem, therefore, that the value of \(\Delta_2\) is close to 11964 \pm 3 and the value of \(F_{1}(\infty) = 25912\).

The actual differences of successive denominators in the normal series may thus be represented:

\[
\begin{align*}
F_{11}(2), \quad F_{12}(3) & = 81(11957'4 - 1'91\xi) = 81(11940'6 - 1'91\xi) + 2\delta, \quad 11959'7; \\
F_{11}(3), \quad F_{12}(4) & = 82(11928'5 - 3'16\xi) = 11960'1, \\
F_{11}(4), \quad F_{11}(5) & = 82(11913'7 - 4'63\xi) = 11960'0 + 3'6, \\
\end{align*}
\]

in which the last column also gives the value of \(\Delta_2\), where \(\xi = -10\).

The first multiple of 81 with addition of 2\(\delta\) suggests (1) a real \(F_{11}(2)\), displaced 2\(\delta\) from 13089, or (2) that there is a normal type \(F_{11}\) about \(\Delta_2\) behind. The latter may well not be a typical \(F_{11}\) line since it makes the exact separation 260 with \(F_{21}\). There

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is a line by LEHMANN at 13096'55, but its collateral displacement cannot be 2δ within any likely limits of even LEHMANN's measurements. As to the second supposition, there is a line at \( n = 12992'53 \) by LEHMANN which gives denominator 2'892050. This gives a difference with \( F_{19}(3) \) of 82 (11943'5 - 1'90\( \xi \)), which with \( \xi = -10 \) would again make \( \Delta_2 = 11962'5 \). It would thus appear that the normal \( F_{11}(2) \) line is 12992, and the system receives a double displacement, first to 13089, and again to 13471. The mantissa is 912483 - 112'6\( \xi \). The addition of 2\( \Delta_2 \) makes it 936403 - 112'6\( \xi \), which with \( \xi = -10 \) is 937529, well within error limits of the same quantity in the case of Ca and Sr. Again we are met with the apparent simultaneous existence of two explanations which cannot be compatible. Is the true explanation that the typical first line is 937300 - 2\( \Delta_2 \), but that the corresponding configuration is not very stable and transforms to one depending on the nearest complete multiple of \( \Delta_2 \). Certainly such instability is indicated in Ba.

Radium.—The discussion for radium is rendered even more uncertain than that for barium, in that the ultra-red region has not been observed, the process of disintegration and re-aggregation has proceeded further, and, in addition, there is some uncertainty about \( \Delta_2 = 37\delta_2 \) adopted.

RUNGE and PRECHT's plates were only sensitive up to 6500 Å.U., and EXNER and HASHEK give only two lines above this, 6642'73 and 6641'38, both of which belong to the F cycle. The number of lines, however, coming within this cycle is very large, but a complete discussion would involve the consideration of the new kinds of relationships referred to under barium, and cannot therefore be undertaken here. It will be sufficient to deal only with some generalities, specially bearing on the series proper, which will also give some further light on the general D series.

There are a large number of triplets with separations in the neighbourhood of 692 and 432, which are roughly in the proper ratio 5 : 3, allowing for the fact that the actual separations must be larger. Those in the table of F lines above are roughly parallel to the BaF, and give a limit somewhere about 24520. VD\( _{10} \) (2) would therefore be about this, and VD\( _{12} \) (2) more than 692 + 432 = 1124 larger. The denominator of VD\( _{12} \) (2) should be a multiple of \( \Delta_2 \). Using the most probable value of \( \Delta_2 = 37\delta_2 \), it is found that the denominator comes out very close to a multiple of 3\( \Delta_2 \). If this be made exact it is found that VD\( _{12} \) (2) = \( F_3(\infty) = 25752'75 + \xi \). The values of \( \delta \) are so large that there can be no ambiguity about the multiples to be chosen to give the separations, viz., 16\( \delta \), 10\( \delta \). These multiples march well with those for Ca, Sr, Ba. The separations resulting are 705'93, 456'69, with \( F_1(\infty) = 24590'13 \) and \( F_2(\infty) = 25296'06 \). If we apply the rule shown in the preceding elements for F\( _{13} \) (2), the denominator is 2'937300 - 2\( \Delta_2 \) = 2'868676. Satellites depending on 3\( \delta \), 2\( \delta \) would give separations 51'44 and 34'20, and the fact that these separations occur in connection with \( n = 17300 \) renders the identification of that for F (3) rather doubtful, a doubt which is increased when we test the allocation by the law indicated above that the
differences of the denominators of $F_{11}(m) - F_{12}(m+1)$ is always the same multiple of $\Delta_\nu$ as is done below. The line 19897 has separation 689.33, and therefore should have a satellite (too faint) 16'60 above it. Allowing for observational errors on 19897 this is $5\delta$ on the denominator. The following scheme will then illustrate the law of formation:

Calculated $F_{12}(2)$ 2'868676

Satellite 3'837028 $9\delta$ $993288 = 29(34251'14 - 9'05\xi), 34296'14$

$17165'94$ 3'843521 - 258'8\xi

$17236'68$ 3'861964 - 262'6\xi $9\delta$ $993464 = 29(34257'4 - 8'8\xi), 34301$

$17300'80$ 3'878913 - 266'1\xi

Satellite 4'825655 4'830492 $2\delta$ $993158 = 29(34246'8 - 13'2\xi), 34312$

$19897'05$ 4'834202 - 515\xi

$21350'92$ 5'818813 - 898'2\xi

In the above the first is the denominator calculated from 937300 - $2\Delta_\nu$. It is affected with an uncertainty of about 400 on the 937300. The line $n = 17165'94$ has a separation 680'84 with 17846'78, and therefore should have a satellite 25'09 above it. Its denominator difference is 6493 behind and $3\frac{1}{2}\delta = 6492$. The line 17236'68 is associated with 17300. Its denominator is 16949 behind that of 17300 and $9\delta = 16695$—the same within limits. The satellite of 19897 is displaced $4\frac{1}{2}\delta$. There is no evidence of a satellite $2\delta$ behind it, but the difference of $29\Delta_\nu$ is made with this suppositions one. It is seen that a value of $\xi$ about $-5$ makes these the same within limits. The corresponding values of $\Delta_\nu$ are appended in the last column. It may be taken that the discussion has established, that the satellite differences in the lines RaD(2) are $16\delta$ and $10\delta$. This is the only result of which there can be certainty.

The Zn Sub-group.—Using the limits given in Table II. above and the corresponding values of $D_1(2)$, the limits $F_1(\infty) = D(\infty) - D_1(2)$ come as follows:

Zn . . . . . 12988'37, with separations 4'38, 3'74.

Cd . . . . . 13022'83 " 18'23, 11'10.

Hg . . . . . 12753'07 " 34'68, 62'04.

3 E 2
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Paschen allot the following for Zn and Cd, viz., in wave numbers:—

\[
\begin{array}{ccc}
6059.50 & 6065.51 & 17.86 \\
ZnF\,(3) & 6062.55 & CdF\,(3) & 6083.37 & 11.98 \\
6064.98 & 6095.35 \\
CdF\,(4) & 8595.57 \\
\end{array}
\]

The line 8595 must be allotted to F\,(4) because it makes the difference 266 with 8872, so that the two are F lines connected with D\,(2) and the companion to the D\,(1) line 267 above it. This relation has already been discussed under the D series above. For Hg he assigns 5814 to F\,(3), 5843 to F\,(2), 5908.68 to F\,(3), 8316.40 to F\,(4), 8409.85 to F\,(4). Now 5843 is double with \( \delta = 4.8 \), corresponding to a displacement \( \delta \), and may well be F\,(21), F\,(22), whilst 5908 is F\,(3). Again, 8316.40 and 8409.85 are separated by 93.45, which is so close to 34.68 + 62.04 as to indicate that they are F\,(4) and F\,(4) and that F\,(4) has not been observed.

With these allocations \( VF\,(3) = 6939.07 \) and \( VF\,(4) = 4436.66 \) and the denominators for the lines calculated from the first lines (except CdF\,(4)) are

\[
\begin{array}{ccc}
Zn. & Cd. & Hg. \\
3.978529 - 287.1\xi & 3.970387 - 285.3\xi & 3.975605 - 286.5\xi \\
4.960809 - 556.4\xi & 4.971938 - 560.3\xi \\
\end{array}
\]

The differences of the mantissae of the two orders in Cd and Hg are

\[
\begin{align*}
\text{Cd.} & \quad 9578 + 271\xi, \quad 21\delta = 9561; \\
\text{Hg.} & \quad 3667 + 274\xi, \quad 2\delta = 3629;
\end{align*}
\]

in which the probable variations of \( \xi \) are small fractions. In fact, the greatest uncertainties are due to observation errors.

There is not much material to throw light on the origin of the F term here, nor in fact is there evidence that the fundamental lines, or the first lines, of the series are

* On the basis of Ritz's combining theory Paschen gives the following allocations (lines in wavelengths):—

\[
\begin{align*}
3011.17 & = S\,(\infty) - VF\,(3) \\
2642.70 & = S\,(\infty) - VF\,(3) \\
2524.80 & = S\,(\infty) - VF\,(3) \\
2799.76 & = S\,(\infty) - VF\,(4) \\
2478.09 & = S\,(\infty) - VF\,(4) \\
2374.11 & = S\,(\infty) - VF\,(4)
\end{align*}
\]

There can be little doubt about the correctness of this allocation. Using the value of \( S\,(\infty) \) in [II] 40139.55 and the wave numbers 33200.19, 35707.02 of the first lines of each set there results \( VF\,(3) = 6939.36, 4432.53 \), which are practically the same as those found direct.
for \( m = 3 \). Any lines corresponding to \( m = 2 \) would have wave numbers about 500, and to \( m = 1 \) negative wave numbers in the neighbourhood of 16000. Now PASCHEN has noted lines which may be treated as the actual lines in question. They depend on terms \( S' (\infty) - VD (2) \) where \( S' (\infty) \) is the limit of his singlet series, and of course \( VD (2) \) is the \( F (\infty) \) of the above. Using sequences for the \( S' \) series of the form \( \mu = 1 + f \), the limits of the series are

\[
\begin{array}{ccc}
\text{Zn.} & \text{Cd.} & \text{Hg.} \\
29019'96 & 28843'40 & 30114'33
\end{array}
\]

The lines in question are

\[
\begin{array}{ccc}
\lambda = 6238'21 & 6325'40 & 5769'45 \\
n = 16025'87 & 15804'98 & 17327'96
\end{array}
\]

If these wave numbers be added to \( F_2 (\infty) \) in each element, there results 29018'62, 28846'14 and 30115'71, i.e., the value of the \( S' (\infty) \) above. The corresponding lines for \( F_1 (\infty) \) do not seem to exist. There is no \( \textit{a priori} \) reason to take \( F_3 (\infty) \) rather than \( F_1 (\infty) \) for Zn. In Cd, however, the case is settled in favour of \( F_3 \), as the other lines exist, viz., \(-15526'84, -15793'05, -15804'98\), giving the differences 266'21, 11'93 corresponding therefore to the companion series to \( D_{1s} (2) \), to \( D_{1s} (2) \) and \( D_{1s} (2) \), \( D_1 \) not appearing. But in Hg \(-17327'96, -17264'98, -17223'97\), with differences 62'98, 41'01 would seem to assign 17327 to the \( F_3 \) term. Nevertheless to get the limit of PASCHEN’s \( S' \) series it is necessary to take \( F_3 (\infty) \).

If these be regarded as the first lines of the \( F \) series, the denominators are Zn, 1'943072 – 33'5\( \xi \); Cd, 1'949840 – 33'9\( \xi \); Hg, 1'908346. In Hg the line \( n = -17121'30 \) would seem to stand in a normal relation to the \( F_1 \), as it comes into line with the others as is seen below. With this the apparent limit with \( F_1 \) would be 29874'37, giving denominator 1'916040 – 32'0\( \xi \). The question now is, are these denominators related in any way to those for \( m = 3 \). The differences of their mantissae are, using our new Hg line

\[
\begin{array}{ccc}
\text{Zn.} & \text{Cd.} & \text{Hg.} \\
34557'254\xi & 20547'252\xi & 59565'254'5\xi \\
= 10(34557'254\xi) & 2(10273'126\xi) & 2(29782'127'2\xi) \\
& 2\Delta_2 & 2\Delta_2
\end{array}
\]

well within errors, it being also remembered that \( \xi \) can only be a fraction. The value of \( \Delta_2 \) for Hg adopted is the corrected one 29765, from \( \delta = 361'85\mu^2 \). This is a striking connection. It shows that the limits for PASCHEN’s singlet series are either \( VF(1) \) or are formed from \( VF(3) \) by deducting 10\( \Delta_2 \) for Zn, 2\( \Delta_2 \) for Cd, and apparently 2\( \Delta_2 \) for a normal type in Hg which then receives some displacement.
Magnesium.—We are now in a better position to take up the consideration of the place Mg is to occupy in the second group of elements, viz., whether it is allied with the Ca or the Zn sub-groups. In the discussion of MgD (p. 356) there was evidence in favour of either view. If it belongs to the former, then the line \( \lambda = 14877 \) is \( D_1(1) \); if to the latter we have Paschen's allocations of 14877 to F(3) and 10812'9 to F(4). Take first the supposition that Mg is analogous to the earths. In this case \( F(\infty) = 39751'08 - 6719'95 = 33031'15, 6719 \) being the wave number of 14877. If the F series is formed on the type of the Ca set the denominator of the first line will be \( 2'937300 - 2\Delta_2 = 2'936474 \). This gives a line \( n = 20312 \). No line has been observed sufficiently near to this to be identified with it. In the other case \( F(\infty) = D_1(2) = 39751'08 - 26044'99 = 13706'09 \). Paschen's allocations then give denominators 3'962183 - 283'35\( \xi \), 4'958710 - 555'7\( \xi \) with a mantissa-difference = 3473 + 272\( \xi \). With \( \xi = 9 \) this is 3717 or 9\( \Delta_2 \). The value \( \xi = 9 \) will upset the difference in Table II. between D(2) and the supposed D(1) which in this case does not exist. It still leaves the difference between the denominators of D(2) and D(3) = 6\( \Delta_2 \). If Mg is completely analogous with the Zn set the combination lines \( S(\infty) - \text{VF}(3) \), \( S(\infty) - \text{VF}(4) \) should exist. They should be at \( n = 32764'92 (67) \), \( +v_1 \), \( +v_2 \), and 35290'72 (87), \( +v_1 \), \( +v_2 \). Now Eder and Valenta give two spark lines of weak intensity at 3050'75, 3046'80, and Saunders a weak are at 3051. The wave numbers in vacuo are 32769'48 and 32811'95 separated by 42'47, which is clearly \( v_1 = 40'90 \). These are therefore the looked for \( S_1(\infty) - \text{VF}(3) \), and \( S_2(\infty) - \text{VF}(3) \), the third \( S_3(\infty) - \text{VF}(3) \) not having been seen. As to the other set, Saunders has observed a line at 2833 giving \( n = 35288'19 \) which is clearly \( S_1(\infty) - \text{VF}(4) \). The existence of these combination lines seems to settle the question in favour of Mg belonging spectroscopically to the Zn group of metals rather than the alkaline earths. It is possible that as a transition element it belongs to both types. Judging from Paschen's various readings it might well be that \( \lambda = 14877 \) is double so that one might be D(1) and the other F(3).

Group III.—In Al and Tl alone have the ultra-red lines been observed, and here the F lines are found in a similar position to those in the Zn groups, and with them Ritz's combinations \( S(\infty) - \text{VF}(3) \) and \( S(\infty) - \text{VP}(3) \). Using the values of D(\( \infty \)) of [II.] the values of \( F(\infty) = \text{VD}(2) \) are 15837'92 for Al and 13064'21 with separation 81'98 (24) for \( F_2(\infty) \) for Tl. For Aluminium Paschen gives \( n = 8882'19 (80) \) and 11392'8 (3'90) for F(3) and F(4), from which result \( \text{VF}(3) = 6955'73 (80) \) and \( \text{VF}(4) = 4445'12 (3'90) \). The combination

\[
n = 41204'14 (3'39) = S_1(\infty) - \text{VF}(3) \text{ gives } \text{VF}(3) = 6957'32 (3'39).
\]

For Thalium Paschen gives \( n = 6118'19 (75) \), 6200'67 (77) for F(3), F(3) and 8622'47 (37), 8706'78 (1'51) for F(4), F(4). These give separations 82'48 \( \pm 75 \pm 77 \), and 84'31 \( \pm 57 \pm 51 \) instead of 81'98 \( \pm 24 \), but the same within limits. He also gives \( n = 34326'21 (1'79), 42321'40 (2'69) \) for \( S(\infty) - \text{VF}(3) \) and 37022'23 (6'85) for
S₁(∞)–VF(4). The possible errors of the latter, however, are so large that they cannot be used to improve the values found from the direct lines. The limit calculated from D(∞) of [II.] and D₁₁(2) is 2565-59, but there is some uncertainty owing to the abnormality of D(2) as explained above under the discussion of the D series. The lines F₁(3) and F₁(4) give 6946-02 (73) for VF(3) and 4441-74 (37) for VF(4).

In the case of In no ultra-red lines have been observed. In K.R.'s list there appears a doublet λ = 2720.10, 2565.59, which shows a separation 2213.32, the true doublet separation being about 2212.38. Its relative position in the spectrum compared with that of Al and Tl point it out as the Ritz combination S₁(∞)–VP₁(3). K.R. also give a line at λ = 2666.33 or n = 3749377 (2.81), which from its position might be S₁(∞)–VF(3). If so, the value of VF(3) is 6960-99 (2-80) and clearly in line with those of Al and Tl. We shall adopt it provisionally. K.R. mark all these lines as doubtful, but the existence of the doublet separation points to their real existence as In lines. Collecting these give the following:—

<table>
<thead>
<tr>
<th></th>
<th>Al</th>
<th>In</th>
<th>Tl</th>
</tr>
</thead>
<tbody>
<tr>
<td>VF(3)</td>
<td>6955.73 + 80p + ξ</td>
<td>6960.99 + 280p + ξ</td>
<td>6946.02 + 73p + ξ</td>
</tr>
<tr>
<td>VF(4)</td>
<td>4445.12 + 390q + ξ</td>
<td>4441.74 + 37q + ξ</td>
<td></td>
</tr>
<tr>
<td>Denom</td>
<td>4'967208 – 2177q – 558ξ</td>
<td>4'969095 – 207q – 559ξ</td>
<td></td>
</tr>
</tbody>
</table>

It is seen that Al and In may be the same within limits. In Tl the uncertainty in D(2) referred to above is such as to raise the limit—and by 1919—if the explanation there given is correct. A rise of 10 would make the denominator for m = 3 the same as for Al and In.

One of the most striking results of this discussion of the F series is the distinct divergence in type between the spectra of the high melting-point elements and those of the low melting point, and at the same time the close resemblance between the individual elements in each division. So close indeed is the resemblance between all the low melting-point elements of Groups II. and III. that the differences between them appear to be almost wholly due to the difference of the limits, or the value of VD(2) and the values of VF(3) are almost the same. To see how closely they agree the denominators to four places of decimals are collected here, and for comparison those of the alkalies.

Mg 3'9621, Zn 3'9785, Cd 3'9703, Eu ?, Hg 3'9756,  
Al 3'9708, Ga ?, In 3'9693, ?, Tl 3'9736,  
Na 3'9979, K 3'9928, Rb 3'9878, Ca 3'9773, ?.

It is seen how closely the elements in each group agree in spite of a very wide difference in atomic weight, and moreover the mantissae in all are very close to unity.
It was shown in [I.] that the F series of the alkalies could be represented by a series of the form \( m + 1 - \alpha (1 - 1/m) \). The same is the case with the Al and Zn groups. As \( \alpha \) is so small and varies so little it can scarcely be a function of the atomic weight. The atomic volumes of the elements are much more even and it may be a function of them as is the case with the \( p \)-sequence. In fact, in the case of the alkalies, the \( \alpha \) are not far from being proportional to \( v, 2v, 2v, 3v \), for the four elements considered, but the data are so inexact and uncertain that it seems not worth while to undertake an exhaustive numerical discussion.

We do not know that the chief lines of these sets are those depending on \( m = 3 \). If lines exist depending on \( m = 2 \) they would all be in extremest red, in fact with wave-lengths comparable with those of electro-magnetic waves capable of being experimentally excited, and it is possible that \( VF(2) \) might be the same for all low melting-point elements and as for He (see [I.]). For \( m = 1 \) we should expect the lines of negative wave number in regions which have been observed and in which no such lines have been seen.

The F series in the high melting-point elements, on the contrary, are profoundly influenced by the atomic weight term. Either the lines observed belong to a different type from those of the others, or they are based on a normal type of aggregation which is modified by collateral and other types of displacement due to the splitting up of the typical aggregations, or to a more complex system of new aggregations.

The notation \( F \) for these series was adopted in [I.] under the idea that the sequence for it was of a more fundamental nature than the others, and that impression is rather strengthened by the present discussion. It has been seen, for instance, how the limits of Paschen's singlet \( S' \) series in the Zn group depend on it. It would be interesting to know whether similar series appear in the alkalies and aluminium group.

*The Value of the Own.*

The further knowledge now gained as to ways in which the own—or the \( \Delta \)—enters in the constitution of spectra, enables a much closer approximation to its actual value to be obtained than was possible from the consideration of the doublet and triplet separations themselves. Amongst the principal aids are (1) the separations themselves, (2) the dependence of the first D denominator on a multiple of \( \Delta \), (3) in the triplet elements, on the collateral relations between associated lines, (4) the satellite separations in the D series, (5) the order separations in the D series which show no satellites, (6) collaterals depending on \( \Delta \). Of these, Nos. (1) and (4) have the great advantage that the values depend only slightly on the exactness of the limit (the value of \( \xi \)), but (4) has the disadvantage that only small multiples of \( \delta \) are involved, and (1) only \( \Delta \) itself. There are also various uncertainties which show themselves when a high order of accuracy is desired—chiefly in the elements of the 3rd group. No. (2) has the great advantage of giving considerable multiples of \( \Delta \),
but they depend to a larger extent than (1) and (4) on the exactness of the limit. This inexactness is, however, in general more than compensated by the largeness of the quantities dealt with. Collateral relations also are capable of giving very exact values, but always subject to uncertainty as to the actuality of the relations indicated by the numerical coincidences. This is less apparent in the F series of the high melting-point elements in Group III., where the relations are largely established by analogy between the different elements involved. No. (5) is affected by the exactness of the limit, and is only useful when the separation is taken between the first two orders and it is a considerable multiple of \(\Delta\), as, for instance, 117\(\Delta\) in Al.

For the special purpose of obtaining as exact a value as possible of the ratio \(\delta/w^2\) it will be better to exclude from consideration Na, Ga, He, Sc, O, S, and Se. Na is excluded on account of the uncertainty as to whether F. and P.'s interferometer measures of the P(2) lines are to be taken as giving the value of \(\nu\) for the S and D series, in which a somewhat larger value is indicated by observers using ordinary methods. Ga is omitted on account of its poor spectroscopic data. He because its \(\nu\), although very accurately determined, is so small that slight errors are very large proportionate ones. O because \(\nu\) is small and the observations not so exact, and Sc, S, and Se because their spectra have not been sufficiently discussed. There remain 17 elements for consideration. In the following the case of each element is considered first, with estimates of its possible error. Then using these possible errors as probable errors, the most probable value of \(\delta/w^2\) is deduced by least squares. The ratio \(\delta/w^2\) is denoted by \(q\).

K. The observations determining \(\nu\) are very bad. The \(\nu\) adopted gives \(\Delta = 2939.\)

\[ D_{12}(2) = 261\Delta \quad \text{and gives} \quad \Delta = 2932.27 \pm 130 - 364\xi, \quad W = 39.097 \pm 0.03, \quad \text{and} \quad \xi \quad \text{is about} \pm 1. \]  

The value of \(q\) from this is 361'944 \pm 11. This is adopted with probable error = '1.

Rb. The only source is from \(\nu\), since there is no light from the \(D_{12}(2)\) as the satellites are doubtful. The value in Table I. is 361'40 \pm 3.6. Probable error taken = '66.

Cs. Table I. gives 361'74 \pm 3.3. \(D_{11}(2)\) is so close to 17\(\Delta\) that it seems justifiable to adopt it. The observations seem to show that \(\xi\) should be 2 \pm 1 in Table II. The denominator is subject also to an observation error of 228. With \(W = 132.823 \pm 0.07\) the consequent value of \(q\) is 362.24 \pm 30, but this value of \(\xi\) makes the former value much less. The relation may be a coincidence, as it ought to be near it, and it will therefore be safer to take the first adopted value, 361'74 \pm 3.3.

Cu and Ag as in Table I., viz., 361'84 \pm 8, 361'81 \pm 2.

Mg. With \(\xi = 2, d\nu = .06, W = 24.362 \pm 0.02, \nu_1 + \nu_2\) gives \(q = 362.36 \pm .66.\)

The actual first D line has been seen to be uncertain, and in any case \(\Delta\) is so small that the actual multiple cannot be obtained. There is an order difference of 6\(\Delta\) between \(m = 2\) and 3, but the observation errors, and those due to \(\xi\), give \(\Delta\) with far less exactness than from \(\nu_1 + \nu_2\). Value therefore adopted, 362'36 \pm 66.
Ca. From $v_1 + v_2$, $\xi = 6$, $dv = .1$, $W = 40'124 \pm .005$, $q = 361'.56 \pm .60$.
The denominator of $D_{13}(2)$ is $691\Delta_2$ gives $q = 361'.84 \pm .1$.
From the discussion of CaF, $q = 361'.77 \pm .2$.
Mantissa difference of CaD (2) and CaD (3) is $99\Delta_2 = 133542 + 100\xi \pm 28$,
$q = 361'.870 \pm .176$, the great uncertainty being due to $\xi$.
The most reliable appears to be that from $D_{13}(2)$, and is included in the others.
Value adopted, 361'.84 \pm .1.
Sr. From $v_1 + v_2$, $\xi = 10$, $dv = .2$, $W = 87'.66 \pm .03$, $q = 361'.63 \pm .56$.
The denominator of $D_{13}(2)$ does not appear as a multiple of $\Delta_2$, whereas that of
$D_{12} = 178\Delta_2$. If this is a real relation, $q = 361'.735 \pm .33$.
From the F collaterals and the denominator of F (2), $q$ cannot be far from 361'.77 \pm .2.
Adopted value, 361'.77 \pm .2.
Ba. From $v_1 + v_2$, with $\xi = -32$, as modified in Table II., and \pm 5 allowed, $dv = .2$
and $W = 137'.43 \pm .06$, $q = 362'.07 \pm .53$.
From the $D_{13}(2)$ collateral is $69\Delta_2$, \hspace{1cm} $q = 361'.968 \pm .36$
From $D_{13}(2)$, as found from the F series, \hspace{1cm} $q = 361'.856 \pm .36$
From the F discussion, \hspace{1cm} $q = 361'.971 \pm .39$.
The most reliable is probably the mean of those depending on $D_{13}$. Adopted
value, 361'.913 \pm .4.
Ra. From $v_1 + v_2$, $\xi = 1$, $dv = .2$, $W = 226'.4 \pm .02$, $q = 361'.846 \pm .66$.
From the F discussion, $q = 361'.94 \pm .11$, but as there is some uncertainty in the
F theory, the limits of error should be greater. Adopted value, 361'.94 \pm .33.*
Zn. From $v_1 + v_2$, with $\xi = 3$, $dv = 0$, $W = 64'.40 \pm .03$, $q = 362'.238 \pm .36$.
From $D_{13}(2)$, $q = 361'.682 \pm .47$, and from the F values lying between, 362'.15 and
361'.87. Value adopted, 362'.01 \pm .25.
Cd. From $v_1 + v_2$, with $\xi = 2$, $dv = .1$, $W = 112'.3 \pm .1$, $q = 362'.36 \pm .66$.
In the $D_{13}(2)$ theory Cd appears to occupy a similar position to that of Sr in the
other sub-group, in that the multiple of $\Delta$ is carried back to $D_{11}$ or $D_{12}$. The most
accurate is that from $v_1 + v_2$. Adopted value, 362'.36 \pm .66.
Eu. From $v_1 + v_2$, with $\xi = 10$, $dv = 4$, $W = 151'.93 \pm .03$, $q = 361'.94 \pm .8$.
From $D_{13}(2)$, $q = 361'.44 \pm 1$. Adopted value, 361'.94 \pm .8.
Hg. From $v_1 + v_2$. There is some uncertainty as to the ratio of $\Delta_2 : \Delta_1$. $\Delta_2 = 41\Delta_2$
best agrees with the transference value from $\Delta_1$ to $\Delta_2$ discussed at the commencement,
and it gives a value of $\Delta_2 = 297.25'.65$ which is in close agreement with the value
found in the F discussion, viz., 297.82 -- 127'.2$\xi$.

* HÖNIGSCHMID ("Sitz. d. k. Akad. Wiss. Wien," November, 1912) has recently made a careful
determination of $W$, and gives 225'.97 in place of 226'.4. This would make the oum 363'.11$\xi^2$--a value
quite inadmissible if the spectroscopic data are reliable. Although they are not good, they can hardly be
so uncertain as this value of $\eta$ would indicate. For what it may be worth, the spectroscopic data would
seem, therefore, to weigh against the acceptance of the new atomic weight.
The former gives with \( \xi = 4, d_\nu = 0, W = 200'3 \pm 3, q = 361'423 \pm 1'61 \), and the latter 362'09 with a large uncertainty owing to 127'2\( \xi \).

\( D_{13}(2) \) gives \( q = 361'50 \pm 1'28 \). The large possible error is due to the uncertainty in the atomic weight of Hg. Adopted value, 361'50 \pm 1'33.

Al. In Al also there is a large possible variation due to the uncertain atomic weight. \( W = 27'10 \pm 0'5 \). From \( \nu, q = 361'88 \pm 1'5 \). There is an order difference 117\( \Delta \) between \( m \) and 3 for the D series (see Table II). This gives \( q = 361'871 \pm 1'30 \). The denominator of \( D(2) \) gives \( q = 361'777 \pm 1'92 \), or if there is a satellite \( D_{13}(2) \) gives 361'717 \pm 1'92. Adopted value, 361'871 \pm 1'33.

In. From \( \nu \), with \( \xi = 1, d_\nu = '25, W = 114'8 \pm 5, q = 361'947 \pm 3'31. \)

From \( D_{13}(2) \), if 22\( \Delta \) = 16\( \delta \), \( q = 361'871 \), but the theory is uncertain. Adopted value, 361'947 \pm 3'33.

Tl. From \( \nu, q = 362'00 \pm 20. \) From \( D_{13}(2), q = 361'913 \pm 6, \) but with somewhat doubtful theory. Value adopted, 362'00 \pm 20.

These values for the 17 elements weighted according to the possible errors now give \( q = 361'890 \). This is the same as our first approximate value, but its probable error is much less. If the determination of the value depended only on questions of errors of calculation and of observation in spectral and atomic weight data, the above number would probably be extremely close to the actual one. It must be remembered, however, that our theory of the constitution is not yet complete. For instance, in [II.] it was seen that the supposition that N was not constant for the \( p \)-sequence, but that the value for the first line was slightly larger explained the introduction of a term in the denominator. A similar explanation might explain the fact that the value of \( q \) appears to deviate from the mean by about the same amount in each group of elements, and if it were justified, the value of \( q \) calculated as above would receive a slight modification. I believe it will be found ultimately that the true value will lie within the limits given by 361'890 \pm 0'05 or 90'4725 \pm 0'125.

If the existence of the ion as a definite proportion of the (atomic weight)? be considered as established, the best and most direct method of determining the value of the factor \( q \) would be from the discussion of an element in which the spectroscopic data are good and in which the atomic weight has been determined with great accuracy. For this purpose we naturally turn to silver. Regarded as the ultimate standard of atomic weight determinations, no error in the atomic weight enters—the value of \( q \) is determined in terms of \( W = 107'88 \). Moreover its separation is large, so that any error of measurement is a small fraction of its total value, and in addition the actual error is extremely small. It is therefore tantalising to find that the lines, \( D(2) \) excepted, are not susceptible of such exact measures as in many others, that the typical series are not well developed, and that in fact there may be a doubt whether the lines generally accepted as the P, S, D series follow laws altogether analogous to those in other groups. In Kayser and Runge's measures four lines are assigned to \( D_{11} (2-5) \) and three to \( D_2 (2-4) \), the possible errors for \( D_{11} \) being much less
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than for \( D_{11} \). Using \( D_{21} \) and \( D_{12} \) and calculating the formula constants from the first three, there results for the \( D_{12} \) series

\[
n = 30644.60 - N / \left( m + 994354 - \frac{934384}{m} \right),
\]

with the large possible variation in \( D(\infty) \) of \( \xi = 12'.23 \). Using this value of the limit and calculating the formula for \( D_{11} \), the line \( D_{11} (5) \) is reproduced with \( O = C = '22, O \) being '5. If, however, the S lines be used with \( \mu = f \) the limit comes to \( S(\infty) = 30614.60 \pm 3'.60 \) and the fourth line is reproduced within limits. \( S(\infty) \) and \( D(\infty) \) cannot be the same within error limits. If \( \mu = 1 + f, S(\infty) = 30633'47 \pm 3 \), and the limits can be the same, viz., with \( \xi = -9.73 \pm 2'.50 \) on \( D(\infty) \), but \( \mu \) cannot be \( 1 + f \) if the doublet usually assigned to \( P \) belongs really to a P series. I had intended to supplement the direct determination of \( \Delta \) from \( \nu \) by a discussion of collaterals, of which both Ag and Cu afford a large number. The doubt however about the form \( \mu = 1 + f \), and the presence of the numerous collaterals, gives a suspicion that the series are related to the F series with limits based on the typical S(\( \infty \)), that they are analogous to the F terms in the high melting-point elements of Group II., and that the doublet usually allocated to the P series is really analogous to the F triplets with negative wave numbers found in Ca and Sr. That discussion is therefore held back for the present.* But certain points not open to doubt and forming a portion of the work of the accurate determination command a place here.

The \( D(2) \) lines are sharp. \( F \) and \( P \) have measured the lines \( D_{11} (2) \) and \( D_{21} (2) \) with their interferometer. Their measures give a separation of 900'3419 \( \pm \) 0.070.

In order to get full advantage of their accuracy, and to avoid the uncertainty due to the last significant figure it is necessary to use logarithmic tables with more than 7 figures. This has been done on the supposition that \( F \) and \( P \)'s errors are not larger than 0.001 A.U., i.e., unity in the seventh significant figure. The old measures are sufficient to show with certainty that the satellite difference is \( 23^3 \), and the old approximation to \( \Delta \) will give \( 23^3 \), with an inappreciable error, whence \( D_{12} \) can be found. Taking \( D(\infty) = 30644.6000 + \xi \) the mantissa of \( D_{11} (2) = 5465'671 \) is 979596'44 \( - 120'59 \xi \) and \( 23^3 = 2421'18 \) \( - 15 \xi \). Hence the mantissa of \( D_{12} \) is the difference or 977175'26 \( - 120'44 \xi \). \( VD_{12} \) calculated from this is 12373'6789 \( + 1'0045 \xi \), giving with \( VD_{21} \) the value \( \nu = 920'4431 + 0045 \xi \) in which the correction for \( \xi \) is only effective if \( \xi > 10 \). The value of \( \Delta \) calculated from this is 27786'80 \( + 1'473 \xi \pm 20 \), in which the uncertainty of \( 20 \) is due to the uncertainty \( 001 \) in \( \lambda \). This value is 4 less than that of Table I. obtained by supposing K.R.'s values for \( D_{12} \) and \( D_{21} \) had no errors. It gives \( \delta = (361'754 \pm 0026 - 0152 \xi) \omega^2 \). With the mean value \( \xi = -9.75 \) suggested above this becomes \( \delta = (361'902 \pm 0026) \omega^2 \).

The foregoing is interesting also because it shows how the application of the laws developed in the present discussion can help towards more accurate determinations of

* The value obtained for \( q \) from the collaterals was 361'708 \( \pm 0026 - 0169 \xi \), which with \( \xi = -10 \), as indicated in the text, gives a value surprisingly close to that deduced from all the elements combined.
quantities involved. For instance, it has enabled us to obtain a value of $\nu$ correct
to about a unit in the sixth significant figure. In the case of Au, the knowledge
is still more fragmentary than in Ag and the value of $\Delta$ has not been determined.
By the application of our new laws, however, it is possible to obtain a good deal
of information based on evidence of weight, and it will be interesting to consider
it shortly here. Although the spectrum of Au shows many analogies with those of
Cu and Ag, no lines have been assigned to the S or D series. There is a strong
doublet in the ultra-violet 2676'05, 2428'06 ($\nu = 3815'28$) analogous to the lines
allocated to the P series in Cu and Ag. There is only one other doublet in K.R.'s
list with the same separation, viz., 6278'37, 5064'75 ($\nu = 3815'54$). This is clearly
analogous to the doublets 5782'30, 5700'39 in Cu and 5545'86, 5276'4 in Ag, which
have the respective doublet separations but which do not belong to the S or D series.
E and H however give an arc line at 4811'81, which gives a separation of 3815'57
with K.R.'s line at 4065'22. This has the appearance of a D set, $D_n$ being at
4792'79 with a satellite separation of 82'47. But if so it is quite out of step with
the progression of the $D_{12}$ lines for Cu and Ag, viz., 5220 (Cu), 5471 (Ag). But
5837'64 gives with the above 4792'79 a separation 3733'43 the same as that between
4792 and 4065, and they are in step with Cu and Ag as $D_n$ (2) and $D_n$ (2), the
fainter satellite $D_{12}$ being unobserved. This would seem the more probable allocation.
In any case, the curious doubling of a D type would have to be explained. There is,
however, here not sufficient data to determine the limits, or the other formulæ
constants or the value of $\Delta$. But it is possible to arrive at a probable estimate by
the following considerations. The limit $D (\infty)$ will probably be in step with those of
Cu and Ag, viz., 31515, 30644, i.e., will be in the neighbourhood of 30000. Now $\Delta$
must give $\nu = 3815'54$ and must itself be a multiple of the unit, in fact if it is similar
to Cu and Ag of $\delta$. Now $W = 197'20$ with an uncertainty of a few units in the
second decimal place. The ratio $q = 361'80^* + y$, where $y$ is probably not greater
than 1 in the first decimal place and it will be regarded as a correction on the '8.
From this it follows that $\delta = 1406'930 \pm 0'07'38y$. The uncertainty '097 due to
the uncertainty in $W$ produces so small an effect that it may be neglected here.
Now $\Delta$ must be a multiple of $\delta$ and must give with the proper value of $D (\infty)$,$
\nu = 3815'54 + 30s$, '30 being the maximum error of $\nu$ and therefore $s$ between $\pm 1$.
This condition gives the following sets of possible limits in the neighbourhood of
$D (\infty) = 30,000$:—

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30819'15 + 1'57s - 6y</td>
<td>76$\delta$</td>
</tr>
<tr>
<td>30542</td>
<td>77$\delta$</td>
</tr>
<tr>
<td>30266</td>
<td>78$\delta$</td>
</tr>
<tr>
<td>29994</td>
<td>79$\delta$</td>
</tr>
<tr>
<td>29724</td>
<td>80$\delta$</td>
</tr>
<tr>
<td>29465'18 + 1'45s - 5'7y</td>
<td>81$\delta$</td>
</tr>
</tbody>
</table>

* The actual calculations were made before the last most probable 361'890 was obtained, but nothing
is to be gained by recalculating to it.
If now the lines 5837.64, 4792.79 be taken as $D_{11}(2)$, $D_{31}(2)$, their wave numbers are 17125.54 ('14), 20858.97 ('22), giving for the wave number of $D_{12}(2)$, or $D_{31}-n$, the value 17043.43 ('22$-30s$). In this '22$-30s$ is the possible observation error in $D_{31}(2)$. The satellite separation of 82.11 must therefore be caused by a denominator difference which is also a multiple of $\delta$. If this be tested, it is found at once that only the first and last of the above set can satisfy this condition, 30819 taking 24$\delta_1$ or 6$\delta$, and 29465 taking 28$\delta_1$ or 7$\delta$. The corresponding values for Cu and Ag are both 23$\delta_1$. The differences between the values calculated from the lines and from the multiples of $\delta_1$ are ($p$ between $\pm 1$ giving the observation error in $D_{11}(2)$).

\[38.4+28s+3.0y+14p-22q \text{ with } 24\delta_1 = 8441\]
\[20+34.3s+4y+17p-26q \text{ } ', \text{ } 28\delta_1 = 9869.\]

It is clear that either can easily be made to vanish well within possible errors, more especially the latter. The limit 30819 is higher than that of Ag instead of lower as might be inferred from the fact that the limit of Ag is lower than that of Cu. The limit 29465 is 1179 below that of Ag, which is itself 931 below that of Cu. This seems a probable order of magnitude, especially when it is remembered that there is a gap in the Periodic Table between Ag and Au. But there is further evidence in favour of the latter. If the lines 6278.37, 5064.75 are collaterals of D (2) as the corresponding lines in Cu and Ag appear to be, 6278.37 should be $D_{11}(2)(x\delta_1)$. With the limit 30819 this cannot possibly be the case. The own $\delta_1 = 351$ is so large that there can be no doubt. If however the limit is 29465, the line is $D_{11}(2)(\Delta + 15\frac{1}{2}\delta)$. Further, with neither limit is the mantissa of D (2) a multiple of $\delta$, and as this is also not the case with Ag or Cu, it may be regarded that in this group either these lines are not of the D type, or possibly like the high melting-point elements of Group II. the first lines correspond to $m = 1$ and not $m = 2$. The actual values of the denominators as found are so close to the same value for all three elements as to suggest the existence of a group constant. If the limit 29465 is used the denominators are as given below, and as is seen they differ from such a constant by very small multiples of $\delta$.

<table>
<thead>
<tr>
<th></th>
<th>Cu.</th>
<th>Ag.</th>
<th>Au.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>978276 (21)</td>
<td>977162 (19)</td>
<td>971409 (26)</td>
</tr>
<tr>
<td>$146 = \delta$</td>
<td></td>
<td>$1263 = 3\delta$</td>
<td>$7034 = 5\delta$</td>
</tr>
<tr>
<td>$978422 (21)$</td>
<td>$978425 (19)$</td>
<td>$978443 (26)$</td>
<td></td>
</tr>
</tbody>
</table>
As a final result the evidence would seem conclusive that $D(\infty)$ for Au is $29465'18 \pm 7$, that $\Delta = 81\delta = 113961 \pm 31'5y$, and that the satellite separation is produced by $28\delta$.

Summary.

It must be confessed that much of the foregoing discussion is of a problematical nature, and that, in fact, some of the suggestions offered are incompatible with one another. This is no objection in a preliminary search for general principles, as the raising of questions is only next in importance to answering them. Nevertheless, some results appear to be well established and others to have considerable evidence in their favour. Amongst the first are—

1. The dependence of the spectrum of an element on its own, a quantity proportional to the square of its atomic weight and which probably does not differ from $\delta_i = 90'4725w^2$ by more than '013$w$ where $w$ denotes one-hundredth of the atomic weight;

2. The direct dependence of the ordinary doublet and triplet separations on multiples of the own;

3. A similar dependence of the satellite separations in the Diffuse—or the 1st associated—series on multiples of the same quantity;

4. The existence of collateral displacement, whereby new lines are formed by the addition or subtraction of multiples of the own. Until, however, the laws which govern the formation of collaterals are more fully known, it is not safe to assume that any displacement indicated by mere numerical coincidence corresponds to the physical change such collateral indicates. Nevertheless, many cases of clear displacement of this kind, involving considerable multiples of $\Delta$, especially in the F series, are given which serve to give more accurate values of the own.

The conditions which govern the various multiples of the own which enter in the various separations have not been determined. It is probable, however, that the multiple for the doublet, or first two of a triplet, in the two sub-groups of the $n^{th}$ group of elements contain $2n+1$ and $2n+2$ respectively as factors.

It is probable that the mantissa of the normal first line of the Diffuse series, the last satellite, when such exists, being considered as the normal, is a multiple of $\Delta$, and it is possible that its magnitude has some general relation of approximation to that of the corresponding F series, which again may depend directly on a group constant.

It is possible that the wave numbers of the lines in the Diffuse and F series may not depend directly on a mathematical function of the order $m$ of the line, and it is probable that this is the case when there are no satellites, the differences now being multiples of the $\Delta$ themselves.

In the discussion of the material it has been attempted to keep the mind as free as
possible from any preconceived theories as to the origin of the vibrations which give the lines. The aim has been to discover relations, which it must be the object of theories to explain. Nevertheless, the way in which multiples of a quantity depending directly on the element enter, and indeed multiples of these multiples, irresistibly suggests that each line is due to a special configuration built up of aggregates of the same kind. Thus, in the Zn group appear multiples of $6\delta_1$, in Mg, of $5\delta_1$, &c. These smaller aggregates peculiar to a group then appear to enter like radicals into more complex aggregates, e.g., in Zn $\Delta_1 = 31\delta_6$. $\Delta_2 = 15\delta_6$, and again, multiples of $\Delta_2$ occur in collaterals. In cases, a certain aggregate, normally to be expected, appears to be affected with instability, a certain number of ouns are expelled or added and we get a stable collateral. In the case of rich spectra and of spark spectra, a very large proportion of the lines appear to be collaterally connected. It suggests systems in which a greater freedom of aggregation is permissible. But there is another way in which the matter may be looked at. The actual multiples may be determined by the number of electrons taking part in the vibrations, and the quantity enters into the formula as the product of this number by a fundamental quantity of the atom. But it is difficult to see how this quantity should depend on the square of the mass. It would almost look as if the gravitational pressure of two atoms always at the same distance produced some change in the configuration of the surrounding aether proportional to the pressure, and that the vibrations were conditioned by this change and by definite numbers of electrons. In any case, the existence of the oun, and the extent in which its influence is shown in a spectrum, point to the conclusion that the positive atom plays an essential part in at least those vibrations emitted which are slow enough for us to observe.

**APPENDIX I.**

*The Value of $\Delta$ in Scandium.*

The value of $\Delta$ as a multiple of $\delta$ in Scandium is of importance in connection with the evidence as to the curious relation, that the $\Delta$'s of the first elements of the two sub-groups in the $n$th group are multiples of $(2n+1)\delta$, and $(2n+2)\delta$. The lines in the visible part have been measured by Fowler,* and lines both in visible and ultraviolet by EXNER and HASHIJK.†

I do not altogether feel full confidence in the allocation suggested below, but it gives related series, even if not the typical ones, and so will serve to determine $\Delta$. The doublet separation is $320 \pm$ a small fraction. There are over 34 doubles with this separation—two, 3613.96, 3572.71 and 3576.52, 3335.88 containing some of the strongest lines in the spectrum. The lines suggested for the S series appear to show

* 'Phil. Trans.,' 202, p. 66.
† 'Spektren der Elemente: Bogenspektren.'
saturates in the first two sets. Further, it appears that the P series take the s-sequence and the S series the p-sequence as is the rule outside the alkalis.

The P Lines.
(Figures in brackets give intensities.)

<table>
<thead>
<tr>
<th>m</th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( n_p )</th>
<th>( n_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>(5) 5717.51</td>
<td>(0) 5721.20</td>
<td>17485.35</td>
<td>1127</td>
</tr>
<tr>
<td>5</td>
<td>(5) 5258.49</td>
<td></td>
<td>19011.70</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(2) 5021.67</td>
<td></td>
<td>19908.26</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(1) 4880.90</td>
<td></td>
<td>20482.40</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>(2) 4791.69</td>
<td></td>
<td>20863.72</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>(2) 4728.95</td>
<td></td>
<td>21140.54</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>(0) 4682.16</td>
<td></td>
<td>21351.78</td>
<td></td>
</tr>
</tbody>
</table>

S Series.

1. (6) 6413.54 (3) 6284.66 15587.78 319.65 15907.43 84.35

2. (15) 3646.46 (16) 3603.1 27423.71 322.39 27746.10 25.81

3. (50) 3642.93 27449.52

Parallel S Series.

1. (3) 5146.43 (0) 5323.94 18457.31 320.65 18777.96 82.60

2. (6) 5392.30 18539.91 30537.33

The S Series.—For a reason to be seen shortly, it is necessary to regard \( S_{11}(1), S_{11}(2), S_{11}(3) \) as the typical series, \( S_{1}(3) \) as not displaced, but the doublets corresponding to 1 and 2 are displaced collaterally to \( S_{1i} \). In other words, we have to do with bodily displacements of the first two doublets and not true satellites. The mean of the doublets gives \( \nu = 320.80 \), and the formula calculated from the three lines is

\[
37949.90 - N \sqrt{\left( m + 1.244902 - \frac{0.26104}{m} \right)^2}.
\]
With the value of \( \nu \) above the value of \( \Delta \) calculates with this limit to \( 7140 = 203\delta \), with \( \delta = 70.34 \). The denominator differences of \( S_1 \) and \( S_{12} \) for \( m = 1 \) and 2 are respectively 4188 and 3964, *i.e.*, close to \( 59\delta = 4150 \) and \( 57\delta = 4010 \), with errors in \( d\lambda \) of '31 and 0, divided between the two sets of lines. It is possible they may be the same within limits of error (59\( \delta \)), when the value of \( d\lambda \) in the second would be '16.

The formula for \( P \), calculated from the first three lines, is

\[
n = 22281.97 - N\left(\frac{m + 828585 - 187372}{m}\right)^2.
\]

This formula gives the following values of O-C for the lines for \( m = 7, 8, 9, 10 \), viz., '55, '85, '26, '186. If the denominator be treated in the same way as \( \text{Al}, \) *i.e.*, deducting the group constant '043761, it may be put into the form

\[
m - 0.043761 + 872346\left(1 - \frac{21480}{m}\right),
\]

which reproduces the '215 constant. Also \( \text{VS}(1) = 22277.80 = P(\infty) \) within error limits.

If \( S_1(3) \) had been taken as \( S_{12}(3) \) this would not have been the case. But \( \text{VP}(1) \) extrapolates to 40717'20 with a denominator '641213, whilst \( S(\infty) = 39749.90 \) with a denominator '699998, and they cannot be the same even approximately if the typical formula holds. The extrapolated value of \( \text{VP}(1) \) requires \( \Delta = 6428 \) to give \( \nu = 320.8 \) or 91\( \delta \), again giving a multiple of 7\( \delta \), and at the same time more in line with other elements as being a multiple of \( \delta \) itself, and a multiple more in step with them. It points to the likelihood that the series chosen for \( S \) is a parallel series to the true \( S \), *i.e.*, the \( \text{VS}(m) \) is correct. If so, using the values of \( \text{VS}(1, 2) \) with \( S(\infty) = \text{the extrapolated limit 40717}, \) we should expect doublets with the first lines at 18440 and 30217. There is a doublet at 18457'31, no line observed at 30217, but the doublet companion expected at 30537 is found at 30537'33. Also 18457 appears as a satellite to a stronger line 18539'91 in a corresponding position 82'60 a-head. The second set of lines above is, therefore, probably the true \( S \) series, the first being a parallel one. With limit 40717'20 and \( \nu = 319.45 \), the mean of all the doublets, the value for \( \Delta \) is 6404 = 91\( \delta \) with \( \delta = 361.89 \). The value of 361'89 is subject to considerable uncertainty owing to uncertainties in the limit value and the atomic weight, and its agreement with the final estimate for the ion is a mere coincidence. The spectrum of \( \text{Sc} \) is a most interesting one, but its discussion must be postponed. The object of touching upon it here is to obtain some indication of the nature of its ion as \( \text{Sc} \) occupies the first place in its sub-group. It would clearly appear that the multiple of the ion in \( \Delta \) contains 7 as a factor, *viz.*, \( 13 \times 7\delta = 52 \times 7\delta \).

The separation of \( P_1(4) \) and \( P_2(4) \) is 11.26, corresponding to a denominator difference of 5607 = 804. This is in fair agreement with the case of other \( P \) series in which the differences for orders below the first are about '8\( \Delta \).

* [II., p. 46.]
APPENDIX II.

The D Series.

<table>
<thead>
<tr>
<th>Na</th>
<th>K.</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>S.</td>
<td>(2) 8196'1</td>
<td>8184'3</td>
</tr>
<tr>
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DR. W. M. HICKS: A CRITICAL STUDY OF SPECTRAL SERIES.

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See Part II., p. 71.

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DR. W. M. HICKS: A CRITICAL STUDY OF SPECTRAL SERIES.

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Analogous Sets in Ca and Sr.

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DR. W. M. HICKS: A CRITICAL STUDY OF SPECTRAL SERIES.

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| "    | 4402.75 
(1) | 4350.49 | 4333.04* |     |
| "    | 4224.11 
(2) | 4179.57 |     | 4166.24 |

(1) \(F_2(5) (\Delta_2)\).
(2) \(F_1(6) (9\Delta_2)\).
* \(D_{22}(4)\) and \(F_2(5)\) not resolved.

<table>
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Na. The first seven doublets were allocated by Rydberg, using the measurements by Angstrom, Thalen, and Liveing and Dewar. They were also so given by K. and R. The remainder were given by Zickendraht. The actual measures given in the list are by the observers indicated by the letters.

K. Both Rydberg and Kayser and Runge interchanged the S and D series, allocating those in the list to the S series. This was first corrected by Ritz.* The mistake is repeated in Kayser's 'Spectroscopie,' Bd. V. The line 6938.8 covers both KD2 (3) and KS, (2). The pair at 4871 were first observed by L.D., but the more recent measurements by S. are inserted.

Rb. Rl. refers to Randall, Re. to Ramage. The lines 3-6 were allocated by Rydberg, and by K.R. The first doublet was observed and allocated by Randall. The second doublet has raised the question of whether Rb possesses satellites. The line 7757 has only been observed by Saunders, who allocates it to RbDn (3), with 7759.5 as

* 'Ann. d Phys.', 12, 1903, p. 444.
satellite. But Eder and Valenta observe 7759.5 and not 7757, whereas the former if a satellite should be fainter than the latter. The question is discussed in [L.] and also in the present communication, and the weight of evidence would appear to be against the existence of satellites in Rubidium. I have entered the third line as 6298.8 instead of K.R.'s value of 6298.7, because the former value agrees with independent observations by E.V., S., and E.H., and gives a better value for \( \nu \).

Cs. The first were observed and allocated by Paschen, the second first by Lehmann, but the measures are those of Paschen. Saunders was the first to draw attention to the satellites.

The last line, 5118, was observed by Hartley, and is clearly the \( D_2(12) \) line. The corresponding \( D_1(12) \) would be in the neighbourhood of 5256.96, or \( D_2(9) \).

Mg, Ca. The lines were all assigned by Rydberg. He also wrongly assigned two lines about 12000 to MgD(1). The measures are by K.R., except the ultra-red in Ca due to Paschen (Ann. d. Phys., 29, p. 655).

Sr. Assigned and observed by K.R., except ultra-red due to Paschen.


Zn. The lines down to \( D_1(6) \) with the exception of the satellites were assigned by Rydberg. The measures, as well as allocations of the others are by K.R.

Cd. Lines to \( D_2(4) \), satellites excepted, assigned by Rydberg. The remainder allocated and all the measures by K.R.

Eu. The Eu spectrum gives evidence of much collateral disturbance, and the unobserved D lines are possibly displaced in this way.

Hg. The \( D_1 \) lines to \( D_{11}(4) \) assigned by Rydberg, to \( D_3(5) \) satellite by K.R. \( D_1(6) \) to \( D_1(16) \) by Milner (Phil. Mag., 6, 20, p. 636).

Al. Rydberg gives the first two doublets without the satellites and assigns 11280 wrongly to \( D(1) \). Further he assigns D lines to the S series, but the observations at his disposal were too inexact. The measures and allocations are by K.R.

In. Rydberg down to \( D_1(5) \) with satellites of first two. The remainder and all the measures by K.R. As in the S series it should be noted how the \( D_2 \) lines are more persistent than the \( D_1 \).

Tl. Rydberg gives all except from \( D_1(10) \), and he gives \( D_2 \) down to \( m = 15 \). The measures given are by K.R. except those below 2105, which are due to Cornu.

The F Lines.

Bergmann in 1908 (Z. S. f. Wiss. Phot., 6, see also Beih., xxxii., p. 956) measured lines in the ultra-red spectra of the Alkalies and observed in Cs a number of doublets which clearly formed a series, and a few lines in Na, K, Rb which were evidently analogous. It was, however, Runge (Astrophys. Journ., 27, p. 158; Phys. Z. S., 9, p. 1) who pointed out the dependence of the limit of the series on
D(2). The lines in the far ultra-red have since been observed by Paschen ('Ann. d. Phys.,' 33, 1910, p. 717). Bergmann's lines began with the order \( m = 4 \), but in the lists above more recent and more accurate measures by others have been inserted. BN. refers to Bergmann, RL. to Randall ('Ann. d. Phys.,' 33, p. 741), S. to Saunders ('Astrophys. Journ.,' 20, p. 188).

In 1905 Fowler ('Astrophys. Journ.,' 21, p. 84) discovered the lines in the F series for Sr, beginning with K.R.'s 4892, adding two sets of his own observations and also the two connected triplets given in the table. But he attempted to combine them all in one formula. The disarranged triplet for \( m = 3 \) is assigned in the text above from K.R. and Lehmann ('Ann. d. Phys.,' 8, p. 647). In the same volume Saunders ('Astrophys. Journ.,' 21, p. 195) added the last line and gave a similar series for Ca, commencing with 4586. The measures inserted in the table, however, are from later observations by Barnes (Br.) ('Astrophys. Journ.,' 30, p. 14). Saunders also suggested a corresponding series for Ba in which the separations are larger than those of the series assigned in the text. It is possible they may form a similar series connected with the enhanced series. The lines for RαF and allied sets are assigned also in the text above.

The F lines for Mg, Zn, Cd, Hg, Al, and Tl are all due to Paschen ('Ann. d. Phys.,' 29, p. 651, and 35, p. 860). In Hg, however, he assigns 17195 to a combination line, whereas in the text above it is assigned to F₁.

[Notes, September 2, 1913.]

Note 1 — Since the present communication was read Messrs. Fowler and Reynolds ('Roy Soc. Proc.,' 89, p. 137) have published more accurate and extended measurements of the series, and other lines of Mg. The limit of the D or S series appears to be somewhat higher than that adopted in the Table II. (p. 344), and the table will require the limit to be slightly raised to bring in multiples of \( \Delta \) for the order differences. With Fowler and Reynolds' limit the mantissae show rising values for the first few orders and then decreasing, an effect which has been explained in the text in similar cases by a collateral displacement in the limit after a particular order. If the limit be taken to be 39757.78 (i.e., 6.7 higher than in the text), there results an order difference of 4\( \Delta \) between \( m = 2 \) and 3, and equal mantissae afterwards down to \( m = 9 \), when there is a sudden change to a rising. If the limit is now raised by unity for these, the mantissae can again be all equal. Now, a displacement of \( \delta \) on the limit makes a change of 1.03, so that
the change can be explained by a collateral change of \((-\delta) D (\infty)\) for \(m = 10-12\). The scheme is then as follows:

\[
\begin{align*}
2.828063 & \quad (20) + 10 \\
4\Delta_x & \\
3.829715 & \quad (79) - 36 \\
4.829715 & \quad (190) + 9 \\
5.829715 & \quad (362) + 58 \\
6.829715 & \quad (626) - 580 \\
7.829715 & \quad (945) - 173 \\
8.829715 & \quad (2300) + 63 \\
9.829715 & \quad (3200) + 1373 \\
\end{align*}
\]

Collaterals with \((-\delta) D (\infty)\):

\[
\begin{align*}
10.8297 & \quad (44) - 16 \\
11.8297 & \quad (57) - 8 \\
12.8297 & \quad (78) - 5 \\
13.8297 & \quad (92) + 8 \\
\end{align*}
\]

The arrangement is seen to be exceedingly simple. The value for \(m = 1\) has been omitted as the evidence seems strong that Mg conforms to the Zn type and has no D(1). If, however, it is retained, it is now \(13\Delta_x\) below that for \(m = 2\). The values of \(\Delta_x\) are so small that Mg can give no positive evidence for any arrangement. This is evidenced by the changed arrangements called for by more accurate values. The measures of Fowler and Reynolds for the D lines are here adduced in order to complete Appendix II.

\[
\begin{align*}
3097.03 & \quad 3093.09 & \quad 3091.19 \\
2851.76 & \quad 2848.54 & \quad 2846.88 \\
2736.63 & \quad 2733.64 & \quad 2732.16 \\
2672.53 & \quad 2669.66 & \quad 2668.24 \\
2632.98 & \quad 2630.14 & \quad 2628.73 \\
2606.73 & \quad 2603.98 & \quad 2602.59 \\
2588.37 & \quad 2585.63 & \quad 2584.32 \\
2575.02 & \quad 2575.30 & \quad 2570.96 \\
2565.00 & \quad 2562.30 & \quad 2560.96 \\
2557.29 & \quad 2554.70 & \quad 2551.22 \\
2551.22 & \quad 2548.56 & \quad 2546.88 \\
\end{align*}
\]

Note 2, p. 357.—The effect of positive collaterals on D(\(\infty\)) for \(m = 5, 6, 7\) is to diminish the separations of the triplets, so that from \(m = 5\) onwards they would show diminishing values. It is interesting to note that the observations bear this out.

Note 3, p. 380.—The F series of the alkalies. In K, the denominator 3·007542 has its mantissa 1·007542, and this is 14725 (250 + 1) above that of K.F (3), and 5\(\Delta\) = 14700. This suggests that if K.F (3) has a denominator \(3 + d\), that of F\(_{11}\) (2) is \(2 + d\), and the 1182·9 (W.N.) is F\(_{12}\) (2) with a satellite difference 2\(\delta\) and 1346·3 is the collateral F\(_{11}\) (2) \(\Delta\). In Rb 3·001138 has its mantissa 13289 (430 + 1) above that of RbF (3) and \(\Delta = 12935\), so that should have a similar arrangement to that in K. and 2156 would correspond to F\(_{11}\) (2) \(\Delta\).

Note 4, p. 383.—CaF (2). The line with wave number 16203·40 is possibly the F\(_{11}\) and F\(_{12}\) combined. If so, F\(_{12}\) is 16202·66 and is \(\cdot 74\) behind 16203·40, and the satellite might have displaced the observation towards itself from 16203·66. The actual separation would easily be 1·00 corresponding to a difference 2\(\delta\), a usual F satellite difference. 16024·72 would be 1·06 on the other side and would correspond to a collateral forming on the violet side.
IX. On the Self-inductance of Circular Coils of Rectangular Section.

By T. R. Lyle, M.A., Sc.D., F.R.S.

Received February 27,—Read March 13, 1913.

As an approximate formula for the calculation of the self-inductance of a coil of rectangular section,

\[ L = 4\pi an^2 \left\{ \log \frac{8a}{r} - 2 \right\} \]

was first given by Maxwell,* where \( a \) is the mean radius and \( r \) the geometric mean distance of the section of the coil from itself, the current being supposed to be uniformly distributed over the section.

In the following paper it will be shown that the same formula will give the self-inductance to any order of accuracy when in it are substituted for \( a \) and \( r \) the mean radius and the G.M.D. respectively, each suitably modified by small quantities which depend on \( a \) and on the section of the coil, provided, of course, that the series for \( L \) is convergent.

Tables will be given by means of which the modified values of \( a \) and \( r \) for any coil of rectangular section can be found, and which, when substituted in the above formula, will give \( L \) correct to the fourth order, uniform current density over the section being assumed.

1. For the purpose of this paper the mutual inductance of two coaxal circles can best be obtained after Weinstein† by substituting in Maxwell's exact elliptic integral formula‡

\[ M = 4\pi \sqrt{ab} \left\{ \left( \frac{2}{k} - k \right) F - \frac{2}{k} E \right\}, \]

the series expressions for \( F \) and \( E \) in terms of the complementary modulus \( k' \).

Thus we obtain

\[ M = 4\pi \sqrt{ab} \left\{ \log \frac{4}{k'} - 2 + \frac{3}{4} k'^2 \left( \log \frac{4}{k'} - 1 \right) + \frac{1}{64} k'^4 \left( 33 \log \frac{4}{k'} - \frac{81}{2} \right) \right\}, \]

which is rapidly convergent when \( k' \), the ratio of the least to the greatest distance between the circles, is small.

* 'Elect. and Mag.,' vol. II., § 706.
† 'Wied. Ann.,' 21, p. 344, 1884.
‡ 'Elect. and Mag.,' vol. II., § 701.

VOL. CCXIII.—A 505. Published separately, January 31, 1914.
Let these two circles be filaments A and B in the rectangular conductor whose section is PQRS, fig. 1. Then, if \( x \) and \( y \) be the co-ordinates of B relative to axes through A parallel to the sides of the rectangle,

\[
k^2 = \frac{x^2+y^2}{x^2+(2a+y)^2},
\]

and on substitution in the above we obtain, as Rosa and Cohen* have done,

\[
M = 4\pi a \left[ \log \frac{8a}{r} \left( 1 + \frac{y}{2a} + \frac{y^2 + 3x^2}{16a^2} - \frac{3y^2 + 6yx^2}{32a^3} + \frac{17y^4 + 42y^2x^2 - 15x^4}{1024a^4} \right) - 2 - \frac{y}{2a} + \frac{3y^2 - x^2}{16a^2} - \frac{y^2 - 6yx^2}{48a^3} - \frac{19y^4 + 534y^2x^2 - 93x^4}{6144a^4} \right] = M_0 \text{ say,}
\]

where \( a \) is the radius of the circle A and \( r^2 = x^2 + y^2 \).

Fig. 1.

If the co-ordinates of the A circle, referred to axes through the centre of the rectangle, be \( X \) and \( Y \), then \( a \) in the above expression for \( M \) becomes \( a + Y \) where \( a \) now and in what follows is the mean radius of the coil.

This substitution is most easily carried out by aid of Taylor's Theorem. Thus the complete expression for \( M \) is given by

\[
M = M_0 + Y \frac{dM_0}{da} + \frac{Y^2}{1.2} \frac{d^2M_0}{da^2} + &c.
\]

2. It is well known that the self-inductance of a single circular conductor with rectangular section for uniform current density is given by

\[ \frac{1}{b^2 c} \int_{-c}^{c} \int_{-b}^{b} \int_{-y}^{y} \int_{-x}^{x} M \, dx \, dy \, dX \, dY, \]

and for a coil of \( n \) turns is \( n^2 \) times this if Maxwell's correction for space between the wires be neglected, \( b \) being the breadth and \( c \) the radial depth of the rectangle.

The evaluation of this definite integral, even for the second order terms, has presented considerable difficulties. The first correct result to this order was that given by Weinstein.† So far as I am aware no one has published a determination of it to the fourth order. By a method indicated in the appendix to this paper the integration can be carried out to the fourth order without difficulty and to still higher orders if desired.

Thus the following expression for \( L \) has been obtained

\[
L = 4\pi a n^2 \left[ \log \left( \frac{8a}{d} \right) + \frac{1}{12} \frac{u + v}{12} \frac{2}{3} (w + w') \right. \\
+ \frac{1}{2^6 \cdot 3 \cdot 5 \cdot 7} \left( (3b^2 + c^2) \log \left( \frac{8a}{d} \right) + \frac{1}{2} b^2 u - \frac{1}{10} c^2 v - \frac{16}{5} b^2 w + \frac{69}{20} b^2 + \frac{221}{60} c^2 \right) \\
+ \frac{1}{2^n \cdot 3 \cdot 5 \cdot 7} \left( -30b^4 + 35b^2 c^2 + \frac{22}{3} c^4 \right) \log \left( \frac{8a}{d} \right) - \frac{115b^4 - 480b^2 c^2}{12} u \\
- \frac{23}{28} c^4 v + \frac{6b^4 - 7b^2 c^2}{21} \cdot 2^6 \cdot w \\
- \frac{36590b^4 - 2035b^2 c^2 - 11442c^4}{2^3 \cdot 3 \cdot 5 \cdot 7} \right],
\]

in which

\( a = \) mean radius, \( b = \) breadth, \( c = \) depth, \( d = \sqrt{b^2 + c^2} = \) diagonal of the rectangle, and

\[
\begin{align*}
 u &= \frac{b^2}{c^2} \log \left( \frac{d^2}{b^2} \right), \\
v &= \frac{c^2}{b^2} \log \left( \frac{d^2}{c^2} \right), \\
w &= \frac{b}{c} \tan^{-1} \frac{c}{b}, \\
w' &= \frac{c}{b} \tan^{-1} \frac{b}{c}.
\end{align*}
\]

3. If \( r \) be the G.M.D. of the rectangle from itself, it is well known that

\[
\log r = \log d - \phi
\]

where

\[
\phi = \frac{u + v + 25}{12} - \frac{2}{3} (w + w').
\]

* 'Elect. and Mag.,' vol. II, § 693. Rosa (see 'Bull. Bureau Standards,' vol. 3, p. 37, 1907) has greatly improved on Maxwell's correction.

† 'Wied. Ann.,' 21, p. 329, 1884.
Substituting for \( d \) in terms of \( r \) and \( \phi \) in the above expression for \( L \), we obtain

\[
L = 4\pi a^2 \left[ \log \frac{8a}{r} - 2 + \frac{d^2}{a^2} \left( p_2 \log \frac{8a}{r} + q_2 \right) + \frac{d^4}{a^4} \left( p_4 \log \frac{8a}{r} + q_4 \right) \right],
\]

where

\[
\begin{align*}
p_2 &= \frac{1}{2^5 \cdot 3} \frac{3b^2 + c^2}{d^2}, \\
p_4 &= \frac{1}{2^2 \cdot 3^2 \cdot 5} \frac{-90b^4 + 105b^2c^2 + 22c^4}{d^4}, \\
q_2 &= \frac{1}{2^2 \cdot 3 \cdot 5} \left\{ \frac{1}{2} \left[ b^2u - \frac{1}{10} c^2v - \frac{16}{5} b^2w - (3b^2 + c^2) \phi + \frac{69}{20} b^2 \right] \right\}, \\
q_4 &= \frac{1}{2^2 \cdot 3 \cdot 5 \cdot d^4} \left\{ \left( 90b^4 - 105b^2c^2 - 22c^4 \right) \phi - \frac{69}{28} b^4 \right\} + \frac{115b^4 - 480b^2c^2 - 120c^4}{2^2 \cdot 3 \cdot 5 \cdot 7}.
\end{align*}
\]

4. If

\[
A = a \left( 1 + m_1 \frac{d^2}{a^2} + m_2 \frac{d^4}{a^4} \right),
\]

\[
R = r \left( 1 + n_1 \frac{d^2}{a^2} + n_2 \frac{d^4}{a^4} + n_3 \frac{d^6}{a^6} \right),
\]

\( m_1, m_2, n_1, n_2, n_3 \) can be determined so that

\[
4\pi a^2 A \left( \log \frac{8A}{R} - 2 \right)
\]

shall differ very little from the value for \( L \) given in § 3.

After substituting for \( A \) and \( R \) in the above and expanding in a series in \( d^2/\alpha^2 \), the first three terms of the expansion are identified with the corresponding terms of \( L \) in the usual way, and in addition, the coefficient of the fourth term of the expansion, that is the coefficient of \( d^2/\alpha^6 \), is equated to zero. The \( n_3 \) term in \( R \) enables this to be done, with the result that a closer agreement is obtained between the proposed formula and that in § 3.

Thus

\[
\begin{align*}
m_1 &= p_2, \quad n_1 = - (p_2 + q_2), \\
m_2 &= p_4, \quad n_2 = -(p_4 + q_4) + \frac{4}{3} (m_1 - n_1)^2, \\
n_3 &= (m_1 - n_1) \left[ m_2 - n_2 - \frac{1}{3} (m_1 - n_1) \right] (m_1 + 2n_1).
\end{align*}
\]

Hence, when \( A \) and \( R \) have been so determined the formula

\[
L = 4\pi A a^2 \left( \log \frac{8A}{R} - 2 \right),
\]

will give the self-inductance of the coil correct to the fourth order.
CIRCULAR COILS OF RECTANGULAR SECTION.

It will be seen that in the application of this formula the coefficient $n_3$ need rarely be used. It becomes important, however, when the mean radius is less than the diagonal of the section and especially in this case when the section is square or nearly so.

It is obvious that in a similar way to the above, series for $A$ and $R$ could be obtained which, when substituted in the proposed formula would make it practically equivalent to $L$, no matter to what order the integration, if performed, had been carried out.

5. In order to render convenient the practical application of the above formula to the determination of self-inductances the following tables* have been prepared.

| Table I. |
| G.M.D. = $r$. $d^2 = b^2 + c^2$. |

<table>
<thead>
<tr>
<th>$\frac{c}{b}$ or $\frac{b}{c}$</th>
<th>$\phi = \log_e \frac{d}{r}$</th>
<th>$\frac{r}{b+c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0·00</td>
<td>1·5</td>
<td>0·223130</td>
</tr>
<tr>
<td>0·025</td>
<td>1·474734</td>
<td>0·223328</td>
</tr>
<tr>
<td>0·05</td>
<td>1·451005</td>
<td>0·223455</td>
</tr>
<tr>
<td>0·10</td>
<td>1·407566</td>
<td>0·223599</td>
</tr>
<tr>
<td>0·15</td>
<td>1·368975</td>
<td>0·223664</td>
</tr>
<tr>
<td>0·20</td>
<td>1·334799</td>
<td>0·223686</td>
</tr>
<tr>
<td>0·25</td>
<td>1·304680</td>
<td>0·223686</td>
</tr>
<tr>
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<td>1·278284</td>
<td>0·223675</td>
</tr>
<tr>
<td>0·35</td>
<td>1·255312</td>
<td>0·223658</td>
</tr>
<tr>
<td>0·40</td>
<td>1·235461</td>
<td>0·223639</td>
</tr>
<tr>
<td>0·45</td>
<td>1·218448</td>
<td>0·223619</td>
</tr>
<tr>
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<td>1·203998</td>
<td>0·223601</td>
</tr>
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<td>0·55</td>
<td>1·191853</td>
<td>0·223584</td>
</tr>
<tr>
<td>0·60</td>
<td>1·181768</td>
<td>0·223570</td>
</tr>
<tr>
<td>0·65</td>
<td>1·173516</td>
<td>0·223558</td>
</tr>
<tr>
<td>0·70</td>
<td>1·166888</td>
<td>0·223548</td>
</tr>
<tr>
<td>0·75</td>
<td>1·161691</td>
<td>0·223540</td>
</tr>
<tr>
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<td>1·157752</td>
<td>0·223534</td>
</tr>
<tr>
<td>0·85</td>
<td>1·154914</td>
<td>0·223530</td>
</tr>
<tr>
<td>0·90</td>
<td>1·153034</td>
<td>0·223527</td>
</tr>
<tr>
<td>0·95</td>
<td>1·151987</td>
<td>0·223525</td>
</tr>
<tr>
<td>1·00</td>
<td>1·151660</td>
<td>0·223525</td>
</tr>
</tbody>
</table>

Table I. contains (1) values of $\phi$, that is of $\log_e \frac{d}{r}$, for different values of the ratio $b/c$ or $c/b$ (2) values of the ratio $r/(b+c)$ for different values of $b/c$. It will be noticed how nearly $r$ the G.M.D. of a rectangle from itself is proportional to the sum of the

* All the tables given in this paper have been calculated with the greatest care by the aid of a "millionaire" calculating machine. Each separate series of numbers, not only the final series but every intermediate series that had to be determined, was calculated at least twice, the end terms and one or two intermediate terms of each series were carefully re-checked, and each series then examined by taking successive differences.

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sides. These figures will enable the G.M.D. for values of \( c/b \) or \( b/c \), intermediate to those given in the table, and consequently the first or important term of \( L \) for such intermediate values to be obtained with great accuracy.

Table II. contains the values of the coefficients \( m_1, m_2, n_1, n_2, n_3 \) for thick coils, that is for ones in which \( b \) is greater than \( c \) for different values of the ratio \( c/b \), and Table III. contains the values of the same coefficients for thin coils, that is for ones in which \( b \) is less than \( c \) for different values of the ratio \( b/c \).

**Table II.**

<table>
<thead>
<tr>
<th>( \frac{c}{b} )</th>
<th>( 10^2m_1 )</th>
<th>( 10^4m_2 )</th>
<th>( 10^2n_1 )</th>
<th>( 10^4n_2 )</th>
<th>( 10^3n_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>3.12500</td>
<td>-9.766</td>
<td>0.78125</td>
<td>-8.647</td>
<td>-6.9</td>
</tr>
<tr>
<td>0.025</td>
<td>3.12370</td>
<td>-9.746</td>
<td>0.69934</td>
<td>-8.179</td>
<td>-8.2</td>
</tr>
<tr>
<td>0.05</td>
<td>3.11980</td>
<td>-9.688</td>
<td>0.61606</td>
<td>-7.663</td>
<td>-9.7</td>
</tr>
<tr>
<td>0.10</td>
<td>3.10437</td>
<td>-9.461</td>
<td>0.44541</td>
<td>-6.505</td>
<td>-12.6</td>
</tr>
<tr>
<td>0.15</td>
<td>3.07916</td>
<td>-9.094</td>
<td>+0.26934</td>
<td>-5.702</td>
<td>-15.7</td>
</tr>
<tr>
<td>0.20</td>
<td>3.04487</td>
<td>-8.604</td>
<td>+0.08919</td>
<td>-3.795</td>
<td>-18.9</td>
</tr>
<tr>
<td>0.25</td>
<td>3.00345</td>
<td>-8.011</td>
<td>-0.09342</td>
<td>-2.326</td>
<td>-22.1</td>
</tr>
<tr>
<td>0.30</td>
<td>2.95298</td>
<td>-7.340</td>
<td>-0.27664</td>
<td>-0.838</td>
<td>-25.2</td>
</tr>
<tr>
<td>0.35</td>
<td>2.89764</td>
<td>-6.614</td>
<td>-0.45856</td>
<td>+0.630</td>
<td>-28.0</td>
</tr>
<tr>
<td>0.40</td>
<td>2.83764</td>
<td>-5.857</td>
<td>-0.63754</td>
<td>+2.045</td>
<td>-30.6</td>
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<tr>
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<td>2.77417</td>
<td>-5.090</td>
<td>-0.81188</td>
<td>3.375</td>
<td>-32.8</td>
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<td>0.50</td>
<td>2.70833</td>
<td>-4.332</td>
<td>-0.98060</td>
<td>4.610</td>
<td>-34.7</td>
</tr>
<tr>
<td>0.55</td>
<td>2.64165</td>
<td>-3.596</td>
<td>-1.14240</td>
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<td>-36.1</td>
</tr>
<tr>
<td>0.60</td>
<td>2.57333</td>
<td>-2.895</td>
<td>-1.29662</td>
<td>6.725</td>
<td>-37.2</td>
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<tr>
<td>0.65</td>
<td>2.50622</td>
<td>-2.237</td>
<td>-1.44274</td>
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<td>-37.8</td>
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<td>-1.58048</td>
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<td>-1.70975</td>
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<td>-38.2</td>
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<td>-1.83060</td>
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<td>9.978</td>
<td>-37.5</td>
</tr>
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</tr>
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<td>-35.9</td>
</tr>
<tr>
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<td>+1.004</td>
<td>-2.23473</td>
<td>10.824</td>
<td>-35.0</td>
</tr>
<tr>
<td>1.05</td>
<td>2.03550</td>
<td>+1.289</td>
<td>-2.31773</td>
<td>10.978</td>
<td>-33.9</td>
</tr>
</tbody>
</table>

It will have been noticed that the coefficient \( m_1 \) and \( m_2 \) are algebraic and can be easily calculated for any value of \( c/b \). Those in the tables are given for convenience.

6. If the formula for \( L \) given in §2 be written in the form

\[
L = 4\pi an^2 \left[ \left( 1 + m_1 \frac{d^2}{\alpha^2} + m_2 \frac{d^4}{\alpha^4} \right) \log \frac{8a}{d} - l_3 + l_1 \frac{d^2}{\alpha^2} + l_2 \frac{d^4}{\alpha^4} \right],
\]

tables giving \( m_1, m_2, l_3, l_1, l_2 \), for different values of \( c/b \) would also render easy the computation of the self inductances of coils. Such tables have been computed from Weinstei's formula by Stefan, but he is in error in thinking that the second order coefficient has the same value for a given value of \( b/c \) in a thin coil as it has for the

same value of \( c/b \) in a thick coil. The second order coefficients he gives are correct for thick coils.

In using the above formula with tables for the computation of \( L \) for values of \( c/b \) or \( b/c \) intermediate to those given in the tables, the value of \( l_0 \) which is part of the large or first order term will have to be obtained by interpolation, whereas in the

<table>
<thead>
<tr>
<th>( \frac{b}{c} )</th>
<th>( 10^4 m_1 )</th>
<th>( 10^4 m_2 )</th>
<th>( 10^4 n_1 )</th>
<th>( 10^4 n_2 )</th>
<th>( 10^4 n_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
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<td>-3.21180</td>
<td>6.073</td>
<td>+0.5</td>
</tr>
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<td>-3.23737</td>
<td>6.134</td>
<td>+0.6</td>
</tr>
<tr>
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<td>6.214</td>
<td>+0.5</td>
</tr>
<tr>
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<td>+0.1</td>
</tr>
<tr>
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<td>-3.31747</td>
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<td>-0.6</td>
</tr>
<tr>
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<td>-3.32107</td>
<td>7.090</td>
<td>-1.7</td>
</tr>
<tr>
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</tr>
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</tr>
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<td>-20.3</td>
</tr>
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<td>-22.8</td>
</tr>
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<td>-25.2</td>
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<td>-27.4</td>
</tr>
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<td>1.85468</td>
<td>2.111</td>
<td>-2.59020</td>
<td>11.171</td>
<td>-29.3</td>
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<td>-2.50019</td>
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<td>1.590</td>
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<td>1.303</td>
<td>-2.32192</td>
<td>10.985</td>
<td>-33.9</td>
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<td>1.004</td>
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<td>10.624</td>
<td>-36.2</td>
</tr>
</tbody>
</table>

The method previously given, the whole of the first order term can be easily got with great accuracy, by making use of the nearly constant ratio of \( r \) to \( b+c \) indicated by the figures given in the third column of Table I.

The coefficients \( l_0, l_1, l_2 \), occurred in the computation of Tables I., II., and III., \( m_1 \) and \( m_2 \) are the same as in these tables, and are in any case algebraic as

\[
m_1 = \frac{1}{2^4 \cdot 3} \frac{3b^2 + c^2}{b^2 + c^2},
\]

\[
m_2 = \frac{1}{2^{16} \cdot 3^2 \cdot 5} \frac{-90b^4 + 105b^2c^2 + 22c^4}{(b^2 + c^2)^2}.
\]

Table IV. gives the values of \( l_0, l_1, l_2 \) for different values of the ratio \( c/b \) for thick coils, and of \( b/c \) for thin coils.
Table IV.

<table>
<thead>
<tr>
<th>For both thick and thin coils.</th>
<th>For thick coils.</th>
<th>For thin coils.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{c}{b} \text{ or } \frac{d}{c})</td>
<td>(l_c)</td>
<td>(10^2l_c)</td>
</tr>
<tr>
<td>0.00</td>
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<td>0.00</td>
</tr>
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</tr>
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<td>0.838309</td>
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<td>0.95</td>
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<tr>
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</tbody>
</table>

7. The only available means of testing the above methods of computing self-inductances and of finding the limit outside which they are practically reliable is to compare the results they give with those given by Lorenz's* exact elliptic integral formula for the self-inductance of a current sheet solenoid, which is

\[
L = \frac{32}{3} \pi \alpha^3 \left[ \frac{2k^2 - 1}{k^3} E + \frac{1 - k^2}{k^3} F - 1 \right],
\]

where \(\alpha\) is the radius, \(d\) the length of the solenoid, and

\[
k^2 = \frac{4\alpha^2}{4\alpha^2 + d^2}.
\]

Thus consider the case of a solenoid whose length is twice its radius. Here

\[
\frac{d}{\alpha} = 2, \quad \frac{c}{b} = 0,
\]

and from Table II.

\[ A = (1 + 4 \times 0.03125 - 16 \times 0.0009766) \alpha, \]

\[ = 1.109375 \alpha. \]

\[ R = (1 + 4 \times 0.0078125 - 16 \times 0.0008647 - 64 \times 0.0000069) \varphi, \]

\[ = 1.016973 \varphi. \]

so

\[ \log \frac{8A}{R} = \log \frac{8\alpha}{d} + \varphi + \log \frac{1.109375}{1.016973}, \]

(where \( \varphi \) is given in Table I.)

\[ = \log 4 + \varphi + 0.086965, \]

\[ = 2.973259, \]

and

\[ 4\pi A \left( \log \frac{8A}{R} - 2 \right) = 4\pi \alpha \times 1.07970. \]

Lorenz's exact formula gives

\[ L = 4\pi \alpha \times 1.08137. \]

Thus the error in this case is 1 part in 650.

When the comparison is made in less extreme cases we find the agreement with the Lorenz formula very close.

Thus when the length of the solenoid is equal to its radius (\( \alpha \)) either of the methods of this paper give

\[ L = 20.7453 \alpha, \]

while Lorenz's formula gives

\[ L = 20.7463 \alpha, \]

showing an error of 1 part in 20,000, and when the length of the solenoid is half the radius we obtain

\[ L = 28.85332 \alpha, \]

as against the Lorenz value

\[ L = 28.85335 \alpha, \]

showing an error of only about 1 part in 1,000,000.
APPENDIX I.

In order to determine \( L \) to the fourth order we have seen that it is necessary to evaluate the definite integral

\[
\int_{-b}^{b} \int_{-c}^{c} \int_{-e}^{e} \int_{-b}^{b} M \, dx \, dy \, dX \, dY
\]

where

\[
M = P + QY + RY^2 + SY^3 + TY^4,
\]

\( P, Q, R, S, \) and \( T \) being functions of \( x \) and \( y \).

If we proceed in the ordinary way by putting in the limits after each integration the expression becomes very cumbersome on account of the nature of some of the functions (log and \( \tan^{-1} \)) with which we have to deal.

By the method to be explained below all the integration will be carried out first and the limits introduced in an easy and symmetrical way at the finish.

1. Dealing first with \( P \), the term independent of \( Y \), if

\[
\iint P \, dx \, dy = \theta (xy),
\]

the result, with limits introduced, of the integrations with respect to \( x \) and \( y \) will be

\[
\theta (x_1y_1) - \theta (x_2y_2) - \theta (x_3y_3) + \theta (x_4y_4),
\]

where

\[
x_1 = \frac{1}{2}b - X, \quad x_2 = -\frac{1}{2}b - X, \\
y_1 = \frac{1}{2}c - Y, \quad y_2 = -\frac{1}{2}c - Y.
\]

We have now to evaluate four definite integrals of which the first is

\[
\int_{-c}^{c} \int_{-b}^{b} \theta (x_1y_1) \, dX \, dY.
\]

Changing the variables to \( x_1 \) and \( y_1 \), and the limits accordingly, this integral is equal to

\[
\int_{-e}^{e} \int_{-b}^{b} \theta (x_1y_1) \, dx_1 \, dy_1
\]

where

\[
\phi (xy) = \iint \theta (xy) \, dx \, dy = \iiint P \, dx^2 \, dy^2.
\]

Dealing in the same way with the three remaining integrals

\[
- \int_{-c}^{c} \int_{-b}^{b} \theta (x_1y_2) \, dX \, dY, \quad - \int_{-c}^{c} \int_{-b}^{b} \theta (x_2y_1) \, dX \, dY, \quad \text{and} \quad \int_{-c}^{c} \int_{-b}^{b} \theta (x_3y_2) \, dX \, dY,
\]
we find that they become
\[
\int_{-c}^{c} \int_{-b}^{b} \theta (x_1 y_2) \, dx_1 \, dy_2, \quad \int_{-c}^{c} \int_{-b}^{b} \theta (x_2 y_1) \, dx_2 \, dy_1, \quad \text{and} \quad \int_{-c}^{c} \int_{-b}^{b} \theta (x_3 y_3) \, dx_3 \, dy_3
\]
respectively, which are equal to
\[
\phi (0, 0) - \phi (0, -c) - \phi (b, 0) + \phi (b, -c),
\]
and
\[
\phi (0, 0) - \phi (0, c) - \phi (-b, 0) + \phi (-b, c),
\]
respectively, where
\[
\phi (xy) \text{ has the same meaning as before.}
\]
Hence, if
\[
\phi (xy) = \int_{-c}^{c} \int_{-b}^{b} P \, dx \, dy^2
\]
\[
\int_{-c}^{c} \int_{-b}^{b} P \, dx \, dy \, dX \, dY
\]
is equal to
\[
4\phi (0, 0) + \sum \phi (\pm b \pm c) - 2\phi (0, c) - 2\phi (0, -c) - 2\phi (b, 0) - 2\phi (-b, 0).
\]
This expression, obtained from the function \( \phi (xy) \) by substituting in it \( b, c, -b, -c, 0, 0 \) in the way indicated, will, in what follows, be designated by
\[
\sum \phi.
\]
2. As an illustration of the above I will indicate the process as applied to the simplest term in \( P \) involving \( \log (x^2 + y^2) \).
Thus to obtain
\[
\int_{-c}^{c} \int_{-b}^{b} \int_{-c}^{c} \int_{-b}^{b} \log (x^2 + y^2) \, dx \, dy \, dX \, dY,
\]
we find by simple integrations that
\[
\phi (xy) = \int_{-c}^{c} \int_{-b}^{b} \log (x^2 + y^2) \, dx \, dy^2 = \left( \frac{x^2 y^2}{4} - \frac{x^4 + y^4}{24} \right) \log (x^2 + y^2)
\]
\[
+ \frac{1}{3} \left( x^2 y \tan^{-1} \frac{y}{x} + x y^2 \tan^{-1} \frac{x}{y} \right) - \frac{25}{24} x^2 y^2.
\]
By inspection it is seen that
\[
\phi (00) = 0,
\]
\[
\sum \phi (\pm b, \pm c) = \left( b^2 c^2 - \frac{b^4 + c^4}{6} \right) \log (b^2 + c^2) - \frac{25}{6} b^2 c^2 + \frac{4}{3} \left( b^2 c \tan^{-1} \frac{c}{b} + b c^2 \tan^{-1} \frac{b}{c} \right),
\]
\[
\phi (b, 0) = \phi (-b, 0) = - \frac{b^4}{24} \log b^2,
\]
\[
\phi (0, c) = \phi (0, -c) = - \frac{c^4}{24} \log c^2.
\]
Hence the definite integral above is equal to

\[ b^2c^2 \left[ \log (b^2+c^2) - \frac{b^2}{6c^2} \log \frac{b^2+c^2}{b^2} - \frac{c^2}{6b^2} \log \frac{b^2+c^2}{c^2} - \frac{25}{6} + \frac{4}{3} \left( \frac{b}{c} \tan^{-1} \frac{c}{b} + \frac{c}{b} \tan^{-1} \frac{b}{c} \right) \right] \]

The above is the well-known definite integral used for determining the G.M.D. of a rectangle from itself.

3. To determine

\[ \iiint YQ \, dx \, dy \, dX \, dY, \]

between the given limits.

If

\[ \theta (xy) = \iint Q \, dx \, dy, \]

the result of the integrations with respect to \( x \) and \( y \) will now be

\[ Y \{ \theta (x_1y_1)-\theta (x_2y_2)-\theta (x_2y_1)+\theta (x_1y_2) \} \]

where \( x_1, y_1, x_2, y_2 \) have the same significations as before.

We have now to evaluate four integrals of the type

\[ \int_{-a}^{a} \int_{-b}^{b} \, \theta (x_1y_1) \, dX \, dY. \]

Proceeding as in § 1, these, affected by their proper signs, become

\[ \int_{a}^{b} \, \int_{c}^{d} \frac{1}{2}(c+y_1) \, \theta (x_1y_1) \, dx_1 \, dy_1 - \int_{-a}^{-b} \, \int_{c}^{d} \frac{1}{2}(c+y_2) \, \theta (x_2y_1) \, dx_2 \, dy_2 \]

\[ + \int_{a}^{b} \, \int_{c}^{d} \frac{1}{2}(c-y_1) \, \theta (x_2y_1) \, dx_2 \, dy_2 - \int_{-a}^{-b} \, \int_{c}^{d} \frac{1}{2}(c-y_2) \, \theta (x_1y_2) \, dx_1 \, dy_2, \]

so that, if in this case

\[ \phi (xy) = \iiint Q \, dx^2 \, dy^2, \]

\[ \phi' (xy) = \int y \, dy \iiint Q \, dx^2 \, dy, \]

and

\[ \Delta \phi = \phi (b, c) + \phi (-b, c) - \phi (b, -c) - \phi (-b, -c) + 2\phi (0, -c) - 2\phi (0, c), \]

then

\[ \iiint YQ \, dx \, dy \, dX \, dY \]

between the given limits is equal to

\[ \frac{c}{2} \Delta \phi - \Sigma \phi' \]

where \( \Sigma \) has the signification given to it in § 1.
4. In a similar way it can be shown that

\[ \int \int \int Y^3 R \, dx \, dy \, dX \, dY \]

between the given limits is equal to

\[ \left( \frac{c}{2} \right)^2 \Sigma \phi - 2 \left( \frac{c}{2} \right) \Delta \phi' + \Sigma \phi'' \]

where, in this case,

\[ \phi (xy) = \int \int \int R \, dx^2 \, dy^2, \]
\[ \phi' (xy) = \int y \, dy \int \int R \, dx^2 \, dy, \]
\[ \phi'' (xy) = \int y^2 \, dy \int \int R \, dx^2 \, dy, \]

and that

\[ \int \int \int Y^3 S \, dx \, dy \, dX \, dY \]

between the given limits is equal to

\[ \left( \frac{c}{2} \right)^3 \Delta \phi - 3 \left( \frac{c}{2} \right)^2 \Sigma \phi' + 3 \left( \frac{c}{2} \right) \Delta \phi'' - \Sigma \phi''', \]

where, in this case,

\[ \phi (xy) = \int \int \int S \, dx^2 \, dy^2, \]
\[ \phi' (xy) = \int y \, dy \int \int S \, dx^2 \, dy, \]
\[ \phi'' (xy) = \int y^2 \, dy \int \int S \, dx^2 \, dy, \]
\[ \phi''' (xy) = \int y^3 \, dy \int \int S \, dx^2 \, dy. \]

The result of integration can now be easily written out for integrations involving higher powers of \( Y \).

5. Before proceeding with the integrations it is advisable to have prepared beforehand a table giving

\[ \int x^n \log (x^2 + y^2) \, dx \]

and

\[ \int x^n \tan^{-1} \frac{y}{x} \, dx \]

from \( n = 0 \) to \( n = 7 \).

If this be done, and the method indicated above followed, the work presents little difficulty and is not very tedious.
APPENDIX II.

*(Added October 1, 1913.)*

Since writing the above I have determined the sixth order term of the series for L.

In order to do this it was necessary to extend to the sixth order Maxwell's series formula (see § 1) for M, the mutual inductance of two unequal coaxal circles which Rosa and Cohen* had already extended to the fifth order.

Thus

\[
M = 4\pi a \left[ \log \frac{8a}{r} \left\{ 1 + \frac{y}{2a} + \frac{3x^2 + y^2}{2^4 \cdot a^2} - \frac{3x^2 y + y^3}{2^5 \cdot a^3} - \frac{15x^4 - 42x^2 y^2 - 17 y^4}{2^{10} \cdot a^4} + \frac{45 x^4 y - 30 x^2 y^2 - 19 y^6}{2^{21} \cdot a^6} + \frac{35 x^6 - 345 x^4 y^2 + 45 x^2 y^4 + 89 y^6}{2^{24} \cdot a^8} \right\} \right. 
- \frac{2 - \frac{y}{2 \cdot a} - \frac{x^2 - 3y^2}{2^4 \cdot a^2} + \frac{6x^2 y - y^3}{2^5 \cdot 3 \cdot a^3}}{2^{21} \cdot 3 \cdot a^4} + \frac{93x^4 - 534x^2 y^2 - 19 y^4}{2^{22} \cdot 3 \cdot 5 \cdot a^5} + \frac{1845x^4 y - 3030 x^2 y^2 - 379 y^6}{2^{23} \cdot 3 \cdot 5 \cdot a^6} 
- \frac{1235x^6 - 17445x^4 y^2 + 12045 x^2 y^4 - 7371 y^6}{2^{25} \cdot 3 \cdot 5 \cdot a^6} \right].
\]

When in M we substitute, as explained in § 1, \(a + Y\) for \(a\), the term of the sixth order in the variables \(x\), \(y\), and \(Y\) becomes equal to \(U\), where

\[
U = p + q Y + r Y^2 + s Y^3 + t Y^4 + u Y^5 + v Y^6,
\]
in which

\[
p = 4\pi a \left\{ \log \frac{8a}{r} \left( \frac{35x^6 - 345 x^4 y^2 + 45 x^2 y^4 + 89 y^6}{2^{14} \cdot a^6} \right) \right. 
- \frac{1235x^6 - 17445x^4 y^2 + 12045 x^2 y^4 - 7371 y^6}{2^{25} \cdot 3 \cdot 5 \cdot a^6} \right] \right\},
\]

\[
q = 4\pi a \left\{ \log \frac{8a}{r} \left( \frac{-45x^4 y + 30 x^2 y^2 + 19 y^4}{2^9 \cdot a^6} \right) \right. \left( \frac{4635 x^4 y - 6510 x^2 y^2 - 1043 y^6}{2^{21} \cdot 3 \cdot 5 \cdot a^6} \right) \right\},
\]

\[
r = 4\pi a \left\{ \log \frac{8a}{r} \left( \frac{-45x^4 + 126 x^2 y^2 + 51 y^4}{2^9 \cdot a^6} \right) \right. \left( \frac{291x^4 - 1362 x^2 y^2 - 157 y^4}{2^{21} \cdot 3 \cdot a^6} \right) \right\},
\]

\[
s = 4\pi a \left\{ \log \frac{8a}{r} \left( \frac{3x^2 y + y^3}{2^5 \cdot a^6} \right) \right. \left( \frac{87x^2 y + 5y^3}{2^7 \cdot 3 \cdot a^6} \right) \right\},
\]

\[
t = 4\pi a \left\{ \log \frac{8a}{r} \left( \frac{3x^2 + y^2}{2^4 \cdot a^6} \right) \right. \left( \frac{87 x^2 - 11 y^2}{2^6 \cdot 3 \cdot a^6} \right) \right\},
\]

\[
u = 4\pi a \cdot \frac{1}{2 \cdot 3 \cdot 5 \cdot a^6}.
\]

CIRCULAR COILS OF RECTANGULAR SECTION.

The term of the sixth order in the series for \( L \) is the value of the integral

\[
\frac{1}{b^c} \int \int \int U \, dx \, dy \, dX \, dY,
\]

between the specified limits, and I have found it to be equal to,

\[
\frac{4\pi}{2^{16} \cdot 3 \cdot 5 \cdot 7 \cdot \alpha^5} \left[ (525b^6 - 1610b^6c^2 + 770b^6c^4 + 103c^6) \log \frac{8a}{d} 
+ \frac{3633}{10} b^6 - 3220b^6c^2 + 2240b^6c^4 \right] u,
\]

\[
- \frac{359}{30} c^6v - 2\left( \frac{5}{3} b^6 - 4b^6c^2 + \frac{7}{5} b^6c^4 \right) w,
\]

\[
+ \frac{2161453}{2^3 \cdot 3 \cdot 5 \cdot 7} b^6 - \frac{617423}{2^2 \cdot 3^2 \cdot 5} b^6c^2 - \frac{8329}{2^2 \cdot 3 \cdot 5} b^6c^4 + \frac{4308631}{2^1 \cdot 3 \cdot 5 \cdot 7} c^6
\]

in which \( u, v, w, \) and \( d \) have the significations assigned to them in § 2.

The method of integration indicated in this paper renders the determination of \( L \) in series form comparatively easy for the special cases of a solenoid \((c/b = 0)\), and a flat circular ring coil \((b/c = 0)\), uniform current density being assumed.

Thus Coffin’s formula\* for a solenoid can be easily obtained, and Rayleigh\† and Niven’s formula for a coil whose axial dimension \((b)\) is zero can be extended to the sixth order, giving

\[
L = 4\pi n^2 a \left[ \left( 1 + \frac{c^2}{2^5 \cdot 3 \cdot \alpha^2} + \frac{11c^4}{2^{10} \cdot 3^2 \cdot 5 \cdot \alpha^4} + \frac{103c^6}{2^{16} \cdot 3 \cdot 5 \cdot 7 \cdot \alpha^6} \right) \log \frac{8a}{c}
\right.
\]

\[
- \frac{1}{2} + \frac{43c^2}{2^7 \cdot 3^2 \cdot \alpha^2} + \frac{c^4}{2^5 \cdot 3 \cdot 5^2 \cdot \alpha^4} + \frac{4298579c^6}{2^{19} \cdot 3 \cdot 5^2 \cdot 7^2 \cdot \alpha^6} \right]
\]

which can also be obtained by putting \( b = 0 \) in the general formula obtained above for \( L \), and remembering that when \( b/c = 0, v = 1, \) and \( u' = 1. \)


3 K 2
X. A Method of Measuring the Pressure Produced in the Detonation of High Explosives or by the Impact of Bullets.

By Bertram Hopkinson, F.R.S.

Received October 17,—Read November 27, 1913.

The determination of the actual pressures produced by a blow such as that of a rifle bullet or by the detonation of high explosives is a problem of much scientific and practical interest but of considerable difficulty. It is easy to measure the transfer of momentum associated with the blow, which is equal to the average pressure developed, multiplied by the time during which it acts, but the separation of these two factors has not hitherto been effected. The direct determination of a force acting for a few hundred-thousandths of a second presents difficulties which may perhaps be called insuperable, but the measurement of the other factor, the duration of the blow, is more feasible. In the case of impacts such as those of spheres or rods moving at moderate velocities the time of contact can be determined electrically with considerable accuracy.* The present paper contains an account of a method of analysing experimentally more violent blows and of measuring their duration and the pressures developed.

If a rifle bullet be fired against the end of a cylindrical steel rod there is a definite pressure applied on the end of the rod at each instant of time during the period of impact and the pressure can be plotted as a function of the time. The pressure-time curve is a perfectly definite thing, though the ordinates are expressed in tons and the abscissae in millionths of a second; the pressure starts when the nose of the bullet first strikes the end of the rod and it continues until the bullet has been completely set up or stopped by the impact. Subject to qualifications, which will be considered later, the result of applying this varying pressure to the end is to send along the rod a wave of pressure which, so long as the elasticity is perfect, travels without change of type. If the pressure in different sections of the rod be plotted at any instant (fig. 1) then at a later time the same curve shifted to the right by a distance proportional to the time will represent the then distribution of pressure. The velocity with which the wave travels in steel is approximately 17,000 feet per second. As the wave travels over any section of the rod, that section successively experiences pressures represented


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by the successive ordinates of the curve as they pass over it. Thus the curve also 
represents the relation between the pressure at any point of the rod and the time, 
the scale being such that one inch represents the time taken by the wave to travel 
that distance which is very nearly \( \frac{1}{200,000} \) of a second. In particular the curve 
giving the distribution of pressure in the rod along its length is, assuming perfect 
estILITY, the same as the curve connecting the pressure applied at the end and the 
time, the scale of time being that just given.

The progress of the wave of stress along the rod is accompanied by corresponding 
strain and therefore by movement. It is easy to show that the same curve which 
represents the distribution of pressure at any moment also represents the distribution 
of velocity in the rod, the scale being such that one ton per square inch of pressure 
 corresponds to about 13 feet per second of velocity. Until the wave reaches any 
section of the rod that section is at rest. It is then, as the wave passes over it, 
accelerated more or less rapidly to a maximum velocity, then retarded, and finally left 
at rest with some forward displacement. In this manner the momentum given to the 
rod by the application of pressure at its end is transferred by wave action along it, 
the whole of such momentum being at any instant concentrated in a length of the rod 
which corresponds, on the scale above stated (one inch = \( \frac{1}{200,000} \) second), to the time 
taken to stop the bullet completely. Consider a portion of the rod to the right of any 
section A (fig. 1) which lies within the wave at the moment under consideration. 
The pressure has been acting on this portion since the wave first reached it, that is 
for a time represented by the length OA and equal to \( \frac{OA}{V} \) where \( V \) is the velocity of 
propagation. The momentum which has been communicated to the part under 
consideration is equal to the time integral of the pressure which has acted across the 
section A, that is to the shaded area of the curve in the figure. The portion of the 
rod to the right of the section is continually gaining momentum at the expense of the 
portion to the left while the wave is passing, the rate of transfer at any instant being 
equal to the pressure.

When the wave reaches the free end of the rod it is reflected as a wave of tension 
which comes back with the same velocity as the pressure wave, and the state of stress 
in the rod subsequently is to be determined by adding the effects of the direct 
and of the reflected waves. Now suppose that the rod is divided at some section, B, 
near the free end (fig. 2), the opposed surfaces of the cut being in firm contact 
and carefully faced. The wave of pressure travels over the joint practically 
unchanged and pressure continues to act between the faces until the reflected 
tension wave arrives at the joint. The pressure is then reduced by the amount 
of the tension due to the reflected wave and as soon as this overbalances at
section B the pressure of the direct wave (which is the moment shown in the figure) the rod, being unable to withstand tension at the joint, parts there and the end flies off. The end piece has then acquired the quantity of momentum represented by the shaded area in the figure, equal to the time-integral of the pressure curve from O to B, less that of the tension wave during the time for which it has been acting, that is from O' to B. The piece flies off with this amount of momentum trapped, so to speak, within it. If it be caught in a ballistic pendulum and its momentum thus measured we have the time integral of the pressure curve between the points B and B' on the pressure-time curve which are such that they correspond to equal pressures on the rising and falling parts of the curve, while the time-interval between them is equal to that required for a wave to travel twice the length of the end piece. By taking end pieces of different lengths and measuring the momentum so trapped in each the area of the pressure-time curve over corresponding intervals can be obtained. In general the precise form of the curve itself cannot be deduced because the points of commencement of the several intervals are not known. Thus a given set of observations would be consistent with any one of the three forms shown in fig. 3 which can be derived from one another by shearing parallel to the base so that the intercept of any line such as AA' is the same on all. But the maximum pressure and the total duration of the impact can always be obtained, and these are the most important elements. The maximum pressure is the limiting value of the average acting on a piece when the piece is very short, and the duration corresponds to twice that length of piece which just catches the whole of the momentum leaving the rod at rest. If the circumstances of the impact are such that the pressure is known to rise or to fall with great suddenness, the curve assumes the form I. or III. and its form may be determined completely from the observations.

This is the basis of the method described in the present paper. A cylindrical rod or shaft of steel is hung up horizontally by four equal threads so that it can swing in a vertical plane remaining parallel to itself. A short piece of rod of the same diameter is butted up against one end being held on by magnetic attraction but otherwise free. A rifle bullet is fired at, or gun-cotton is detonated near, the other end; the short piece flies off and is caught in a box suspended in a similar manner to the long rod. Suitable recording arrangements register the movement both of the long rod and of the box, and the momentum in each is calculated in the usual way as for a ballistic pendulum. Sufficient magnetic force to hold the end-piece in position is provided by putting a solenoid round the rod in the neighbourhood of the joint. The slight force required to separate the piece from the rod under these conditions may be neglected in comparison with the pressures and tensions set up, since these amount to several tons on the square inch, and, practically speaking, the joint will transmit the pressure wave unchanged but will sustain no tension.
Pressure Produced by the Impact of Lead Bullets.

The pressure which should be produced by the impact of a lead bullet can be predicted theoretically, and the study of this pressure was made rather with a view to checking the method than in the hope of discovering any new facts. At velocities exceeding 1000 feet per second lead behaves on impact against a hard surface practically as a perfect fluid.

The course of the impact is shown in fig. 4. The base of the bullet at the moment of striking is at A; a little later it is at B. Assuming perfect fluidity the base of the bullet knows nothing of the impact at the nose and continues to move forward with unimpaired velocity. Hence the time elapsing between the two positions shown in the figure is \( \frac{AB}{V} \). The momentum which has been destroyed up to this time is to a first approximation that of the portion of the bullet which has been flattened out, namely that portion shown shaded in the dotted figure. Knowing the distribution of mass along the length this is easily calculated. This simple theory is subject to some qualifications due partly to want of perfect fluidity, and partly to the fact that the sections of the bullet are not brought right up to the face and there stopped dead, as is assumed in the theory, but are more or less gradually retarded or deflected in the region of curved steam-lines at C. These corrections are, however, most conveniently introduced when comparing the theory with the experimental results.

The bullets used were of two patterns, one the ordinary service form (Mark VI.) and the other a soft-nosed bullet supplied on the market for sporting purposes. Both are of lead, encased in nickel. Sections of the bullets are shown in fig. 5.* Sample bullets were sawn into sections, and the sections weighed. The distribution of weight along the length thus determined

* The soft-nosed bullet (lower figure) has four longitudinal saw-cuts in the nickel casing; the section is taken through two of these cuts.
is shown in the curve fig. 5. The bullets were almost precisely alike both in regard to total weight (0'0306 lbs.) and distribution of weight along the length.

Most of the experiments were made with the service cartridge, in the service rifle, giving an average velocity of 2000 feet per second. These cartridges were very uniform, the range of variation in velocity being under one per cent. Some experiments were also made with cartridges giving velocities of about 1240 feet per second and 700 feet per second respectively.

The rod against which the bullet was fired was in most cases of steel containing C, 0'4 per cent.; Mn, 1'05 per cent. Its breaking strength was 37 tons per square inch with 24 per cent. elongation over 8 inches. The end of the rod was heated to a white heat in the forge and quenched and would then stand a large number of shots without serious damage. In some cases tool steel hardened, and tempered blue, was used, but it was found difficult to get the temper exactly right. The pieces butted to the end of the rod were usually of mild steel. For recording the movement of the rod and of the box in which the piece was caught each was fitted with a pencil which moved over a horizontal sheet of paper and the length of the mark was measured.

Assuming that the bullet strikes the rod fairly in the centre, and that the fragments are shot out radially, the total momentum recorded in rod and piece should be equal to the momentum of the bullet, which at 2000 feet per second is 61'2 lb. feet per second units. In fact, considerable variations were found in the total momentum. For instance, in 110 shots fired at a 1-inch rod, the maximum total recorded was 76,
might vary widely. This is to be expected if the explanation just given of the irregularities is correct. For instance a cup-shaped cavity in the rod such as is formed after a large number of shots will give a high value for the momentum, but if not too pronounced it will not seriously affect the form of the relation between pressure and time.

The results have accordingly been reduced by taking in every case the percentages of the total momentum found in the piece. The following table gives details of one set of experiments. It was found that there was no systematic difference between the service bullets and the soft bullets, and the results for both types are included in the table:

Rod, 1 inch diameter, 43 inches to 50 inches long. 2000 feet per second.

<table>
<thead>
<tr>
<th>Length of piece.</th>
<th>Number of shots</th>
<th>Percentage of total in piece.</th>
<th>Total momentum in rod and piece.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>19</td>
<td>11.6  9.8  10.9</td>
<td>63  58  60</td>
</tr>
<tr>
<td>1.0</td>
<td>25</td>
<td>24.0  20.4  22.1</td>
<td>66  58  62</td>
</tr>
<tr>
<td>2.0</td>
<td>8</td>
<td>46.0  40.6  43.2</td>
<td>73  60  65</td>
</tr>
<tr>
<td>3.0</td>
<td>26</td>
<td>63.0  58.0  61.0</td>
<td>66  59  62</td>
</tr>
<tr>
<td>3.5</td>
<td>6</td>
<td>71.0  69.0  70.4</td>
<td>71  65  67</td>
</tr>
<tr>
<td>4.0</td>
<td>6</td>
<td>82.0  79.0  79.7</td>
<td>67  50  62</td>
</tr>
<tr>
<td>5.0</td>
<td>11</td>
<td>93.0  94.5  93.5</td>
<td>76  59  67</td>
</tr>
<tr>
<td>6.0</td>
<td>9</td>
<td>99.0  --  97.6</td>
<td>69  63  66</td>
</tr>
</tbody>
</table>

The mean percentages given in the third column of the table are plotted against length of piece in fig. 6. As the wave travels 2.04 inches in $10^{-5}$ seconds, 1 inch length of piece represents $0.98 \times 10^{-5}$ seconds.* The slope of this curve represents pressure, and as already explained the maximum pressure is represented by the slope at the origin. This is 22 per inch, and assuming an average total momentum of 61.2 units the corresponding pressure is

$$\frac{0.22 \times 61.2 \times 10^6}{32.2 \times 0.98} = 42,600 \text{ lbs. or } 19.0 \text{ tons.}$$

It will also be noticed that the impact is practically complete in $6 \times 10^{-5}$ seconds, 97.5 per cent. of the total being then accounted for in the piece.

According to the simple theory, which regards each element of the bullet as coming up to the end of the rod with its velocity $v_0$ unimpaired and there suffering instant

* The value of $E$ for the mild steel of which the pieces were made was found to be $3.00 \times 10^7$ lbs. per square inch. The density was 482 lbs. per cubic foot. Both determinations are probably right within 1 per cent. The velocity of propagation $\sqrt{\frac{E}{\rho}}$ is 17,000 feet per second.
stoppage, the pressure at any time is \( \lambda v_e^2 \) where \( \lambda \) is the mass per unit length at the section which is undergoing stoppage at the time. The pressure-time curve, calculated in this way, is shown in fig. 7, in which the ordinates are proportional to the values of \( \lambda \). This is the same curve as that giving the distribution of mass along the length of the bullet, the abscissa scale being such that the length of which the impact is complete is equivalent to the time required by the bullet to travel its own length (1.25 inches) at a velocity of 2000 feet per second. This is \( 5.2 \times 10^{-5} \) seconds. The maximum pressure corresponds to the maximum value of \( \lambda \) (0.35 lbs. per foot) and is

\[
\frac{0.35 \times 2000 \times 2000}{32.2} = 43,500 \text{ lbs.}
\]

which is \( 2\frac{1}{2} \) per cent. in excess of the value found by experiment. This difference is no more than can be accounted for by errors of observation.

The momenta which should according to theory be taken up by various lengths of piece are readily calculated from this curve. For instance, that corresponding to a 3-inch piece is the area ABCDE. The following table shows the results so obtained with the corresponding observed values. The momenta are reckoned as percentages of the total:

<table>
<thead>
<tr>
<th>Length of piece</th>
<th>Percentage momentum in piece</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Calculated.</td>
</tr>
<tr>
<td>inches</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>65</td>
</tr>
<tr>
<td>4</td>
<td>84</td>
</tr>
<tr>
<td>5</td>
<td>98.5</td>
</tr>
<tr>
<td>6</td>
<td>100</td>
</tr>
</tbody>
</table>

3 L 2
The differences between the calculated and observed figures in this table are probably rather outside experimental errors. Especially is this the case as regards the 5-inch and 6-inch pieces. The impact seems to last appreciably longer than it ought.

The Effect of the Rigidity of the Bullet.

In the simple theory it is assumed that the bullet is absolutely fluid. In fact, it possesses a certain rigidity, partly because of the nickel casing and partly because of the viscosity of the lead the effects of which may be quite appreciable at such high speeds of deformation. The general effect of rigidity may be represented by saying that any section of the bullet requires to be subjected to an end-pressure $P$ before it begins to deform at all, and this pressure must act across the section CC (fig. 4) where deformation is just beginning and where, if the bullet were really fluid, there would be no pressure. To a first approximation, $P$ will be proportional to the area of the cross-section of the bullet which is undergoing deformation, that is to $\lambda$ the mass per unit length in the plane CC. The pressure $P$ is added to that due to the destruction of momentum, making a total pressure $P + \lambda v^2$ where $\lambda$ is the mass per foot of the section of the bullet in the plane CC, $v$ the velocity of that section. Further, the part of the bullet behind CC is being continually retarded by the pressure $P$, with the result that the hinder parts do not come up with unimpaired velocity $v_0$, as they would if the bullet were quite fluid, but with a diminishing velocity.

The general effect of this is obvious. In the early stages of the impact there has not been time for much retardation, and the pressure will be increased above the theoretical value by nearly the amount $P$. As the hinder parts come up, however, with less and less velocity, the fluid pressure term diminishes until the pressure falls below the theoretical value in spite of the rigidity term $P$. Applying this correction to a pressure curve such as that in fig. 7 in which the maximum pressure occurs somewhat late in the impact, it will be seen that the general effect will be to reduce that maximum, and also to make it flatter. Furthermore, since the tail of the bullet takes longer to reach the end of the rod, the impact will be prolonged beyond the theoretical time.

It is easy to get a rough idea of the magnitude of these effects. Assume that the bullet is cylindrical and of mass $\lambda$ per unit length and that the deforming pressure is constant. Let $x$ be the length of the bullet behind the plane CC (fig. 4). This portion is moving as a rigid body with acceleration $\ddot{x}$ and its equation of motion is

$$\lambda x \ddot{x} = -P,$$

which integrates in the form

$$\frac{1}{2} \dot{x}^2 = -\frac{P}{\lambda} \log x + \text{const.}$$

If $l$ be the length of the bullet and $v_0$ its velocity on striking, and if we neglect the
small distance between the plane CC and the end of the rod, the constant of integration is

\[ \frac{1}{2}v_0^2 + \frac{P}{\lambda} \log l, \]

and we have

\[ 1 - \frac{x^2}{v_0^2} = \frac{2P}{\lambda v_0^2} \log \frac{l}{x}. \]

From this \( \dot{x} \) can be plotted in terms of \( x \), and thence in terms of \( t \). The total pressure \( P + \lambda \dot{x}^2 \) is then plotted in terms of the time.

As an example, take \( \lambda = 0.35 \) lbs. per foot, \( l = 1.05 \) inches which correspond to a bullet having the same mean density diameter and total mass as those used in the experiments. The pressure required to stop such a bullet at 2000 feet per second, if fluid, would be constant and equal to 43,500 lbs. If \( P \) be taken as \( \frac{1}{2}P_0 \) of this, or 2170 lbs., and the curve plotted as described, it will be found that when \( x = 0.3l \) the hydrodynamical pressure \( \lambda \dot{x}^2 \) has dropped 12 per cent. making, after allowing the addition of 5 per cent. for the rigidity, a nett drop of 7 per cent. Furthermore, the momentum still left after a fluid bullet would have been completely set up is about 4 per cent. of the whole.

If corrections of this amount were applied to the calculated figures in the last section, the effect would be to make the observed maximum pressure about 4 per cent. too high, while the observed time of impact would be still slightly too long. It was found that to crush the cylindrical part of the service bullet in a testing machine required an end pressure of about 1800 lbs., but the nickel casing failed by buckling, whereas in the impact it apparently bursts and is torn into strips along the length of the bullet. The pressure required to deform the bullet in the latter case, after rupture is once started, is probably less than 2000 lbs. Thus, while the difference between the observed and calculated times of impact may undoubtedly be referred in part to rigidity, it is unlikely that the whole can be accounted for in this way.

Discussion of Errors Inherent in the Method of Experiment.

In calculating the pressure from the momentum in the piece which is thrown off the end of the rod it is assumed that the pressure wave transmitted along the rod represents exactly the sequence of pressures applied at the end, that it travels along the rod and through the joint without change of type, and that it is perfectly reflected at the other end. These assumptions are correct if the wave is long compared with the diameter of the rod, and if the pressure is uniformly distributed over the end, but are subject to certain qualifications in so far as these conditions are not fulfilled.

(a) Effect of Length of the Rod.—The mathematical theory of the longitudinal oscillations of a cylinder shows that a pressure wave of simple harmonic type is propagated without change, but the velocity of propagation depends on the wavelength. Because of the kinetic energy involved in the radial displacements, which is
negligible when the wave is long compared with the diameter, the velocity diminishes with the wave-length. If the wave-length be $\frac{2\pi}{\gamma}$, and if the radius of the cylinder be $a$, the velocity is $\sqrt{\frac{\gamma}{\rho}}(1-\frac{1}{4}\pi^2\gamma^2a^2)$ correct to the square of $\gamma a$. In a wave of any form, the simple harmonic components move with different velocities, and the wave accordingly changes its form as it progresses.

Rough calculation of this effect on waves generally similar in form to that produced by the blow of the bullet, but of periodic character, showed that the change should not be very serious with rods of the lengths and diameters used in these experiments. It was, however, thought advisable to check this inference by direct experiment, and trials were therefore made with a rod 15 inches long and 1 inch diameter. The small mass of this rod precluded its use as a ballistic pendulum suspended in the ordinary way, it was therefore arranged to slide in bearings and to compress a spring buffer. Difficult questions arose as to the precise allowance which should be made for the kinetic energy given to the spring (which was of considerable mass) by the rod, and no attempt was therefore made to get an accurate measure of the total momentum. Instead of taking the fraction of this total which was trapped in the piece, the absolute values of the momenta so trapped were taken in a series of shots, in each of which, from the accuracy of the aiming and the absence of cupping in the end, it might be assumed that the total momentum was approximately equal to the average. The results are shown in the following table and are compared with the corresponding figures obtained with the long rod:

**Rod, 1 inch diameter. 2000 feet per second.**

<table>
<thead>
<tr>
<th>Length of piece</th>
<th>Number of shots</th>
<th>Momentum given to piece.</th>
<th>Mean</th>
<th>Maximum</th>
<th>Minimum</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>inches</td>
<td></td>
<td></td>
<td>Short rod (15 inches).</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>7</td>
<td>6.5</td>
<td>6.8</td>
<td>6.4</td>
<td>6.7</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>13.3</td>
<td>13.9</td>
<td>12.8</td>
<td>13.5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>26.5</td>
<td>26.8</td>
<td>26.2</td>
<td>26.4</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>49.3</td>
<td>51.2</td>
<td>48.6</td>
<td>48.8</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>60.2</td>
<td>61.3</td>
<td>59.1</td>
<td>57.2</td>
<td></td>
</tr>
</tbody>
</table>

It is clear from these figures that there is no systematic difference between the results obtained with the two rods. The change, if any, between the forms of the wave when at 15 inches and at 45 inches from the end consists in a shearing of the

DETONATION OF HIGH EXPLOSIVES OR BY THE IMPACT OF BULLETS.

whole curve as in the manner illustrated in fig. 3. Such a change of form—analogous to the change preparatory to breaking which a wave experiences as it advances into shallower water—would not be detected by these experiments, and it is not impossible that it occurs to some extent.

(b) Reflection and Effect of the Joint.—The simple harmonic pressure-wave which is propagated without change of type is accompanied by a distribution of shearing-stress across the section. This shearing-stress depends on the square of the ratio \( \gamma a \), and is small. That it plays no important part in these experiments is shown by the fact that if there be a joint in the long rod the results are unaltered. Such a joint transmits the pressure, but stops the shearing-stress part of the wave. As might be expected, it was found that the faces of the joint must be a carefully scraped fit if the wave is to pass it unaltered.

The small magnitude of the shearing-stress is the foundation of the assumption that the wave is perfectly reflected at the free end. Strictly accurate reflection is not possible. A reflected wave which is exactly the same as the incident wave, except that the signs of all the stresses are reversed, will when combined with the incident wave give no normal force over the free end. The shearing-stresses corresponding to the two waves do not, however, neutralise each other, but are added, hence accurate reflection can only be brought about by the application of a distribution of shear over the free end. The shear required is, however, of the order \( \gamma a^2 \) and the experiment with the joint shows that its effects may be neglected.

(c) Effect of the Diameter of the Rod.—The pressure exerted by the bullet is confined to a comparatively small area in the centre of the end; whereas the pressure-wave travelling without change of type implies a nearly uniform distribution of pressure over the section. The question of the nature of the wave developed under such conditions seemed quite intractable mathematically, but from general considerations it appeared probable that it would not differ greatly from that of the wave originated by a uniform pressure distribution. In order to test this point

2000 feet per second.

<table>
<thead>
<tr>
<th>Length of piece</th>
<th>Percentage of momentum in piece</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \frac{4}{8} ) inch.</td>
</tr>
<tr>
<td>inches</td>
<td></td>
</tr>
<tr>
<td>0·5</td>
<td>10·8</td>
</tr>
<tr>
<td>1·0</td>
<td>21·1</td>
</tr>
<tr>
<td>2·0</td>
<td>—</td>
</tr>
<tr>
<td>3·0</td>
<td>61·3</td>
</tr>
<tr>
<td>4·0</td>
<td>79·5</td>
</tr>
<tr>
<td>5·0</td>
<td>92·5</td>
</tr>
<tr>
<td>6·0</td>
<td>—</td>
</tr>
</tbody>
</table>
comparative tests were made with rods of \( \frac{3}{4} \) inch, 1 inch, and \( 1\frac{1}{2} \) inch diameter. The lengths of the rods were roughly 48 inches, 43 inches, and 30 inches, respectively. The results are exhibited in the table on p. 447, in which the figures for the 1-inch rod are the same as those already given.

It will be seen that the diameter of the rod has no appreciable effect up to a length of 4 inches, but that for greater lengths the large rod gives appreciably lower values. In other words the apparent maximum pressure is not much affected by the diameter, and is presumably correctly given by all three rods, while the duration of the blow is largely overestimated by the \( 1\frac{1}{2} \)-inch rod, and presumably somewhat overestimated by the other two, though as they are in substantial agreement on this point the error cannot be very large. It may be surmised that some at any rate of the difference between the observed and calculated times of impact is due to this cause, though, as already pointed out, the rigidity of the bullet is competent to account for part of it.

**Experiments at Lower Velocities.**

Measurements were also made with cartridges giving velocities of about 1240 feet per second and 700 feet per second respectively, the same types of bullet being used. The results for the 1240 feet per second cartridges are exhibited in the following table, which corresponds to that already given on p. 442 for the 2000 feet per second cartridges:

| Rod, 1 inch diameter, about 40 inches long. Velocity of bullets 1240 feet per second. (Mean of 5 shots: maximum 1257, minimum 1229.) |

<table>
<thead>
<tr>
<th>Length of piece</th>
<th>Number of shots</th>
<th>Percentage of total in piece</th>
<th>Total momentum in rod and piece</th>
</tr>
</thead>
<tbody>
<tr>
<td>0·5</td>
<td>1</td>
<td>12·9</td>
<td>12·3</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>26·7</td>
<td>25·8</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>38·4</td>
<td>37·5</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>51·6</td>
<td>50·6</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>63·0</td>
<td>61·5</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>67·7</td>
<td>67·5</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>89</td>
<td>81·5</td>
</tr>
</tbody>
</table>

The mean total momentum registered (37 shots) is 37·7 units; the calculated total is \( 1240 \times 0.0306 = 38 \) units.

The percentage figures are plotted in fig. 6 (curve marked "1240 feet per second").
The percentage of momentum trapped by short pieces is 13 per inch, and the corresponding maximum pressure for the normal velocity of 1240 feet per second is

\[
\frac{0.13 \times 38}{32.2 \times 0.98 \times 10^{-5}} = 15,700 \text{ lbs.}
\]

The maximum pressure which should be exerted by a perfectly fluid bullet having the same mass and velocity is

\[
\frac{0.35 \times (1240)^2}{32.2} = 16,700 \text{ lbs.}
\]

The time taken by the bullet to travel its own length is \(8.4 \times 10^{-4}\) seconds. Thus if the bullet were perfectly fluid, the whole momentum should be trapped in a piece 9 inches long, whereas in fact only 86 per cent. is so trapped. The errors inherent in the method of experiment, which have been discussed in the last section, will all be less at the lower velocity. On the other hand the rigidity of the bullet will be relatively more important and probably suffices to account for much of the difference between the theoretical and observed times of impact.

The 700 feet per second bullets showed a maximum pressure of 5450 lbs., as compared with 5320 lbs. calculated. \(54\frac{1}{2}\) per cent. of the momentum was trapped by a 9-inch piece. It was not possible to experiment with longer pieces, so that the time of impact in this case could not be determined.

It should be observed here that just after the piece has been shot off it tends to pull the rod after it by magnetic attraction, which of course still continues after the joint is broken, though it diminishes rapidly as the distance between piece and rod widens. The effect of this is to give more momentum to the rod and less to the piece than they would respectively possess as the effect of the blow alone. By measuring the amount of the magnetic pull when the piece is held at different distances from the rod, the current in the solenoid being the same as that used in the impact experiment, it is possible to estimate the amount of this effect. With 2000 feet per second bullets it is quite negligible, but when the velocities are lower particularly with long pieces, it necessitates a correction. This correction has been applied in the figures given above for the 1240 feet per second and 700 feet per second bullets.

**Detonation of Gun-Cotton.**

It is well-known that a charge of 1 lb. gun-cotton will shatter a mild steel plate 1 inch thick or more, if it be detonated in firm contact with it. The fracture is quite "short," like that of cast-iron, though the broken pieces are usually more or less deformed. Typical fractures of this kind obtained on plates of very good mild steel are illustrated in figs. 8, 9, 10, and 11. Figs. 8 and 9 are photographs of a plate 1\(\frac{1}{4}\) inches thick originally quite flat. It was broken by a slab of gun-cotton weighing 1 lb. which covered the section of the plate AB and was detonated in contact with
that which became the convex face (lower face in fig. 9). Fig. 10 is a view of the broken edge of one of the two fragments. The plate shown in fig. 11 was a flat piece of boiler plate 1\(\frac{1}{4}\) inch thick. A slab of 1 lb. of gun-cotton was detonated against that which is the under side in the figure and the two pieces subsequently fitted together again and photographed. Thinner plates—e.g., 1 inch thick—are usually cracked in two places, one at each edge of the gun-cotton slab, and the portion covered by the slab is blown out of the plate, sometimes whole and sometimes shattered into pieces. The fact that no tamping is necessary suggests that the duration of the process of detonation is of the same order as the time taken by sound to travel an inch or less in air, so that during the conversion of the cotton into gas there is not time for much expansion.* If this be so, the maximum pressure

* The velocity of detonation of long trains of gun-cotton has often been measured and is variously estimated at 18,000 to 20,000 feet per second. If the same velocity obtained in the small primers they would be completely converted into gas in about \(2 \times 10^{-6}\) secs.
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developed must be that which would be reached if the cotton were fired in a closed chamber of a volume not greatly exceeding that of the slab. The pressure is then dissipated with great rapidity by the expansion of the gas, which is resisted only by its own inertia and that of the surrounding air.

Experiments on the detonation of gun-cotton have been made by the method described in this paper. It has only been possible hitherto to use quite small charges and the results are a very rough approximation, but as they throw light on a matter

Fig. 12.

of which little is known I have thought it worth while to give them. Briefly, the conclusion is that the pressure at a point distant \( \frac{3}{4} \) of an inch from the surface of one ounce of dry gun-cotton (a cylindrical "dry primer" about 1\( \frac{1}{4} \) inch diameter and 1\( \frac{1}{4} \) inches long), when detonated with fulminate, has fallen to less than \( \frac{1}{3} \) of the maximum value within \( 2 \times 10^{-5} \) seconds. At least, 80 per cent. of the blow has been delivered within that time. Over an interval of \( 10^{-5} \) seconds round about the time of maximum pressure the average pressure is about 30 tons per square inch, and the
actual maximum is probably of the order of 40 tons per square inch. At a point on the surface the maximum pressure is at least twice as great, 80 tons per square inch.*

The arrangements are shown in fig. 12.

The gun-cotton cylinder A is fixed by short splints of wood opposite the end of the shaft B, which is of mild steel 1½ inches diameter and from 15 to 30 inches long. This shaft is suspended as a ballistic pendulum with a pencil and paper for recording its movement. The end piece C, from ½ to 6 inches long, is held on by magnetic attraction. The faces of the joint are a scraped fit. In line with the shaft is the box D, which is also suspended as a pendulum and provided with a recording pencil. Some part of the momentum given to the box is due to the blast from the gun-cotton; this

* The pressure developed by the explosion of gun-cotton in a vessel which it completely fills does not appear to have been measured. From measurements made with charges of lower density Sir Andrew Noble estimates that it would be about 120 tons per square inch (‘Artillery and Explosives,’ p. 345). Allowing for the partial expansion during the process of detonation, this agrees fairly well with the pressure here determined.

† In these cases the air space between the gun-cotton and the end of the shaft was 1 inch. In all the others it was ½ inch.
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was estimated from experiments in which there was no piece on the end of the shaft. Separate experiments were also made to determine the effect of the blast on the supports of the shaft. The momentum accounted for by the blast is in each case deducted from the total recorded momentum to get the nett momentum due to the blow on the end of the shaft. This correction in the case of the box amounted to about 8.3 units with a 15-inch shaft, and 1.2 units with a 30-inch shaft. The correction for the blast on the supports of the shaft was 5 units.

The table on p. 452 gives the results of all the trials made with the gun-cotton about \( \frac{3}{4} \) inch from the end of the shaft.

The total impulse of the blow when the air space is \( \frac{3}{4} \) inch varies from about 35 to 70 units, the average being about 46 units. The percentages absorbed by the different end-pieces are, however, more nearly constant, and from them a rough approximation to the pressure wave transmitted by the rod in an average case may be constructed. As already explained the precise form of this curve depends on the way in which the pressure rises, but it may be assumed in this case that the pressure reaches its maximum in a time that is short even in comparison with the duration of the blow. Assuming an average total momentum of 45 units, fig. 13 has been constructed. The area of the parallelogram marked 1 represents the momentum given to a 1-inch piece, the width of this parallelogram is \( 10^{-5} \) seconds and the height is the average pressure acting during the first \( 10^{-5} \) seconds. The parallelogram marked 2 represents the excess of the momentum given to the 2-inch piece over that given to the 1-inch piece and its height is the average pressure acting during the second \( 10^{-5} \) seconds. The dotted curve gives the same average pressures over the successive intervals of time. It is obviously largely conjectural, but it gives a rough idea both of the maximum pressure and of the duration of the blow.

The chief difficulty experienced hitherto in measuring by this method the pressures developed in the detonation of gun-cotton has been the permanent deformation of the end of the rod by the blow. No steel has yet been discovered which will stand, without flowing or cracking, the detonation of gun-cotton in contact with it, and even when a
cushion of air $\frac{3}{4}$ inch thick is interposed some flow takes place.* In consequence of this, the pressure wave which emerges and is propagated elastically cannot be quite the same as the wave of applied pressure. It is easy to see that the general effect of the setting up of the end must be to deaden the blow, that is to reduce the maximum pressure and prolong its duration. In fig. 14, A is the (conjectural) curve representing

![Fig. 14.](image)

the pressure applied to the end of the rod. If the rod were perfectly elastic, the pressure across a section 2 inches from the end would be represented on the same time base by the curve B, which is the same as A, but moved $10^{-5}$ seconds to the right. The momentum in the end 2 inches at any time is the difference between the areas of the curves up to that time. For instance at $2 \times 10^{-5}$ seconds it is represented by the shaded area under curve A. But if the end be not completely elastic, the higher pressures developed over section B will be less than those acting on the end at corresponding times. Thus the record of pressure over the section 2 inches from the end will be a curve such as B' and the momentum in the end two inches at any time will be greater than it would be if the end were elastic by the difference between the areas of curves B and B' which is double shaded in the figure. This extra momentum is transferred to the remainder of the rod later on, causing the curve B' to rise above B. The curve B' represents the wave of pressure actually sent along the rod. It is this curve which is determined by the method which has been described, and it is evident that that method under-estimates the maximum pressure and over-estimates the duration of the blow.

A few experiments were made with the gun-cotton touching the end of the shaft. The average total momentum given to the shaft and piece in this case is about 90 units or roughly twice as great as that transmitted through $\frac{3}{4}$-inch air-space. Of this total about 80 per cent. is caught in a piece 4 inches long, and about 50 per cent. in a piece 1 inch long. When the gun-cotton is at a distance of $\frac{3}{4}$-inch these figures are 90 and 60 respectively. The apparent duration of the pressure is therefore rather greater at the surface of the explosive. The setting up of the end of the shaft is, however, much more marked when the gun-cotton is in contact and it may

* This is when the steel is in the form of a shaft, so that there is no lateral support of the part subjected to pressure. It is, of course, possible to make a plate with hardened face which will withstand the attack of gun-cotton on a portion of the face.
be that the distribution of the pressure in time is not materially different in the two cases. If that were so, the maximum pressure developed on the surface of the gun-cotton would be 80 or 100 tons per square inch.

It is hoped that by the use of special steels it may be possible to give greater precision to these estimates of the amount and duration of the pressure produced by the detonation of gun-cotton in the open. Meanwhile the information already obtained as to the order of magnitude of these quantities is sufficient to throw some light on the nature of the fractures produced. The general result obtained may be expressed by saying that a gun-cotton cylinder 1½ inches × 1½ inches produces at its surface, when detonated, pressure of the average value of 100,000 lbs. per square inch lasting for \( \frac{1}{50,000} \) second. Probably figures of the same sort of magnitude will describe the blow produced by the detonation of a slab 1½ inches thick, one of whose faces is in contact with a steel plate. It may be that the pressure is greater and the duration correspondingly less, but this does not affect the point that the pressure is an impulsive one in its effect on the plate. That is, the effect of the pressure is to give velocity to the parts of the plate with which the gun-cotton is in contact but the pressure disappears before there has been time for much movement to take place. For instance, if the plate be 1 inch thick (mass 0.28 lbs. per square inch) a pressure of 100,000 lbs. per square inch acting on it for \( \frac{1}{50,000} \) second will give a velocity of about 230 feet per second, and while the pressure is being applied it will move 0.028 inches.

The parts of the plate not covered by the gun-cotton are left behind and the strain set up by the forced relative displacement is the cause of the shattering of the plate. The magnitude of this strain, and of the consequent stress, depends (speaking generally), on the relation between the velocity impressed on the steel by the explosion and the velocity of propagation of waves of stress into the material. For instance, if the section AB (fig. 15) be given instantaneously a velocity of 200 feet per second and this velocity be maintained, the state of the plate after the lapse of \( \frac{1}{100,000} \) second will be that represented diagrammatically by fig. 15. The section AB has moved forward relatively to the remainder by 0.002 feet. As soon as this section started moving a wave of shear stress started out from A into the parts of the plate to the left which had been left at rest by the blow. This wave travels in steel at 11,000 feet per second and will therefore in \( \frac{1}{100,000} \) second get to C when AC = 0.11 feet. To the left of C the metal has not moved, the wave not having reached it; therefore the average shear in the section AC is

\[
\frac{0.002}{0.11} = 0.018.
\]

Under forces of this duration even mild steel has nearly perfect elasticity up to very
high stresses.* If it maintained its elasticity and continuity the shearing stress
would be of the order $0.018 \times 12 \times 10^6$, or say 220,000 lbs. or 100 tons per square inch.
This illustration is of course very far from representing the actual effect of suddenly
giving velocity to a portion of a plate, the real distribution of stress would be far
more complicated, but it gives an idea of the magnitude of the stresses which may be
expected to arise. In static tests on mild steel, the material begins to flow as
soon as the shearing stress exceeds about 10 tons per square inch and no stress
materially greater than this can exist. But when the metal is forcibly deformed at a
sufficiently high speed the shearing stress is increased by something analogous to
viscosity and the tensile stress which accompanies it may be sufficient to break down
the forces of cohesion and tear the molecules apart. Thus the steel is cracked, though
in ordinary static tests it can stretch 20–30 per cent. without rupture, just as pitch,
which can flow indefinitely if given time, is cracked by the blow of a hammer. The
essence of the matter is the forcible straining of the substance at a velocity so high
that it behaves as an elastic solid rather than as a fluid, thus experiencing stresses
which are measured by the strain multiplied by the modulus of elasticity. The effect
of gun-cotton on mild steel shows that in this material a rate of shear of the order
1000 radians per second is sufficient to cause cracking.

The most probable account of the smashing of a mild steel plate by gun-cotton is,
then, that the plate is cracked before it has appreciably deformed, the cracks being
caused by relative velocity given impulsively to different parts of the plate. Bending
of the broken pieces occurs after the plate has cracked and the pieces have
separated from one another. It is due to relative velocity in different portions of each
piece which still persists after the initial fracture, and is taken up as a permanent set
in each piece. In this connection the fracture shown in fig. 11 is instructive. It
will be noticed that the general bend of the plate, after the pieces have been fitted
together, is opposite to that which might at first sight be expected as the result of
the blow in the middle. Inspection of such fractures leads to the conclusion just
stated as to their history. The experiments on gun-cotton pressures described in this
paper, though lacking in precision, supply I think the missing link in an explanation
which is otherwise probable, namely, sufficient evidence that the blow may be regarded
as an impulsive force communicating velocity instantaneously.

Most of the experimental work described in this paper was done by my assistant,
Mr. H. Quinney. I also received valuable help in the earlier stages from
Mr. A. D. Browne, of Queens' College, and from my brother Mr. R. C. Hopkinson,
Trinity College. To these gentlemen I wish to express my obligation for aid without
which it would hardly have been possible to carry out a research of this character. I
have also to thank Sir Robert Hadfield, Mr. W. H. Ellis, and Major Strange for
providing steel plates and shafts.

XI. Gravitational Instability and the Nebular Hypothesis.

By J. H. Jeans, M.A., F.R.S.

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Introduction.

§ 1. A consideration of the processes of cosmogony demands an extensive knowledge of the behaviour of rotating astronomical matter. What knowledge we have is based upon the researches of Maclaurin, Jacobi, Poincaré, and Darwin. These researches refer solely to matter which is perfectly homogeneous and incompressible, although it is, of course, known that the primordial astronomical matter must be far from homogeneous and probably highly compressible as well. The question of how far we are justified in attributing to real matter the behaviour which is found to be true for incompressible and homogeneous matter is obviously one of great importance.

§ 2. There are a priori reasons for expecting that there will be wide differences between the two cases. Consider first a sphere of homogeneous incompressible matter devoid of rotation. This will be stable if every small displacement increases (or, at least, does not decrease) its potential energy. The sphere has a number of independent possible small displacements which can be measured by the number of harmonics which can be represented on its surface. The spherical configuration is known to be stable because it can be shown that every one of these displacements increases the potential energy.

Contrast this case with the corresponding one in which the matter is compressible. The number of possible small displacements in this latter case is measured by the sum of the numbers of harmonics which can be represented on all the spherical surfaces inside the sphere. Let R be the radius of the outer surface; let \( r, r', r'', \ldots \) be the radii of all the spheres which can be drawn inside this outer sphere, and let \( r_n, r'_n, r''_n \ldots \) \( R_n \) be the number of independent harmonics which can be represented on these spheres. To prove that the sphere is stable it is now necessary to prove that every one of the \( r_n + r'_n + r''_n + \ldots \) \( R_n \) possible displacements increases the potential energy. If we argue by analogy from the case of an incompressible sphere we are, in effect, merely considering \( R_n \) of these displacements and neglecting the much greater number \( r_n + r'_n + r''_n + \ldots \). Furthermore, in these neglected displacements, the nature of the displacement is essentially different from that in the
R\textsubscript{a} displacements, so that there appears to be no justification at all for an argument from analogy.

In each of the neglected displacements, the change in the potential energy will consist of two terms. There will be a change in the elastic energy of the compressible material, and this can be easily shown to involve an increase in the potential energy. There will, in addition, be a change in the gravitational energy, and this can be shown\textsuperscript{*} to involve invariably a \textit{decrease} in the energy. If \( W, E, G \) denote the total, the elastic, and the gravitational potential energies,

\[ \delta W = \delta E + \delta G, \]

in which \( \delta G \) is invariably negative. The condition for stability is that for every possible displacement \( \delta E \) shall be numerically greater than \( \delta G \).

It might naturally be thought that by considering a system in which the matter was, so to speak, very gravitational or very little elastic we could have \( \delta E \) small or \( \delta G \) great, and so should have instability of the spherical configuration. But it must be remembered that the gravitation and the elasticity of the matter are not independently at our disposal. The action of the gravitational forces tends to concentrate the matter and so involves that the elasticity becomes large in the equilibrium configuration. If we consider a system in which the elasticity is artificially kept small, as, for instance, by adding an additional repulsive field of force to annul, or partially annul, the gravitational field, we can easily construct systems for which a spherical configuration is unstable,\textsuperscript{†} but, short of this, it appears to be a general law that the elastic and gravitational agencies must march together in such a way that \( \delta E \) is always numerically greater than \( \delta G \),\textsuperscript{‡} so that every natural spherical system is stable.

The nearest approach in nature to the artificial repulsive field imagined above is found in the influence of rotation. This influence may be represented by the superposition of the usual repulsive field of centrifugal force of potential \(-\frac{1}{2} w^2 (x^2 + y^2)\). The field is not spherical, and so the figures of equilibrium obtained under its influence cannot be spherical. But it can be regarded as made up of a spherical part \(-\frac{2}{3} w^2 r^2\) and a superposed harmonic disturbance \( \frac{2}{3} w^2 P v^2 \). The first term is certainly a spherical repulsive field, and will, of course, tend to annul the concentrating influence of gravitation. The problem which requires study is that of how far, or in what circumstances, the presence of rotation can disturb the otherwise general law that \( \delta E \) is always greater than \( \delta G \).

The problem is one of enormous complexity and great generality. It will hardly be expected that the present paper will contain anything approaching a general

\‡ Cf. below, §§ 11, 22.
solution, and it may as well be stated at once that it does not. All I have been able
to do is to grope after general principles by solving a problem here and a problem
there as seemed needful to illuminate a possible path towards a general theory, and
the present paper is confined to a very few of the special problems I have considered,
but I have selected those which seemed to have most bearing on the general question
in hand.

Medium in which the Pressure is a Function of the Density.

§ 3. In the most general astronomical medium the pressure is, of course, not a
function of the density. The relation between pressure and density varies from
point to point, partly on account of inequalities of temperature and partly on account
of variations of chemical constitution. But no general theory can be expected to
apply to the most general heterogeneous mass of matter possible, and before any
general theory can be deduced we must have material from which to deduce it.

§ 4. The simple system from which we shall start will be a system in which the
matter is homogeneous as regards its properties, so that at all points the pressure
and density will be connected by the same relation. It will be seen later (§ 15) how
it is possible, in at least one important respect, to escape from this limitation.

For the present we assume the pressure and density to be connected by the
relation

\[ p = f(\rho) \]  

at every point. We take the centre of gravity of the rotating mass to be the origin,
and the axis of rotation to be the axis of \( z \). The equations of equilibrium are

\[ \frac{\partial p}{\partial x} = \rho \frac{\partial V}{\partial x} + w^2 \rho x, \]
\[ \frac{\partial p}{\partial y} = \rho \frac{\partial V}{\partial y} + w^2 \rho y, \]
\[ \frac{\partial p}{\partial z} = \rho \frac{\partial V}{\partial z}, \]

in which \( V \) is the potential of the whole gravitational field of force. In virtue
of relation (1), these equations have the common integral

\[ \int \frac{dp}{\rho} = V + \frac{1}{2} w^2 (x^2 + y^2) + C. \]  

or

\[ \phi (\rho) = V + \frac{1}{2} w^2 \sigma^2 + C, \]

in which \( \sigma^2 \) stands for \( x^2 + y^2 \), and \( \phi (\rho) \) for \( \int \frac{dp}{\rho} \), which is by hypothesis a function of

\[ 3 N 2 \]
\( \rho \) only. There is further the relation of Poisson,
\[
\nabla^2 V = -4\pi \rho, \quad \ldots \quad \ldots \quad \ldots \quad (4)
\]
so that on operating on (3) with \( \nabla^2 \) we obtain
\[
\nabla^2 \phi (\rho) + 4\pi \rho = 2w^2, \quad \ldots \quad \ldots \quad \ldots \quad (5)
\]
the differential equation which must be satisfied by \( \rho \) in any configuration of equilibrium under a rotation \( w \).

§ 5. In general a solution of equation (5) will involve negative and zero values of \( \rho \). In the physical problem \( \rho \) will be limited as to values, and this limitation will determine the physical boundary of the rotating mass.

Let \( V_m \) denote the gravitational potential at any point in space of the finite mass determined in this way. We have found a configuration of equilibrium under a potential \( V \), the potential of the mass is \( V_m \), so that for equilibrium we require an additional field of potential \( V - V_m \). We can say that the configuration found will be a true configuration of equilibrium under an external field of force of potential \( V_0 \) such that
\[
V = V_0 + V_m, \quad \ldots \quad \ldots \quad \ldots \quad (6)
\]
And, inasmuch as \( \nabla^2 V_m = -4\pi \rho = \nabla^2 V \), it is clear that \( \nabla^2 V_0 = 0 \), so that the external field has poles only at the origin or at infinity. The condition that any solution shall lead to a configuration of equilibrium for a mass rotating free from external influence is, of course, \( V_0 = 0 \).

§ 6. The simplest solution of equation (5) is obviously that in which \( \rho \) is a function of \( r \) only, but it is clear from (3) that this cannot give a free solution except when \( w = 0 \).

§ 7. The next simplest form of solution is that in which \( \rho \) is a function of \( z \) and \( \varpi \) only, and this can give a free solution. It includes, of course, as a particular case the system of Maclaurin spheroids. For this class of solutions every section at right angles to the axis of \( z \) is circular, and in any such section the lines of equal density are circles. The density at any point is of the form \( \rho = f (\varpi, z) \).

Let \( \theta \) denote colatitude measured from \( \Omega z \), and let \( \psi \) be azimuth measured from the plane of \( xz \). The most general configuration which can be obtained by displacement of that just considered will have a law of density of the form
\[
\rho = f_0 (\varpi, z) + \sum_{i=1}^{\infty} f_i (\varpi, z) \cos s \psi.
\]

It is easily seen that the separate cosine terms lead to independent displacements, and we shall for the moment only consider the displacement of the first order, for which the law of density is
\[
\rho = f_0 (\varpi, z) + f_1 (\varpi, z) \cos \psi, \quad \ldots \quad \ldots \quad \ldots \quad (7)
\]
where \( f_1 (\varpi, z) \) is a small quantity of the first order.
The boundary being a surface of constant pressure must also be a surface of constant density, say \( \sigma \). The equation of the boundary is accordingly

\[
\phi_0(\sigma, z) + f_1(\sigma, z) \cos \psi = \sigma.
\]

The whole mass inside this boundary may be regarded as composed of coaxial rings of matter as follows. Inside the figure of revolution \( f_0(\sigma, z) = \sigma \), we suppose there to be a series of rings of density given by (7), while the surface inequality can be regarded as represented by the presence of rings on this figure of revolution of density proportional to \( \cos \psi \).

On integration the potential \( V_m \) at any external point is seen to be of the form

\[
V_m = \chi_0 + \chi_1 \cos \psi,
\]

where \( \chi_0, \chi_1 \) are functions of \( \sigma \) and \( z \) only.

Suppose now that the surface is so nearly spherical that spherical harmonic analysis may be used with reference to it, then, since \( V_m \) is a solution of Laplace's equation at all external points, and is also of the form (9), it must be of the form

\[
V_m = \frac{A_0}{r} + \sum_{\mu} \left( \frac{A_\mu}{r^{\mu+1}} \right) P_\mu^1(\psi) \cos \psi,
\]

where \( \mu = \cos \theta \), and \( P_\mu^1(\psi) \) is the usual tesseral harmonic \( \frac{\partial}{\partial \theta} P_\mu(\psi) \). Moreover, since the centre of gravity of the mass is supposed to coincide with the origin, \( A_1 \) must vanish.

We have, from equation (3), if \( V = V_m \),

\[
V_m = \phi(\rho) - \frac{1}{2} w^2 \rho^2 - C
\]

\[
\phi \left\{ f_0(\sigma, z) \right\} + f_1(\sigma, z) \cos \psi \phi' \left\{ f_0(\sigma, z) \right\} - \frac{1}{2} w^2 \rho^2 (1 - P_2) - C
\]

at all internal points. Equating these two expressions, we must have at the boundary

\[
\frac{A_0}{r} = \phi \left\{ f_0(\sigma, z) \right\} - \frac{1}{2} w^2 \rho^2 - C,
\]

\[
f_1(\sigma, z) \phi' \left\{ f_0(\sigma, z) \right\} = \sum_{\mu} \left( \frac{A_{\mu}}{r^{\mu+1}} \right) P_\mu^1(\psi),
\]

or, neglecting small quantities of the second order,

\[
f_1(\sigma, z) \phi' (\sigma) = \sum_{\mu} \left( \frac{A_{\mu}}{r^{\mu+1}} \right) P_\mu^1(\psi).
\]

Hence either \( f_1(\sigma, z) \) vanishes at the boundary or is of at least the second order of harmonics.

It follows that if there can be a configuration of equilibrium which differs from the configuration of revolution \( \rho = f_0(\sigma, z) \), by a displacement proportional to the first
harmonic, this configuration must be one in which \( f_1(\omega, z) \) vanishes at the boundary, so that the boundary must be a figure of revolution about the axis of rotation.

It now follows from (5) that \( V \) must be a function of \( \omega \) and \( z \) only on the boundary, and hence also (since \( V \) is harmonic) at all external points. It follows that \( \partial V / \partial n \), and hence also \( \partial \rho / \partial n \), are functions of \( \omega, z \) only at the boundary. Whence again, by equations (4) and (5), it follows that \( \partial^2 V / \partial n^2 \) and \( \partial^2 \rho / \partial n^2 \) are functions only of \( \omega \) and \( z \) at the boundary. And, by successive differentiation of equations (4) and (5), it is seen that all the differential coefficients of \( V \) and \( \rho \) are functions only of \( \omega \) and \( z \) at the boundary.

It can be seen from this* that the configuration must be one of revolution throughout. In other words, there can be no configuration of equilibrium which differs from the configuration of revolution by first harmonic terms only.

LAPLACE'S Law.

§ 8. I have not found that any progress worth recording can be made with the general relation \( p = f(\rho) \), so that progress can only be hoped for by examining special cases.

The case that suggests itself as most important is that of the gas law \( p = \kappa \rho \), satisfied in a perfectly gaseous nebula at uniform temperature. The difficulty is that such a nebula extends to infinity in all directions, and so cannot rotate as a rigid body. Or rather, when it is caused to rotate, it throws off its equatorial portions and the remainder rotates in the shape of an elongated spindle of infinite length. In this connection I have worked out the purely two-dimensional problem of a rotating gaseous cylinder of infinite length. The results are too long to be worth printing; it will, perhaps, suffice to record that the analysis bears out in full the conclusions arrived at in this paper.

The law which is most amenable to mathematical treatment is LAPLACE's law

\[
p = c(\rho^2 - \sigma^2),
\]

or, as it is more convenient to write it,

\[
p = \frac{2\pi}{\kappa^2} (\rho^2 - \sigma^2), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)
\]

in which \( c, \rho, \kappa, \) and \( \sigma \) are constants, \( \sigma \) being the value of the density at the free

* I have not succeeded in obtaining a rigorous proof of this. It might be objected that nothing in the above argument precludes first harmonic terms proportional to such a function as \( e^{-1/\sqrt{f(\omega, z)}} \), where \( f(\omega, z) = 0 \) is the equation of the boundary. The pure mathematician may not, although the astronomer will, be influenced by the consideration that such functions never occur in natural problems. If such a function did occur, it would involve an extremely fantastic relation between \( p \) and \( \rho \).
surface. This law has the merit that the case of an incompressible fluid is covered by the special value $c = \infty$ or $\kappa = 0$, the density now having the value $\sigma$ throughout.

There is the \textit{\emph{a priori}} objection to the law that its form precludes first harmonic displacements (cf. below, §11). This objection would be fatal were it not that we have seen that first harmonic displacements are in any case of no importance. This being so, the objection falls to the ground, and I have thought it worth working out this law as far as possible.

§ 9. Using the relation (12), we have in place of the more general equations (3), (4), and (5), the particular equations

\[ \frac{4\pi}{\kappa^2} \rho = V + \frac{1}{2} \omega^2 \sigma^2 + C, \quad \cdots \cdots \cdots \cdots \quad (13) \]

\[ \nabla^2 V = -4\pi \rho, \quad \cdots \cdots \cdots \cdots \quad (14) \]

\[ \frac{4\pi}{\kappa^2} \nabla^2 \rho + 4\pi \rho = 2w^2. \quad \cdots \cdots \cdots \cdots \quad (15) \]

On putting

\[ \rho - \frac{u^2}{2\pi} = \chi, \quad \cdots \cdots \cdots \cdots \quad (16) \]

this last equation reduces to

\[ (\nabla^2 + \kappa^2) \chi = 0. \quad \cdots \cdots \cdots \cdots \quad (17) \]

\textit{No Rotation.}

§ 10. When there is no rotation \( w = 0, \chi = \rho \) and the equation becomes

\[ (\nabla^2 + \kappa^2) \rho = 0. \]

The general solution is

\[ \rho = \Sigma A_n r^{-\nu_i} J_{n+\nu_i}(\kappa r) S_n, \quad \cdots \cdots \cdots \cdots \quad (18) \]

while the particular solution giving a spherical boundary is

\[ \rho = A_0 r^{-\nu_i} J_{\nu_i}(\kappa r) = \frac{A_0}{\sqrt{2\pi} \kappa} \sin \kappa r, \quad \cdots \cdots \cdots \cdots \quad (19) \]

the last being, of course, the well-known solution which occurs in \textit{Laplace's theory} of the figure of the earth. It will now be shown that this configuration is stable for all displacements.

§ 11. Let \( r = \alpha \) be the free surface corresponding to the simple solution (19).

Consider an adjacent solution

\[ \rho = A_0 r^{-\nu_i} J_{\nu_i}(\kappa r) + A_0 r^{-\nu_i} J_{n+\nu_i}(\kappa r) S_n, \quad \cdots \cdots \cdots \cdots \quad (20) \]

and let the corresponding free surface be

\[ r = \alpha + b S_n. \quad \cdots \cdots \cdots \cdots \quad (21) \]
On substituting this value for \( r \) in \( (20) \), neglecting squares of \( b \) and equating corresponding harmonic terms, we obtain

\[
\sigma = A_0 a^{-\frac{1}{2}} J_{\frac{1}{2}} (\kappa \alpha) = \frac{A_0}{\sqrt{2\pi}} \frac{\sin \kappa \alpha}{\alpha}, \quad \ldots \quad \ldots \quad (22)
\]

\[
A_0 a^{-\frac{1}{2}} J_{\frac{n+1}{2}} (\kappa \alpha) = -A_0 b \frac{d}{d\alpha} \left\{ a^{-\frac{1}{2}} J_{\frac{1}{2}} (\kappa \alpha) \right\} = \kappa A_0 b a^{-\frac{1}{2}} J_{\frac{1}{2}} (\kappa \alpha), \quad \ldots \quad \ldots \quad (23)
\]

whence

\[
b_\sigma = A_0 a^{-\frac{1}{2}} J_{\frac{n+1}{2}} (\kappa \alpha) \frac{J_{\frac{1}{2}} (\kappa \alpha)}{\kappa J_{\frac{1}{2}} (\kappa \alpha)}, \quad \ldots \quad \ldots \quad (24)
\]

By integration, the value of \( V_m \) at a point on the sphere \( r = \alpha \) is found to be

\[
V_m = \frac{4\pi}{\alpha^2} \int_0^\alpha A_0 p^{-\frac{1}{2}} J_{\frac{1}{2}} (\kappa \alpha) r^2 dr - \frac{4\pi}{(2n+1) \alpha} \int_0^\alpha \frac{A_0 S_n}{\alpha} p^{-\frac{1}{2}} J_{\frac{n+1}{2}} (\kappa \alpha) r^{2n+2} dr + \frac{4\pi \alpha_0 \sigma_0 b}{2n+1} S_n
\]

\[
= \frac{4\pi A_0}{\alpha^2} \alpha^{-\frac{n}{2}} J_{\frac{n+1}{2}} (\kappa \alpha) + \frac{4\pi A_0}{(2n+1) \alpha^{n+1}} \alpha^{-\frac{n+1}{2}} J_{\frac{n+1}{2}} (\kappa \alpha) S_n
\]

\[
+ \frac{4\pi}{(2n+1) \alpha^{n+1}} A_n \alpha^{n+1} J_{\frac{n+1}{2}} (\kappa \alpha) J_{\frac{1}{2}} (\kappa \alpha) S_n,
\]

while the value of \( V \), as given by equation \( (13) \), is

\[
V = \frac{4\pi A_0}{\kappa^2} \alpha^{-\frac{1}{2}} J_{\frac{1}{2}} (\kappa \alpha) + \frac{4\pi A_0}{\kappa^2} \alpha^{-\frac{1}{2}} J_{\frac{n+1}{2}} (\kappa \alpha) S_n + \text{cons.}
\]

If we put

\[
V - V_m = v = v_0 + v_1 S_n,
\]

we obtain, after some reduction,

\[
v_n = \frac{4\pi A_0}{(2n+1) \kappa} \left\{ J_{\frac{n-1}{2}} (\kappa \alpha) - \frac{J_{\frac{n+1}{2}} (\kappa \alpha) J_{\frac{1}{2}} (\kappa \alpha)}{J_{\frac{1}{2}} (\kappa \alpha)} \right\}, \quad \ldots \quad \ldots \quad (25)
\]

In general, this gives the value of \( A_n \) which determines the tide raised by a field of potential \( v_n \left( \frac{r}{\alpha} \right)^n S_n \) proportional to \( S_n \). We notice that when \( n = 1 \), \( v_n = 0 \) or \( A_n = \infty \) independently of the values of \( \kappa, \alpha \). This merely expresses the obvious fact that there can be no equilibrium at all so long as the fluid is acted on by a force proportional to a harmonic of the first order.

If it is possible for there to be a configuration of equilibrium when \( v_n = 0 \), other than that given by \( A_n = 0 \), this configuration will of course determine a point of bifurcation in the series of symmetrical configurations. The points of bifurcation are accordingly given by \( v_n = 0 \), or by

\[
J_{\frac{n-1}{2}} (\kappa \alpha) \frac{J_{\frac{n+1}{2}} (\kappa \alpha) J_{\frac{1}{2}} (\kappa \alpha)}{J_{\frac{1}{2}} (\kappa \alpha)}.
\]

For brevity in printing, let us introduce the function \( u_n \) defined by

\[
u_n = J_{\frac{n+1}{2}} (\kappa \alpha) = -\frac{d}{d(\kappa \alpha)} \log \{(\kappa \alpha)^{-\frac{n+1}{2}} J_{\frac{n+1}{2}} (\kappa \alpha)\}.
\]

\[
\ldots \quad \ldots \quad (27)
\]
Near $\kappa \alpha = 0$, $u_n = \frac{\kappa \alpha}{2n+1}$; the value of $u_1$ is $\frac{1}{\kappa \alpha} - \cot \kappa \alpha$, and successive $u$'s satisfy the difference-equation

$$u_{n+1} = \frac{2n+1}{\kappa \alpha} \cdot \frac{1}{u_n}.$$  \hspace{1cm} (28)

With the help of these properties it is easy to draw approximate graphs of the curves $y = u_n$. Such a graph, for values of $\kappa \alpha$ up to the first zero of $u_1 (\kappa \alpha = 4.49)$ is represented in fig. 1, in which the vertical scale is $2\frac{1}{2}$ times the horizontal scale.

![Graph of the functions $u_n$.](image)

In terms of these functions, the points of bifurcation are given by $u_n = u_1$. It is at once evident that there is no root of this equation for values of $\kappa \alpha$ less than $\pi$, and therefore (cf. equation (19)) no point of bifurcation at all so long as $\rho$ is restricted to being always positive. It follows that the spherical configuration is stable for all displacements.

**Small Rotation.**

§ 12. When the fluid experiences a slight rotation $w$, the spherical configuration is of course slightly flattened. The appropriate solution of equation (17) is

$$x = \rho - \frac{\omega^2}{2\pi} = A_0 \rho^{-\frac{1}{3}} J_{\frac{1}{3}}(\rho \sigma) + A_2 \rho^{-\frac{2}{3}} J_{\frac{2}{3}}(\rho \sigma) P_2,$$  \hspace{1cm} (29)
where \( P_2 \) is the second zonal harmonic about the axis of rotation as \( \theta = 0 \). Assuming the free surface to be
\[
r = \alpha + b P_2, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (30)
\]
the equations analogous to (22), (23), and (24) are found to be
\[
\sigma - \frac{w^2}{2\pi} = A_2 \alpha^{-\frac{1}{2}} J_{\nu_0} (\kappa \alpha), \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (31)
\]
\[
A_2 \alpha^{-\frac{1}{2}} J_{\nu_0} (\kappa \alpha) = \kappa A_0 b \alpha^{-\frac{1}{2}} J_{\nu_0} (\kappa \alpha), \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (32)
\]
\[
b\sigma = A_2 \alpha^{-\frac{1}{2}} \frac{J_{\nu_0} (\kappa \alpha) J_{\nu_0} (\kappa \alpha)}{(1 - \frac{w^2}{2\pi\sigma}) J_{\nu_0} (\kappa \alpha)}, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (33)
\]
Let \( v \) be given by
\[
v = V + \frac{1}{2} w^2 \sigma^2 - V_m = V - V_m + \frac{3}{2} w^2 \sigma^2 (1 - P_2),
\]
then, instead of equation (25), we have
\[
v = \frac{4\pi A_2 \alpha^\nu_0}{5\kappa} \left\{ J_{\nu_0} (\kappa \alpha) - \frac{J_{\nu_0} (\kappa \alpha) J_{\nu_0} (\kappa \alpha)}{(1 - \frac{w^2}{2\pi\sigma}) J_{\nu_0} (\kappa \alpha)} \right\} P_2, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (34)
\]
in which constant terms are omitted, and the value is taken on the sphere \( r = \alpha \). For a configuration of equilibrium under no external field of force we must have \( V = V_m \), and therefore \( v \) in equation (34) equal to \(-\frac{3}{2} w^2 \sigma^2 P_2 \). Neglecting squares of \( w^2 \), and therefore omitting the factor \(1 - \frac{w^2}{2\pi\sigma}\) in the denominator, the equation becomes
\[
\frac{1}{2} w^2 \sigma^2 = \frac{4\pi A_2 \alpha^\nu_0}{5\kappa} J_{\nu_0} (\kappa \alpha) (v_1 - v_2), \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (35)
\]
giving \( A_2 \) in terms of \( w^2 \), when \( w^2 \) is small. It will be readily verified that this equation is identical with that obtained by THOMSON and TAIT ('Nat. Phil.' \( \S \) 824, equation (14)).

§ 13. We next examine the solution
\[
\chi = \rho \frac{w^2}{2\pi} = A_0 \rho^{-\frac{1}{2}} J_{\nu_0} (\kappa r) + A_2 \rho^{-\frac{1}{2}} J_{\nu_0} (\kappa r) P_2 + A_n \rho^{-\frac{1}{2}} J_{\nu_0} (\kappa r) S_n, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (36)
\]
which is appropriate to a mass of fluid having a rotation \( \omega \) given by equation (35), and acted on by a field of force of potential \( \omega S_n \). By analysis exactly similar to that just given, we obtain at \( r = \alpha \)
\[
v_m = \frac{4\pi A_2 \alpha^\nu_0}{(2n+1) \kappa} \left\{ J_{\nu_0} (\kappa \alpha) - \frac{J_{\nu_0} (\kappa \alpha) J_{\nu_0} (\kappa \alpha)}{(1 - \frac{w^2}{2\pi\sigma}) J_{\nu_0} (\kappa \alpha)} \right\} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (37)
\]
AND THE NEBULAR HYPOTHESIS.

This gives $A_n$ for the general tide raised by the field $v_n S_n$. The condition for a point of bifurcation is $v_n = 0$, or

$$u_n = \left(1 - \frac{w^2}{2\pi\sigma}\right) u_1, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (38)$$

Thus the points of bifurcation, if any, are still determined by the intersections of the graphs in fig. 1, except that the graph of $u_1$, must be supposed decreased vertically in the ratio $1 - \frac{w^2}{2\pi\sigma}$ to 1.

We may, if we please, imagine that we start with very small rotation, and allow the rotation progressively to increase, this increase being accompanied in imagination by a greater and greater flattening of the graph of $u_1$.

It is clear that under all circumstances the curve which will first be intersected by the flattened graph of $u_1$ will be the graph of $u_2$. It is further clear that the requisite value of $w^2$ is least when $\kappa \alpha = 0$, and progressively increases as $\kappa \alpha$ increases, at any rate up to $\kappa \alpha = \pi$.

This means that in the first place the circular vibration will invariably become unstable through a vibration proportional to a second harmonic, so that the first point of bifurcation reached will be one such that the spheroidal form gives place to an ellipsoidal form. If the rotation is so small that the problem may be treated as a statical one, there will be no question as to there being an actual exchange of stabilities at the point of bifurcation, for clearly $v_n$ changes sign at this point. Thus for rotation greater than that at the point of bifurcation, the spheroidal form will be definitely unstable, and the ellipsoidal form definitely stable, at least until the next point of bifurcation is reached.

Our result shows, in the second place, that the masses which become ellipsoidal for the smallest values of $w^2$ are those for which $\kappa \alpha$ is smallest. To put it briefly, the mass which is most unstable when it begins to rotate is the incompressible mass—a somewhat unexpected result.

For any value of $\kappa \alpha$, the value which $w^2$ must have for the spheroidal form to become unstable is (cf. equation (38))

$$1 - \frac{w^2}{2\pi\sigma} = u_3/u_1, \ldots \ldots \ldots \ldots \ldots \ldots \ldots (39)$$

and when $\kappa \alpha = 0$, the value of $u_3/u_1 = \frac{3}{\pi}$ (cf. § 4).

Thus our equations would make the spheroidal mass of incompressible fluid first become unstable when $\frac{w^2}{2\pi\sigma} = \cdot 400$, but these equations have only been obtained on the supposition that $\frac{w^2}{2\pi\sigma}$ is so small that its squares may be neglected, a supposition which is now seen à posteriori to be hardly admissible. Probably the results obtained are qualitatively true, but quantitatively unreliable. In point of fact the...
first point of bifurcation for an incompressible mass, instead of being given by \\
\frac{w^2}{2\pi\sigma} = .400, is known to be given by the widely different value \frac{w^2}{2\pi\sigma} = 1871.

Our analysis has nevertheless proved rigorously the point which is really most important, namely, that there can be no point of bifurcation at all for quite small values of \frac{w^2}{2\pi\sigma}. At the same time, since the question of when and how a rotating mass first becomes unstable is one of considerable importance, I have attempted to obtain a more reliable investigation than the preceding. I have found that the accuracy is not greatly improved by taking the analysis as far as squares of \frac{w^2}{2\pi\sigma}, while the labour of working with a general power series would be appalling. I have, therefore, reluctantly been compelled to give up hopes of carrying the rigorous solution of the problem further in this direction, but have thought it worth while to examine the analogous problem for rotating cylindrical masses. All the essential physical features of the natural three-dimensional problem appear to be reproduced in the simpler cylindrical problem, so that it seems legitimate to hope that an argument by analogy may not lead to entirely erroneous result.

*Cylindrical Masses in Rotation.*

§14. The fundamental equations are, of course, the first two of the equations already written down in §3. The third equation does not occur, since \frac{\partial}{\partial z} = 0. The equations have, as before, the integral (13) leading to the differential equation (15) for \(\rho\). The most general solution possible will be

\[
\rho = \frac{w^2}{2\pi} + \sum_{n} A_n J_n(\zeta r) \cos(n\theta - \epsilon), \ldots \ldots \ldots (40)
\]

in which \(r\) now stands for \(\sqrt{x^2+y^2}\). No matter how great the rotation, there is always a special circular solution

\[
\rho = \frac{w^2}{2\pi} + A_0 J_0(\zeta r), \ldots \ldots \ldots \ldots \ldots (41)
\]

this being analogous to the spheroidal figures of equilibrium investigated in §12.

Let us examine the deformed solution

\[
\rho = \frac{w^2}{2\pi} + A_0 J_0(\zeta r) + A_n J_n(\zeta r) \cos n\theta, \ldots \ldots \ldots \ldots \ldots (42)
\]

in which \(A_n\) is supposed small, but there are no restrictions on the value of \(\frac{w^2}{2\pi}\). If the free surface \(\rho = \sigma\) is supposed given by \(\text{cf.} \text{ equations (21) and (30)}\)

\[
r = a + b \cos n\theta,
\]
then, as in equations (31) and (32), \( \alpha \) and \( b \) must satisfy
\[
\sigma = \frac{\omega^2}{2\pi} + A_\phi J_0(\alpha\alpha). \quad \ldots \ldots \ldots \ldots (43)
\]
whence
\[
A_n J_n(\alpha\alpha) = -A_\phi b e J_\alpha(\alpha\alpha) = A_\phi b c J_1(\alpha\alpha),
\]

The potential of the mass, \( V_m \), can be regarded as arising from a distribution of density \( \rho \) inside the cylinder \( r = \alpha \), together with a surface density \( b\sigma \cos n\theta \) spread over the surface of the cylinder.

The first part of the potential, evaluated at \( R, \Theta \), is
\[
C - \left[ \log \left\{ r^2 + R^2 - 2rR \cos(\theta - \Theta) \right\} \right] \left[ \frac{\omega^2}{2\pi} + A_\phi J_0(\alpha r) + A_n J_n(\alpha r) \cos n\theta \right] r \, dr \, d\theta
\]
\[
= C - 2 \left[ \log R - \sum_{k=1}^{\infty} \frac{r^k}{sR^k} \cos s(\theta - \Theta) \right] \left[ \frac{\omega^2}{2\pi} + A_\phi J_0(\alpha r) + A_n J_n(\alpha r) \cos n\theta \right] r \, dr \, d\theta
\]
\[
= A_n \int_0^\alpha 2\pi \frac{r^n}{R^n} J_n(\alpha r) r \, dr \, \cos n\Theta + \text{terms independent of } \Theta
\]
\[
= \frac{2\pi}{\kappa n} A_n J_{n+1}(\alpha\alpha) a^{n+1} R^n \cos n\Theta + \text{terms independent of } \Theta.
\]

The potential of the surface distribution is
\[
\frac{2\pi b \alpha^{n+1}}{nR^n} \cos n\Theta,
\]
so that, at \( r = \alpha \),
\[
V_m = \left\{ \frac{2\pi}{\kappa n} A_n J_{n+1}(\alpha\alpha) + \frac{2\pi b \alpha}{n} \right\} \cos n\theta + \text{terms independent of } \theta,
\]
while, by equation (5),
\[
V = \frac{4\pi}{\kappa^2} \left[ \frac{\omega^2}{2\pi} + A_\phi J_0(\alpha r) + A_n J_n(\alpha r) \cos n\theta \right] - \frac{1}{2} \omega^2 r^2.
\]

If, as before, we express the tide-generating potential \( V - V_m \) in the form \( v_0 + v_n \cos n\theta \), we obtain for the value of \( v_n \), at \( R = \alpha \),
\[
v_n = A_\phi \left\{ \frac{4\pi}{\kappa^2} J_n(\alpha\alpha) - \frac{2\pi}{\kappa n} J_{n+1}(\alpha\alpha) \right\} - \frac{2\pi b \alpha}{n}
\]
\[
= \frac{2\pi}{\kappa n} A_n \left\{ J_{n-1}(\alpha\alpha) - \frac{J_n(\alpha\alpha) J_1(\alpha\alpha)}{1 - \frac{\omega^2}{2\pi\sigma}} \right\}, \quad \ldots \ldots \ldots (45)
\]

It will be seen that this equation is exactly analogous to the former equation (37),
but with the important difference that the present equation is true for all values of \( w^2 \), without limit. The points of bifurcation are given by \( v_n = 0 \), or

\[
    u_{n-\frac{1}{2}}(\kappa a) = \left(1 - \frac{w^2}{2\pi}\right) u_{\frac{1}{2}}(\kappa a),
\]

which again is exactly analogous to the former equation (38). The graphs of the functions \( u_{\frac{1}{2}}, u_{\frac{3}{2}}, \ldots \) will be found to lie as in fig. 2, and we may again imagine that

![Fig. 2. Graphs of the functions \( u_{n+\frac{1}{2}} \).](image)

points of bifurcation are sought by flattening the curve \( u_{\frac{1}{2}} \) until it intersects the other curves.

It is clear that, under all circumstances, the first curve to be intersected will be the curve \( u_{\frac{1}{2}} \), corresponding to a displacement proportional to \( \cos 2\theta \). Thus, as before, when the circular form becomes unstable, it gives place to a form of elliptic cross-section, which is stable. Moreover, the smaller \( \kappa a \) is the lower the value of \( w^2/2\pi \) for which the circular form becomes unstable.

These results are true without any regard to the value of \( w^2 \), so that they confirm the results stated, but not rigorously proved, in § 5. The numerical calculations which follow will make the matter clearer.

If \( \bar{\rho} \) denotes the mean density of the rotating mass, the total mass per unit length is given by

\[
    \bar{\rho} \pi a^2 = 2\pi \int_0^a \rho r \, dr = 2\pi \left[ \frac{w^2a^2}{4\pi} + A_0 \frac{a^2J_1(\kappa a)}{\kappa} \right],
\]
giving, on substitution from equation (43),

\[ \bar{\rho} = \frac{w^2}{2\pi} + \frac{2}{\kappa\alpha} \frac{J_1(\kappa\alpha)}{J_0(\kappa\alpha)} \left( \sigma - \frac{w^2}{2\pi} \right) = \frac{w^2}{2\pi} + \left( 1 + \frac{J_2(\kappa\alpha)}{J_0(\kappa\alpha)} \right) \left( \sigma - \frac{w^2}{2\pi} \right), \]

whence, for the ratio of \( \bar{\rho} \) to \( \sigma \), we have the general formula

\[ \frac{\bar{\rho}}{\sigma} = 1 + \frac{J_2(\kappa\alpha)}{J_0(\kappa\alpha)} \left( 1 - \frac{w^2}{2\pi\sigma} \right). \]

For the particular configuration which occurs at a point of bifurcation,

\[ 1 - \frac{w^2}{2\pi\sigma} = \frac{J_0(\kappa\alpha)}{J_1(\kappa\alpha)} \frac{J_2(\kappa\alpha)}{J_1(\kappa\alpha)}, \]

so that

\[ \frac{\bar{\rho}}{\sigma} = 1 + \left( \frac{J_2(\kappa\alpha)}{J_1(\kappa\alpha)} \right)^2 = 1 + u_n^2, \]

whence we obtain

\[ \frac{w^2}{2\pi\bar{\rho}} = \frac{u_n - u_n}{u_n(1 + u_n)}. \]

In the following table I have calculated the values of \( w^2/2\pi\sigma \) and of \( w^2/2\pi\bar{\rho} \) for which cylinders of different radii (\( \alpha \)) and compressibility (\( \kappa \)) first become elliptical in cross section:

<table>
<thead>
<tr>
<th>( \kappa\alpha )</th>
<th>( u_{n\alpha} )</th>
<th>( u_{n\alpha} )</th>
<th>( \frac{w^2}{2\pi\sigma} )</th>
<th>( \frac{w^2}{2\pi\bar{\rho}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.500</td>
<td>0.500</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1261</td>
<td>0.063</td>
<td>0.503</td>
<td>0.500</td>
</tr>
<tr>
<td>1</td>
<td>0.2582</td>
<td>0.126</td>
<td>0.510</td>
<td>0.502</td>
</tr>
<tr>
<td>1.75</td>
<td>0.4040</td>
<td>0.192</td>
<td>0.525</td>
<td>0.506</td>
</tr>
<tr>
<td>2</td>
<td>0.5751</td>
<td>0.261</td>
<td>0.546</td>
<td>0.511</td>
</tr>
<tr>
<td>2.4048</td>
<td>2.575</td>
<td>0.612</td>
<td>0.762</td>
<td>0.554</td>
</tr>
<tr>
<td>3</td>
<td>1.304</td>
<td>0.829</td>
<td>1.000</td>
<td>0.593</td>
</tr>
<tr>
<td>3.8317</td>
<td>0.000</td>
<td>1.433</td>
<td>2.099</td>
<td>0.687</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( 1.000 )</td>
</tr>
</tbody>
</table>

It will be seen that the general result is fully confirmed, that incompressible masses are the first to become unstable, and that the more compressible the mass is, the greater is the rotation required for it to depart from a symmetrical configuration.

*Rotating nearly Spherical Mass with High Internal Temperature.*

§ 15. We now leave the artificial two-dimensional problem and return to the real problem in three dimensions discussed in § 13.

The coefficient \( \kappa \) was there assumed to have the same value throughout the mass, as of course it would if the matter were homogeneous and of uniform temperature.
throughout. But to represent natural astronomical conditions there is no question
that \( \kappa \) ought to increase on passing from the centre to the surface, thus representing
a mass in which the temperature is highest inside and falls towards the surface.

We are in this way led to study the question of stability when \( \kappa \) is a function of \( r \).
It would be difficult to say precisely what function ought to be chosen if we were
trying to represent natural conditions as faithfully as possible. It appears, however,
that no continuous function will lead to equations which admit of integration. The
only case which appears to be soluble is that in which the matter, before rotation,
may be treated as if formed of a series of different layers, each being homogeneous
and at a uniform temperature in itself, but the temperature varying from layer to
layer. To represent this we take different values of \( \kappa \) in the different layers, \( \kappa \) being
smallest in the interior.

There is no limit to the number of layers which can be treated analytically, but
the assumption of a great number of layers naturally leads to highly complicated
formulæ which are capable of conveying their meaning only after laborious numerical
calculations. Both in order to obtain comprehensible results and to simplify the
argument, the layers will, in what follows, be supposed to be only two in number.
They may conveniently be referred to as the core and the crust. It will be found
possible to generalize the results obtained so as to apply to any number of layers.

§ 16. We accordingly suppose that there is an interior core of radius \( a \), in which
the coefficient of compressibility has the uniform value \( \kappa \), and that outside this is the
crust of external radius \( c \), in which the coefficient is \( \kappa' \). It is again necessary to
suppose the rotation to be so small that \( \nu^2 \) may be neglected.

As in § 3, the density \( \rho \) must satisfy

\[
(\nabla^2 + \kappa^2) \left( \rho - \frac{\nu^2}{2\pi} \right) = 0 \quad \ldots \quad (47)
\]

throughout the core, and the same equation with the appropriate value of \( \kappa \) throughout
the crust. The most general solution of equation (47) is

\[
\rho = \frac{\nu^2}{2\pi} + \sum_{n=0}^{\infty} \left\{ A_n r^{-1/2} J_{n+1/2}(kr) + B_n r^{-1/2} J_{-(n+1/2)}(kr) \right\} S_n. \quad \ldots \quad (48)
\]

In the former problem all the terms in \( B_n \) could be omitted because \( \rho \) had to be
finite at the origin. In discussing the solution for the crust these terms must be
retained. The solution can, however, be put in a more concise form.

Let the constants \( A_n, B_n \) be replaced by new constants \( C_n, \theta_n \) given by

\[
A_n = C_n \cos \theta_n, \quad B_n = C_n \sin \theta_n,
\]

and let us introduce a function \( J_{n+1/2}(x, \theta) \) defined by

\[
J_{n+1/2}(x, \theta) = J_{n+1/2}(x) \cos \theta + J_{-(n+1/2)}(x) \sin \theta. \quad \ldots \quad (49)
\]
Then the solution (48) may be replaced by

$$\rho = \frac{u^2}{2\pi} + \sum_0^\infty C_n r^{-\frac{n}{2}} J_{n+\frac{1}{2}} (kr, \theta_k) S_n \ldots \ldots . \quad (50)$$

which is formally analogous to (7).

The following properties of the function $J_{n+\frac{1}{2}} (x, \theta)$ may readily be verified, and will be required later:

$$\frac{d}{dx} \{ x^{n+\frac{1}{2}} J_{n+\frac{1}{2}} (x, \theta) \} = x^{n+\frac{1}{2}} J_{n-\frac{1}{2}} (x, -\theta) \ldots \ldots \quad (51)$$

$$\frac{d}{dx} \{ x^{-(n+\frac{1}{2})} J_{n+\frac{1}{2}} (x, \theta) \} = -x^{-\frac{n+1}{2}} J_{n+\frac{1}{2}} (x, -\theta) \ldots \ldots \quad (52)$$

$$J_{n+\frac{1}{2}} (x, \theta) + J_{n-\frac{1}{2}} (x, \theta) = \frac{2n+1}{x} J_{n+\frac{1}{2}} (x, -\theta) \ldots \ldots \quad (53)$$

There is a ready rule for writing down the values of these functions. In the first place, we have

$$J_{\frac{1}{2}} (x) = \frac{\sin x}{\sqrt{\frac{1}{2} \pi x}}, \quad J_{-\frac{1}{2}} (x) = \frac{\cos x}{\sqrt{\frac{1}{2} \pi x}},$$

so that

$$J_{\frac{1}{2}} (x, \theta) = \frac{\sin (x+\theta)}{\sqrt{\frac{1}{2} \pi x}}.$$

Now let $\phi (x+\theta)$ be used to denote a general function made up of circular functions of $x+\theta$ and of algebraic functions of $x$. Then $J_{\frac{1}{2}} (x, \theta)$ is of the form $\phi (x+\theta)$, and any number of differentiations with respect to $x$, or of multiplications by powers of $x$, will still leave it in this form. It follows from (52) that $J_{\frac{1}{2}} (x, -\theta), J_{\frac{3}{2}} (x, \theta), \&c.$, will be of this form. Hence we have the general law

$$J_{n+\frac{1}{2}} (x, \theta) = \phi \{ x + (-1)^n \theta \} \ldots \ldots \ldots \quad (54)$$

in which the functional form of $\phi$ is at once given by

$$\phi (x) = J_{n+\frac{1}{2}} (x).$$

[For instance,]

$$J_{\frac{1}{2}} (x) = \left( \frac{3}{x^2} - 1 \right) \sin x - \frac{3}{x} \cos x,$$

so that

$$J_{\frac{1}{2}} (x, \theta) = \phi (x+\theta) = \left( \frac{3}{x^2} - 1 \right) \sin (x+\theta) - \frac{3}{x} \cos (x+\theta).$$

§ 17. We proceed to carry out analysis similar to that of § 13. Suppose that under a tide-generating potential $v S_n$, and a rotation $w$, the core assumes a configuration such that its boundary $r = \alpha$ becomes deformed into

$$r = \alpha + b S_n + \beta P_n \ldots \ldots \ldots \ldots \quad (55)$$
while the outer surface becomes

\[ r = c + dS_n + \delta P_x \]  
\[ (56) \]

Let us suppose the densities in the two layers to be

\[ \rho = \frac{u^2}{2\pi} + A_\alpha r^{-\alpha} J_{\nu_1}(\kappa r) + A_\beta r^{-\beta} J_{\nu_1}(\kappa r) + A_\gamma r^{-\gamma} J_{\nu_1}(\kappa r) P_2 \]  
\[ (57) \]
in the core, and

\[ \rho = \frac{u^2}{2\pi} C_\alpha r^{-\alpha} J_{\nu_1}(\kappa r, \alpha) + C_\beta r^{-\beta} J_{\nu_1}(\kappa r, \beta) S_n + C_\gamma r^{-\gamma} J_{\nu_1}(\kappa r, \gamma) P_2 \]  
\[ (58) \]
in the crust. The boundary between the two layers must clearly be an equipotential, and therefore a surface of constant pressure and density. Let \( \sigma, \sigma' \) be the densities at this boundary in the core and crust respectively.

On replacing \( r \) by \( a + \beta S_n + \beta P_x \) in equations (57) and (58), the values of \( \rho \) must become \( \sigma \) and \( \sigma' \) respectively. This leads to the relations

\[ \frac{\sigma - u^2}{2\pi} = A_\alpha a^{-\alpha} J_{\nu_1}(\kappa a), \]  
\[ (59) \]

\[ \frac{\sigma' - u^2}{2\pi} = C_\alpha a^{-\alpha} J_{\nu_1}(\kappa a, a), \]  
\[ (60) \]

\[ b \left( \frac{\sigma - u^2}{2\pi} \right) = A_\beta a^{-\beta} J_{\nu_1}(\kappa a) J_{\nu_1}(\kappa a), \]  
\[ (61) \]

\[ b \left( \frac{\sigma' - u^2}{2\pi} \right) = C_\beta a^{-\beta} J_{\nu_1}(\kappa a, a) J_{\nu_1}(\kappa a, a). \]  
\[ (62) \]

From similar analysis applied to the outer boundary, if \( \sigma_0 \) is now the density at this boundary,

\[ \frac{\sigma_0 - u^2}{2\pi} = C_\alpha c^{-\alpha} J_{\nu_1}(\kappa c, a), \]  
\[ (63) \]

\[ d \left( \frac{\sigma_0 - u^2}{2\pi} \right) = C_\gamma c^{-\gamma} J_{\nu_1}(\kappa c, a) J_{\nu_1}(\kappa c, a). \]  
\[ (64) \]

Similar equations, of course, connect the coefficients which depend on the rotation.

The value of \( V_m \) at a point on the sphere \( r = c \) can now be written down, as in §11, and is found to be

\[ V_m = \frac{4\pi}{c} \left[ \frac{u^2}{2\pi} + \frac{C_0}{c} \left\{ c^{-\alpha} J_{\nu_1}(\kappa c, -\alpha) - \alpha^{-\alpha} J_{\nu_1}(\kappa c, -\alpha) \right\} + \frac{A_3}{\kappa} \alpha^{-\gamma} J_{\nu_1}(\kappa a) \right] \]  
\[ + \frac{4\pi S_n}{(2n+1)c^{n+1}} U_n, \]  
\[ (65) \]
in which the rotational terms proportional to the second harmonic are omitted and
$U_n$ is given by

$$U_n = \frac{C^2}{k^2} \left\{ \epsilon^{n+\frac{4}{3}} J_{n+\frac{2}{3}} (\epsilon', \beta) - \alpha^{n+\frac{4}{3}} J_{n+\frac{2}{3}} (\alpha', -\beta) \right\} + \frac{A^n}{k^2} \alpha^{n+\frac{2}{3}} J_{n+\frac{2}{3}} (\alpha') + d \sigma \beta^{n+2} + b (\sigma - \sigma') \alpha^{n+2}. \quad \text{(66)}$$

The value of $V$ at $r = c$ is, from equation (13),

$$V = \frac{4\pi}{k^2} \left\{ \frac{w^2}{2\pi} + C \rho \alpha^{n+\frac{2}{3}} J_{n+\frac{2}{3}} (\epsilon, \alpha) + C \rho \beta^{n+\frac{2}{3}} J_{n+\frac{2}{3}} (\epsilon, \beta) \right\} S_n \left\{ \frac{1}{2} w^2 (x^2 + y^2) + \text{cons.} \right\}. \quad \text{(67)}$$

whence, evaluating $V - V_\infty$ and picking out the coefficient of $S_n$, we find as the value of $v_n$ at $r = c$,

$$v_n = \frac{4\pi}{k^2} \left\{ C \rho \alpha^{n+\frac{2}{3}} J_{n+\frac{2}{3}} (\epsilon, \beta) - \frac{4\pi}{(2n + 1) c^{n+1}} U_n \right\}. \quad \text{(68)}$$

As before, the points of bifurcation, if any, are given by $v_n = 0$.

§ 18. It is now necessary to consider the boundary conditions which must be satisfied at the junction of the two layers. The condition of continuity of material, i.e., that the inner surface of the crust shall coincide with the outer surface of the core, has been expressed in equation (55), $b$ and $\beta$ being the same for both core and crust. There is an equation of continuity of pressure expressed by

$$\frac{\sigma^2 - \sigma_0^2}{\kappa^2} = \frac{\sigma^2 - \sigma_0^2}{\kappa'^2}, \quad \text{(69)}$$

which $\sigma_0$ is now used to represent the density associated with zero pressure. Finally, there is a condition of continuity of normal force and this requires careful discussion.

Let $M_1$ denote the mass of matter actually forming the core and let $V_1$ denote its potential at any point outside the core. Let $M_2$ denote the mass which would replace the core if the solution (58) for the crust were extended to the centre and let $V_2$ denote the potential of this mass at any external point. It will be noticed that if solution (58) were extended to the origin, it would give an infinite density $\rho$ at the origin and also an infinite value of $V$ in virtue of equation (5). On the other hand, it is readily found, by direct integration, that $V_2$ the potential of the mass $M_2$ is finite at every point, including the origin. It follows that $V$ can only be the potential of this imaginary arrangement of matter when it is acted on by certain external forces of which the potential becomes infinite at the origin. Let $V_3$ represent the potential of these forces. The value of $V_3$ is readily found, for it must satisfy

$$\nabla^2 V_3 = 0,$$

and must coincide with $V$ or with $\frac{4\pi \rho}{\kappa^2}$ to within an additive constant at the origin. Thus $V_3$ is the limit of the right-hand side of equation (67) when $r = 0$.
r replacing c. This is found to be

\[ V_3 = \frac{4\pi}{\kappa^2} \left[ \frac{C_0 (\frac{1}{2} \pi)^{1/4}}{r} \sin \alpha + \frac{C_n (\frac{1}{2} \pi)^{1/4}}{r} \frac{1}{2} \cdot 3 \cdot 5 \ldots (2n-1) (-1)^n \sin \beta S_n \right]. \]

The condition now to be satisfied is clearly that

\[ \frac{\partial}{\partial r} V_1 = \frac{\partial}{\partial r} (V_2 + V_3) \]

at all points on the boundary \( r = \alpha + bS_n \). This requires that \( V_1 - V_2 - V_3 \) shall vanish, to within a constant, at all points outside this boundary, and therefore, in particular, at \( r = c \). It will be readily seen that

\[ V_1 - V_2 = V_m - (V_m)_{n=0} \]

while \( V_3 \) is exactly the value of the terms in \( V_m \) that involve \( \alpha \), when \( \alpha \) is put equal to zero. The conditions sought are, therefore, simply that all the terms in \( \alpha \) which occur in \( V_m \) shall vanish at every point of the sphere.

§ 19. We may now equate the coefficients of the separate harmonics, and obtain

\[ \frac{C_0}{\kappa} J_{\nu/2} (k' \alpha, -\alpha) = A_\nu J_{\nu/2} (k \alpha), \quad \ldots \ldots \ldots \ldots \ldots (70) \]

\[ \frac{C_n}{\kappa} J_{n+\nu/2} (k' \alpha, -k) + b \left( \sigma' - \frac{w^2}{2\pi} \right) \alpha^{\nu/2} = A_n J_{n+\nu/2} (k \alpha) + b \left( \sigma - \frac{w^2}{2\pi} \right) \alpha^{\nu/2}. \ldots \ldots (71) \]

On account of the simplifications made possible by these equations, equation (68) may be put in the form

\[ r_n = \frac{4\pi}{\kappa^2} \left[ C_n \left\{ J_{n-\nu/2} (k' \alpha, -\beta) - J_{n+\nu/2} (k' \alpha, \beta) J_{\nu/2} (k \alpha, -\alpha) \right\} \right] \]

\[ = \frac{4\pi}{\kappa^2} \left[ \frac{C_n}{2n+1} \left\{ J_{n-\nu/2} (k' \alpha, -\beta) - J_{n+\nu/2} (k' \alpha, \beta) J_{\nu/2} (k \alpha, -\alpha) \right\} \right]. \ldots \ldots \ldots (72) \]

The elimination of \( A_n \) and \( C_0 \) from (59), (60), and (70) gives

\[ \left( \sigma' - \frac{w^2}{2\pi} \right) J_{\nu/2} (k' \alpha, -\alpha) = \left( \sigma - \frac{w^2}{2\pi} \right) \frac{J_{\nu/2} (k \alpha)}{k' J_{\nu/2} (k' \alpha, \alpha)} \ldots \ldots \ldots (73) \]

while similarly the elimination of \( A_n \) and \( C_n \) from (61), (62), and (71) gives

\[ \left( \sigma' - \frac{w^2}{2\pi} \right) \left\{ 1 + \frac{J_{n+\nu/2} (k' \alpha, -\beta) J_{\nu/2} (k' \alpha, -\alpha)}{J_{n+\nu/2} (k' \alpha, \beta) J_{\nu/2} (k' \alpha, \alpha)} \right\} = \left( \sigma - \frac{w^2}{2\pi} \right) \left\{ 1 + \frac{J_{n+\nu/2} (k \alpha) J_{\nu/2} (k \alpha)}{J_{n+\nu/2} (k \alpha) J_{\nu/2} (k \alpha)} \right\}. \ldots \ldots \ldots (74) \]

For brevity we introduce a function \( u_n (x, \theta) \), a generalisation of the \( u_n \) of § 11, the
new function being defined by
\[ u_n(x, \theta) = -\frac{\partial}{\partial x} \log \{x^{-n} J_{n-\nu_n}(x, \theta)\} = \frac{J_{n+\nu_n}(x, -\theta)}{J_{n-\nu_n}(x, \theta)}. \quad (74a) \]

Equations (73) and (74) now become
\[ \frac{\sigma' - \frac{u^2}{2\pi}}{\kappa'} u_1(\kappa' \alpha, \alpha) = \frac{\sigma - \frac{u^2}{2\pi}}{\kappa} u_1(\kappa \alpha), \quad (75) \]
\[ (\sigma' - \frac{u^2}{2\pi}) \left\{ 1 + u_1(\kappa' \alpha, \alpha) u_{n+1}(\kappa' \alpha, \beta) \right\} = \left( \sigma - \frac{u^2}{2\pi} \right) \left\{ 1 + u_1(\kappa \alpha) u_{n+1}(\kappa \alpha) \right\}, \quad (76) \]
while (72) becomes
\[ v_n = \frac{4\pi c^{1/2}}{2n+1} \frac{C_n}{\kappa'} J_{n-\nu_n}(\kappa' \beta, -\beta) \left\{ 1 - \frac{1}{\frac{u^2}{2\pi \sigma \rho}} u_n(\kappa' c, -\beta) \right\} \left\{ 1 - \frac{1}{\frac{u^2}{2\pi \sigma \rho}} u_1(\kappa' c, \alpha) \right\}, \quad (77) \]
so that points of bifurcation are given by
\[ u_n(\kappa' c, -\beta) = \left( 1 - \frac{u^2}{2\pi \sigma \rho} \right) u_1(\kappa' c, \alpha). \quad (78) \]

Again, if the rotation may be treated as small, there will invariably be a change of stability at these points of bifurcation, since \( v_n/C_n \) changes sign on passing through one of them.

§ 20. It is at once clear that the method can be extended to a mass consisting of any number of layers—the only difficulty occurs in the numerical computations at the end. At each boundary between two consecutive layers there will be equations of continuity precisely similar to (75) and (76), while the final value of \( v_n \) will be given by an equation exactly similar to (77), which it will be seen involves only quantities associated with the outer boundary.

The procedure in any particular case will be to start, so to speak, with the innermost core of the system. Equation (75) is linear in \( \cos \alpha \) and \( \sin \alpha \), so that \( \tan \alpha \) is uniquely determined. Leaving out of account systems in which the density is, in any part, negative, this will be found to be adequate to determine \( \alpha \) uniquely. Equation (76) now becomes a linear equation in \( \cos \beta \) and \( \sin \beta \), from which \( \beta \) can be determined uniquely. In this way, passing from layer to layer, we can determine the various values of \( \alpha, \beta \) for the different layers. Finally, the \( \alpha \)'s and \( \beta \)'s being known, equation (78) can be regarded either as an equation for \( u^2 \) or as an equation for \( c \), i.e., it can be regarded either as determining the highest rotation for which the symmetrical configuration is stable for a given value of \( c \), or as determining the largest value of \( c \) for which the mass is stable under a given rotation. If the value of \( \frac{u^2}{2\pi \sigma \rho} \) obtained by the first method is not small, the result will be inaccurate; if the value for \( c \) obtained by the second method is so great that the density is in places
negative, the result will be of no interest except as proving stability for smaller values of \( c \).

§ 21. It may be well to take a general survey of the equations before giving special calculations. For simplicity we again consider two layers only, core and crust. From (75) and (76) it is clear that, when \( \alpha = 0 \) or \( \kappa = \kappa' \) (involving \( \sigma = \sigma' \)), the values of \( \alpha \) and \( \beta \) vanish. Broadly speaking, the more distinct the core is from the crust, the larger \( \alpha \) and \( \beta \) are. Equation (78), of course, differs only from the corresponding equation previously found, by the presence of the terms \( \alpha \) and \( \beta \). The effect of these terms is seen on noticing that, in the notation already used, \( u_1 (\kappa' \alpha, \alpha) \) is of the form \( \phi (\kappa' \alpha - \alpha) \). Thus, to allow for the effect of the core on the term \( u_1 (\kappa' \alpha, \alpha) \), we have to leave the algebraic part of the function unaltered, but to change all the trigonometrical arguments from \( \kappa' \alpha \) to \( \kappa' \alpha - \alpha \). Speaking very broadly, the general effect on the graph of \( u_1 \) (cf. fig. 1) is a compromise between leaving the graph unaltered and moving it bodily a distance \( \alpha \) along the axis. Similar statements apply to the graph of \( u_n \).

Thus, while rotation as before is represented by flattening the graph of \( u_1 \) in fig. 1, the presence of a core is represented by a distortion of the graphs which may, with some truth, be thought of as bodily movements parallel to the axis. These bodily movements may cause new intersections between the graph \( u_1 \) and the other graphs, and the points of intersection will represent points of bifurcation at which the symmetrical configuration will become unstable.

**No Rotation.**

§ 22. The case that may properly be inspected first is that of no rotation. The equations reduce to

\[
\frac{\sigma'}{\kappa} u_1 (\kappa' \alpha, \alpha) = \frac{\sigma}{\kappa} u_1 (\kappa \alpha), \tag{79}
\]

\[
\sigma' \left\{1 + u_1 (\kappa' \alpha, \alpha) u_{n+1} (\kappa' \alpha, \beta)\right\} = \sigma \left\{1 + u_1 (\kappa \alpha) u_{n+1} (\kappa \alpha)\right\}, \tag{80}
\]

and, the equation for points of bifurcation,

\[
u_n (\kappa' \alpha, -\beta) = u_1 (\kappa' \alpha, \alpha). \tag{81}\]

When \( n = 1 \), it is seen that \( \beta = -\alpha \) is a solution of (80), and must therefore (§ 20) be the only solution. To verify that \( \beta = -\alpha \) is a solution, replace \( \beta \) by \(-\alpha\) in (80) and it becomes

\[
\sigma' \left\{1 + \frac{J_{\kappa \alpha}}{J_{\kappa' \alpha} (\kappa' \alpha, \alpha)} \right\} = \sigma \left\{1 + \frac{J_{\kappa \alpha}}{J_{\kappa \alpha} (\kappa \alpha)} \right\}, \tag{82}
\]

which is seen to be identical with (79) (cf. equations (53) and (74\(\alpha\)). Equation (81) now reduces to an identity, so that every configuration is formally a point of bifurcation. The interpretation is, of course, the same as that of § 11, the displacement for which \( n = 1 \) is a rigid body displacement, and so requires no force to
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There is, of course, nothing of the nature of a change of stability, for $v \sqrt{C \alpha}$, instead of changing sign, remains permanently zero. The consideration of $n = 1$ is of no value except that it provides a check on the result of a rather involved series of analytical processes.

When there is homogeneity between core and crust the non-rotating system has been found to be stable for all displacements. To examine whether this is altered by the presence of the crust, it is natural to test first the extreme case in which the difference between the core and crust is as great as possible. Let us make the core so hot that its density is zero, so that $\kappa$ has to be zero in order that the internal pressure may be maintained (cf. equation (12)).

Putting $\sigma = 0$, equations (79) and (80) reduce to

$$u_i (\kappa' \alpha, \alpha) = 0, \quad u_{n+1} (\kappa' \alpha, \beta) = \infty, \quad \ldots \quad (83)$$

or, by equation (74a),

$$J_{n+1} (\kappa' \alpha, -\alpha) = 0, \quad J_{n+1} (\kappa' \alpha, \beta) = 0, \quad \ldots \quad (84)$$

whence (equation (49)) $\alpha, \beta$ are given by

$$\tan \alpha = \frac{J_{n+1} (\kappa' \alpha)}{J_{n+1} (\kappa' \alpha)}; \quad \tan \beta = -\frac{J_{n+1} (\kappa' \alpha)}{J_{n+1} (\kappa' \alpha)} \ldots \quad (85)$$

The values of $\alpha, \beta$ corresponding to a few values of $\kappa \alpha$ are given below—

<table>
<thead>
<tr>
<th>$\kappa \alpha$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-0.12</td>
<td>0.59</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-0.51</td>
<td>0.15</td>
<td>0.21</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-1.00</td>
<td>0.48</td>
<td>0.39</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-1.53</td>
<td>0.91</td>
<td>0.46</td>
<td>0.22</td>
</tr>
<tr>
<td>5</td>
<td>-2.07</td>
<td>1.40</td>
<td>0.86</td>
<td>0.9</td>
</tr>
<tr>
<td>6</td>
<td>-2.63</td>
<td>1.92</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-3.19</td>
<td>2.45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-4.32</td>
<td>3.54</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>-6.02</td>
<td>5.21</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The case which is most favourable to the occurrence of points of bifurcation with positive values of $\rho$ is when $\sigma$ falls to zero at the outer boundary. Let us accordingly examine this case. We have (equation (63))

$$J_{n+1} (\kappa' \alpha, \alpha) = 0, \quad \ldots \quad (86)$$

so that

$$\kappa' \alpha = \pi - \alpha, \quad \ldots \quad (87)$$
And in virtue of (86), the equation giving points of bifurcation (going back to equation (72)) is

\[ v_n = \frac{4\pi \sigma_0}{2n+1} \frac{C_n}{\kappa} J_{n-\frac{1}{2}}(\kappa' c, -\beta). \]  

(88)

so that points of bifurcation of order \( n \) are given by

\[ J_{n-\frac{1}{2}}(\pi - \alpha, -\beta) = 0. \]  

(89)

When \( n = 2 \), this becomes

\[ \tan(\pi - \alpha + \beta) = \pi - \alpha; \]

when \( n = 3 \), it is

\[ \tan(\pi - \alpha - \beta) = \frac{3(\pi - \alpha)}{3 - (\pi - \alpha)^2}. \]

On treating these equations numerically it is found that they can never be satisfied. We conclude that the non-rotating mass is stable for all displacements, subject, of course, to the condition that the density shall be everywhere positive.

\[ \text{Slow Rotation.} \]

§ 23. We consider next the stability of a rotating mass of the type under consideration, in which we are limited to \( \frac{w^2}{2\pi} \) being small compared with the density of the main mass. If we suppose that \( \sigma \), the density of the core at its outer boundary, is equal to \( \frac{w^2}{2\pi} \), we shall have a case—somewhat artificial of course—in which the density of the core is very small compared with that of the crust, and in which the equations are not too complex to admit of treatment.

We accordingly assume that \( \sigma = \frac{w^2}{2\pi} \), and the equations (75), (76), and (77) (or (72)) reduce to the same equations as in the case of no rotation (equations (83)). Thus \( \alpha, \beta \) have the same values as before, being given by the table on p. 479.

If we suppose that at the outer boundary of the crust the density falls to the small value \( \sigma_0 = \frac{w^2}{2\pi} \), then the value of \( c \), the radius of the outer boundary, is, as before, given by equations (86) or (87), and the value of \( v_n \) is still given by equation (88). Thus the analysis is exactly the same as in the case of no rotation, and there are no points of bifurcation.

It follows that, when \( \sigma_0 \) does not have this special value assigned to it, the only hope of finding points of bifurcation rests upon the gravitational tendency to instability which arises from the presence of the small layer of crust in which \( \rho \) has a value less than \( \frac{w^2}{2\pi} \). Let us pass at once to the examination of the extreme case in which \( \sigma_0 = 0 \).
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Denoting as before the density of the crust at its inner surface \((r = a)\) by \(\sigma'\), and putting \(\sigma''\), the density at the outer surface \((r = c)\) equal to zero, we have

\[
C_0\alpha^{-\frac{1}{n}} J_{\frac{\alpha}{n}} (\omega \alpha, \alpha) = \sigma' - \frac{w^2}{2\pi},
\]

\[
C_0\alpha^{-\frac{1}{n}} J_{\frac{\alpha}{n}} (\omega' \alpha, \alpha) = -\frac{w^2}{2\pi}
\]

whence, on elimination of \(C_0\),

\[
\frac{2\pi \sigma'}{w^2} = 1 - \frac{c \sin (\omega \alpha + \alpha)}{a \sin (\omega' \alpha + \alpha)}
\]

Equation (72) still gives

\[
J_{n-\frac{1}{n}} (\omega' \alpha, -\beta) = 0,
\]

as the condition for points of bifurcation, and when \(n = 2\) (the only case which appears to be worth examining), this reduces to

\[
tan (\omega \alpha + \beta) = \omega' \alpha,
\]

in which \(\beta\) is given from the table on p. 479. The procedure is to find \(\omega' \alpha\) from equation (91), and hence calculate the value of \(w^2/2\pi \sigma'\) from equation (90). The results for a few values of \(\omega \alpha\) are given in the table following (the last column is explained later):

<table>
<thead>
<tr>
<th>(\omega \alpha)</th>
<th>(\omega' \alpha)</th>
<th>(c/a)</th>
<th>(w^2/2\pi \sigma')</th>
<th>(w^2/2\pi \theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.489</td>
<td>257 27</td>
<td>\infty</td>
<td>\infty</td>
</tr>
<tr>
<td>1</td>
<td>4.475</td>
<td>256 54</td>
<td>4.475</td>
<td>0.2221</td>
</tr>
<tr>
<td>2</td>
<td>4.713</td>
<td>269 21</td>
<td>2.356</td>
<td>0.2269</td>
</tr>
<tr>
<td>3</td>
<td>5.350</td>
<td>308 16</td>
<td>1.793</td>
<td>0.2159</td>
</tr>
<tr>
<td>4</td>
<td>6.153</td>
<td>352 37</td>
<td>1.538</td>
<td>0.1821</td>
</tr>
<tr>
<td>5</td>
<td>7.026</td>
<td>402 40</td>
<td>1.405</td>
<td>0.1570</td>
</tr>
<tr>
<td>6</td>
<td>7.944</td>
<td>455 7</td>
<td>1.324</td>
<td>0.1365</td>
</tr>
<tr>
<td>7</td>
<td>8.886</td>
<td>509 9</td>
<td>1.269</td>
<td>0.1198</td>
</tr>
</tbody>
</table>

The obvious remark must at once be made that probably all the values for \(w^2/2\pi \sigma'\) are too large for results obtained by the neglect of \(w^4\) to be accurate. But apart from absolute accuracy there is an obvious tendency for the value of \(w^2/2\pi \sigma'\) to fall off as \(\omega \alpha\) increases—for lower values of \(c/a\) the symmetrical configuration becomes unstable for lower and lower values of \(w^2/2\pi \sigma'\). For \(\omega \alpha = 100\), the value of \(w^2/2\pi \sigma'\) is 0.0104.

§24. Against this, it must be noticed that the value of \(w^2/2\pi \sigma'\) is of very slight importance; what we are concerned with is the ratio of \(w^2/2\pi\) to the mean density of the whole mass. For a very rough calculation, we may assume the mean density of
the crust to be $\frac{1}{2}\pi'$, whence it follows that the mean density of the whole mass will be roughly equal to a density $\theta$ defined by

$$\theta = \frac{1}{2} \sigma' \frac{c^2 - a^2}{c^3},$$

and the value of $w^2/2\pi\theta$ will be approximately the same as the quantity $w^2/2\pi\rho$ which is computed from observations of binary stars. Values of $w^2/2\pi\theta$ are given in the last column of the table on the preceding page; the value 0.40 corresponding to $\kappa'\alpha = 0$ (no core) being inserted from the result of the previous analysis (§ 13). As before, the numbers are not numerically accurate, but their general trend may be expected to reveal the general trend of the true series of numbers. It at once appears that the values of $w^2/2\pi\theta$ are surprisingly steady: there is certainly no rapid decrease in their amount as $\kappa'\alpha$ increases.

Summary and Conclusion.

§ 25. The problem we have had under consideration has been that of testing whether the behaviour of a rotating mass of compressible heterogeneous matter differs very widely from that of the incompressible homogeneous mass which has been studied by Maclaurin, Jacobi, Poincaré, and Darwin. The result obtained can be summed up very briefly by saying that the ideal incompressible mass has been found to supply a surprisingly good model by which to study the behaviour of the more complicated systems found in astronomy. The problem especially under consideration has been that of determining the amount of rotation at which configurations of revolution (e.g., spheroids) first become unstable. In so far as we have been able to examine the question, it appears probable that the compressible mass will behave, at least up to this point, in a manner almost exactly similar to the simpler incompressible mass, and results obtained for the latter will be nearly true, both qualitatively and quantitatively, for the former. The compressible mass, set into rotation, will apparently pass through a series of flattened configurations very similar to the Maclaurin spheroids; it will then, for just about the same amount of rotation (as measured by $w^2/\rho$), leave the symmetrical form and assume a form similar to the Jacobian ellipsoids. Beyond this stage our analysis has not been able to deal with the problem. Indeed, strictly speaking, our analysis has hardly been able to carry this far. A question of importance has been whether the quasi-spheroidal form for a compressible mass does not become unstable for a much smaller value of $w^2$ than the incompressible mass, and whether the instability does not set in in a different way. These questions we have been able to answer, with, I think, a very high degree of probability, in the negative. The whole matter is of necessity one of probability only, and not of certainty, for the general heterogeneous compressible mass is not amenable to analysis until a great number of simplifying assumptions have been made.

It was first pointed out that a compressible mass has an infinitely greater number
of vibrations than an incompressible one, and as the mass is only stable when every vibration individually is stable, it might be thought that a compressible mass had more chance of being unstable—or would become unstable sooner—than the corresponding compressible mass. This has on the whole been found not to be the case, and on looking through the analysis the reason can be seen.

A vibration in a compressible mass may be regarded loosely as a system of waves; the distance from one point of zero displacement to the next may be regarded as a sort of wave-length of the vibration. The stability or instability of a vibration depends on which is the greater—the gain in elastic energy or the loss in gravitational energy when the vibration takes place. But as between a vibration of great wave-length and one of short wave-length there is this important distinction: for equal maximum amplitudes the gravitational disturbance caused by the disturbance of great wave-length is much greater than that caused by the disturbance of short wave-length, since the elements of the latter very largely neutralise one another. Thus the change in gravitational energy is enormously the greatest for disturbances of great wave-length, while it is easily seen that the changes in elastic energy are approximately the same. It follows that if the mass becomes unstable it will be through a vibration of the greatest possible wave-length, i.e., a wave-length about equal to the diameter of the mass. This general prediction is amply verified in the detailed problems that have been discussed. When we reflect that the vibrations of greatest wave-length are exactly those which are common both to compressible and incompressible masses, we see readily that, in this respect at least, compressibility is likely to make but little difference.

The vibrations of greatest wave-length are put in evidence, both in the compressible and incompressible mass, by the displacement of the surface. A vibration in which the displacement is proportional to a zonal harmonic \( P_n \) may be thought of as having a wave-length approximately equal to \( \pi a / n \). In accordance with the principle that vibrations of great wave-lengths are most effective towards instability, we should expect the lowest values of \( n \) to give the vibrations which first become unstable, and this is, in fact, found to be the case. But here a very real distinction enters between the compressible and the incompressible mass. In the incompressible mass vibrations of order \( n = 1 \) are non-existent, the displacement being purely a rigid body displacement; in the compressible mass vibrations of order \( n = 1 \) can certainly occur, and so might reasonably be expected to be the first to become unstable.

It is in point of fact known that the incompressible mass becomes unstable through vibrations of orders 2, 3, ... in turn; it is found that the compressible mass also becomes unstable through vibrations of orders 2, 3, ... in turn, the vibrations of order 1 failing completely to produce instability. The reason for this apparent anomaly can, I think, be traced in the following way. In a displacement of order 1 any spherical layer of particles will after displacement be spread uniformly over another sphere excentric to the first. The gravitational force produced by this
sphere of particles both before and after displacement is exactly nil at a point inside the sphere. Thus the gravitational field set up by a displacement of order 1 neutralises itself in a way not contemplated in the general argument outlined above. Also the vibrations of order 1 and of greatest wave-length in the interior are not available, for they represent solely a rigid body displacement.

The question of vibrations of order 1 is treated in §§ 3–7; it is shown that they may be disregarded, and we pass to the consideration of vibrations of orders 2, 3, ..., expecting (as, in fact, is found to be the case) that instability will first set in through a vibration of order 2.

It is only possible to discuss special cases, and the one which is most amenable to analysis is that in which the pressure and density are connected by Laplace's law, 

\[ p = c (\rho^2 - \sigma^2) \]

It is first proved (§§ 8–11) that, for a mass of such matter at rest, the spherical form is stable for all displacements. Later (§§ 15–22) it is shown that this is true when \( c \) varies inside the mass; it is true even up to the case which is the most likely to be unstable, in which the matter in the interior is of negligible density and the main part of the mass is collected in a surface crust—a sort of astronomical soap-bubble.

We proceed next to examine for what amount of rotation these figures will become unstable, treating first the case in which \( c \) is the same throughout the mass. Imagining \( c \) and \( \sigma \) to vary we can get a variety of types of mass. The surprising result is obtained (by something short of strict mathematical proof) that the figure which is the first to become unstable (as \( w^2/2\pi\bar{p} \) increases uniformly for them all) is the perfectly incompressible one—gravitational instability appears to act in the unexpected direction, at any rate when the degree of rotation is measured by \( w^2/2\pi\bar{p} \), \( \bar{p} \) being the mean density. As it was not possible to obtain strictly accurate figures in this case, the result was checked by considering the artificial, but physically analogous, problem of rotating cylinders of Laplacian matter, in which it was possible to obtain perfectly exact results (§ 14). The result was confirmed, and the additional information was obtained that the value of \( w^2/2\pi\bar{p} \) remains surprisingly steady through quite a wide range of compressibility (vide table on p. 471).

The physical reason for this can, I think, be understood as follows. The more compressible the matter is the more it tends to concentrate near the centre, i.e., in just those regions where the "centrifugal force" obtains, so to speak, least grip on it. Incompressibility neutralises the gravitational tendency to instability, but tends to compel the matter to place itself so that the rotational tendency to instability can act at the best advantage.

The similar problem is next investigated (§§ 23, 24) when \( c \) varies inside the mass; in particular, the limiting case of a soap-bubble-like mass is considered. Again the surprising result emerges that the value of \( w^2/2\pi\bar{p} \) needed to establish instability of the symmetrical configuration is just about the same as before (vide table, p. 481). The matter is now constrained to remain, so to speak, on the rim of a fly-wheel where
the centrifugal force can act at the best advantage and gravitational instability has full scope. If $\rho$ is the mean density of the crust, $w^2/2\pi \rho$ must obviously be less than before. But if $\bar{\rho}$ is the mean density of the whole mass, $\rho/\bar{\rho}$ is also much smaller. These two quantities march with approximately equal steps, so that $w^2/2\pi \bar{\rho}$ remains almost unaltered.

Thus we have the general result that for all the varied types of mass that have been considered the spheroidal or quasi-spheroidal form always becomes first unstable for just about the same value of $w^2/2\pi \bar{\rho}$. If, from the point of view of discovering new processes in nature, the present investigation has been somewhat barren, at least we may reflect that the work of Darwin and Poincaré has been shown, to some extent, to have an enhanced value, in that it seems to apply to the real bodies of nature and not merely to mathematical abstractions.
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